vanometer indicates a clockwise (viewed from above) current in the solenoid. Is the person inserting the magnet or pulling it out?

Conceptual Example 31.6 Application of Lenz's Law

A metal ring is placed near a solenoid, as shown in Figure 31.15a. Find the direction of the induced current in the ring (a) at the instant the switch in the circuit containing the solenoid is thrown closed, (b) after the switch has been closed for several seconds, and (c) at the instant the switch is thrown open.

Solution (a) At the instant the switch is thrown closed, the situation changes from one in which no magnetic flux passes through the ring to one in which flux passes through in the direction shown in Figure 31.15b. To counteract this change in the flux, the current induced in the ring must set up a magnetic field directed from left to right in Figure 31.15b. This requires a current directed as shown.

(b) After the switch has been closed for several seconds, no change in the magnetic flux through the loop occurs; hence, the induced current in the ring is zero.

(c) Opening the switch changes the situation from one in which magnetic flux passes through the ring to one in which there is no magnetic flux. The direction of the induced current is as shown in Figure 31.15c because current in this direction produces a magnetic field that is directed right to left and so counteracts the decrease in the field produced by the solenoid.

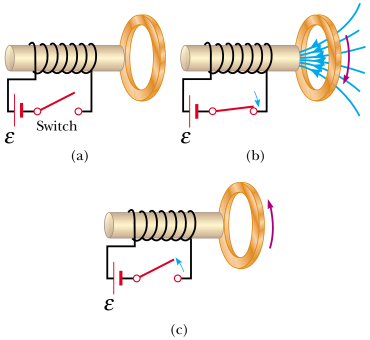


Figure 31.15

Conceptual Example 31.7 A Loop Moving Through a Magnetic Field

A rectangular metallic loop of dimensions ℓ and w and resistance R moves with constant speed v to the right, as shown in Figure 31.16a, passing through a uniform magnetic field **B** directed into the page and extending a distance $3w$ along the x axis. Defining x as the position of the right side of the loop along the x axis, plot as functions of x (a) the magnetic flux through the area enclosed by the loop, (b) the induced motional emf, and (c) the external applied force necessary to counter the magnetic force and keep v constant.

Solution (a) Figure 31.16b shows the flux through the area enclosed by the loop as a function x. Before the loop enters the field, the flux is zero. As the loop enters the field, the flux increases linearly with position until the left edge of the loop is just inside the field. Finally, the flux through the loop decreases linearly to zero as the loop leaves the field.

(b) Before the loop enters the field, no motional emf is induced in it because no field is present (Fig. 31.16c). As the right side of the loop enters the field, the magnetic flux directed into the page increases. Hence, according to Lenz's law, the induced current is counterclockwise because it must produce a magnetic field directed out of the page. The motional emf $-B\ell v$ (from Eq. 31.5) arises from the magnetic force experienced by charges in the right side of the loop. When the loop is entirely in the field, the change in magnetic flux is zero, and hence the motional emf vanishes. This happens because, once the left side of the loop enters the field, the motional emf induced in it cancels the motional emf present in the right side of the loop. As the right side of the loop leaves the field, the flux inward begins to decrease, a clockwise current is induced, and the induced emf is $B\ell v$. As soon as the left side leaves the field, the emf decreases to zero.

(c) The external force that must be applied to the loop to maintain this motion is plotted in Figure 31.16d. Before the loop enters the field, no magnetic force acts on it; hence, the applied force must be zero if v is constant. When the right side of the loop enters the field, the applied force necessary to maintain constant speed must be equal in magnitude and opposite in direction to the magnetic force exerted on that side: $F_B = -I\ell B = -B^2\ell^2 v/R$. When the loop is entirely in the field, the flux through the loop is not changing with time. Hence, the net emf induced in the loop is zero, and the current also is zero. Therefore, no external force is needed to maintain the motion. Finally, as the right side leaves the field, the applied force must be equal in magnitude and opposite

in direction to the magnetic force acting on the left side of the loop.

From this analysis, we conclude that power is supplied only when the loop is either entering or leaving the field. Furthermore, this example shows that the motional emf induced in the loop can be zero even when there is motion through the field! A motional emf is induced only when the magnetic flux through the loop *changes in time.*

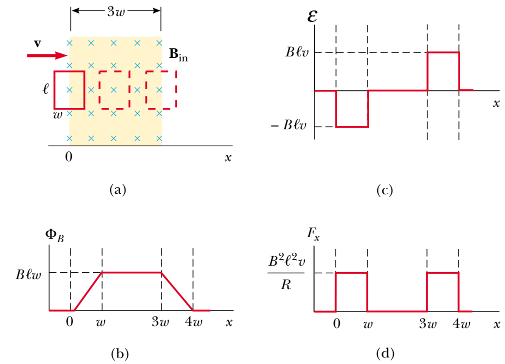


Figure 31.16 (a) A conducting rectangular loop of width w and length ℓ moving with a velocity **v** through a uniform magnetic field extending a distance $3w$. (b) Magnetic flux through the area enclosed by the loop as a function of loop position. (c) Induced emf as a function of loop position. (d) Applied force required for constant velocity as a function of loop position.

31.4 Induced EMF and Electric Fields

12.8 We have seen that a changing magnetic flux induces an emf and a current in a conducting loop. Therefore, we must conclude that **an electric field is created in the conductor as a result of the changing magnetic flux.** However, this induced electric field has two important properties that distinguish it from the electrostatic field produced by stationary charges: The induced field is nonconservative and can vary in time.

We can illustrate this point by considering a conducting loop of radius r situated in a uniform magnetic field that is perpendicular to the plane of the loop, as shown in Figure 31.17. If the magnetic field changes with time, then, according to Faraday's law (Eq. 31.1), an emf $\mathcal{E} = -d\Phi_B/dt$ is induced in the loop. The induction of a current in the loop implies the presence of an induced electric field **E**, which must be tangent to the loop because all points on the loop are equivalent. The work done in moving a test charge q once around the loop is equal to $q\mathcal{E}$. Because the electric force acting on the charge is $q\mathbf{E}$, the work done by this force in moving the charge once around the loop is $qE(2\pi r)$, where $2\pi r$ is the circumference of the loop. These two expressions for the work must be equal; therefore, we see that

$$q\mathcal{E} = qE(2\pi r)$$

$$E = \frac{\mathcal{E}}{2\pi r}$$

Using this result, along with Equation 31.1 and the fact that $\Phi_B = BA = \pi r^2 B$ for a

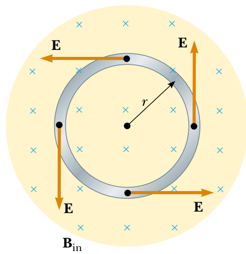


Figure 31.17 A conducting loop of radius r in a uniform magnetic field perpendicular to the plane of the loop. If **B** changes in time, an electric field is induced in a direction tangent to the circumference of the loop.

circular loop, we find that the induced electric field can be expressed as

$$E = -\frac{1}{2\pi r}\frac{d\Phi_B}{dt} = -\frac{r}{2}\frac{dB}{dt} \qquad \textbf{(31.8)}$$

If the time variation of the magnetic field is specified, we can easily calculate the induced electric field from Equation 31.8. The negative sign indicates that the induced electric field opposes the change in the magnetic field.

The emf for any closed path can be expressed as the line integral of $\mathbf{E} \cdot d\mathbf{s}$ over that path: $\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{s}$. In more general cases, E may not be constant, and the path may not be a circle. Hence, Faraday's law of induction, $\mathcal{E} = -d\Phi_B/dt$, can be written in the general form

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \qquad \textbf{(31.9)}$$

Faraday's law in general form

It is important to recognize that **the induced electric field E in Equation 31.9 is a nonconservative field that is generated by a changing magnetic field.** The field **E** that satisfies Equation 31.9 cannot possibly be an electrostatic field for the following reason: If the field were electrostatic, and hence conservative, the line integral of $\mathbf{E} \cdot d\mathbf{s}$ over a closed loop would be zero; this would be in contradiction to Equation 31.9.

EXAMPLE 31.8 Electric Field Induced by a Changing Magnetic Field in a Solenoid

A long solenoid of radius R has n turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I = I_{max} \cos \omega t$, where I_{max} is the maximum current and ω is the angular frequency of the alternating current source (Fig. 31.18). (a) Determine the magnitude of the induced electric field outside the solenoid, a distance $r > R$ from its long central axis.

Solution First let us consider an external point and take the path for our line integral to be a circle of radius r centered on the solenoid, as illustrated in Figure 31.18. By symmetry we see that the magnitude of **E** is constant on this path and that **E** is tangent to it. The magnetic flux through the area enclosed by this path is $BA = B\pi R^2$; hence, Equation 31.9 gives

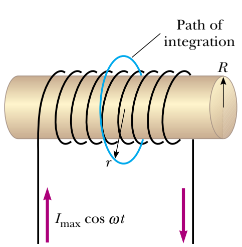


Figure 31.18 A long solenoid carrying a time-varying current given by $I = I_0 \cos \omega t$. An electric field is induced both inside and outside the solenoid.

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}(B\pi R^2) = -\pi R^2 \frac{dB}{dt}$$

$$(1) \qquad \oint \mathbf{E} \cdot d\mathbf{s} = E(2\pi r) = -\pi R^2 \frac{dB}{dt}$$

The magnetic field inside a long solenoid is given by Equation 30.17, $B = \mu_0 nI$. When we substitute $I = I_{max} \cos \omega t$ into this equation and then substitute the result into Equation (1), we find that

$$E(2\pi r) = -\pi R^2 \mu_0 n I_{max} \frac{d}{dt}(\cos \omega t) = \pi R^2 \mu_0 n I_{max} \omega \sin \omega t$$

$$(2) \qquad E = \frac{\mu_0 n I_{max} \omega R^2}{2r} \sin \omega t \qquad (\text{for } r > R)$$

Hence, the electric field varies sinusoidally with time and its amplitude falls off as $1/r$ outside the solenoid.

(b) What is the magnitude of the induced electric field inside the solenoid, a distance r from its axis?

Solution For an interior point ($r < R$), the flux threading an integration loop is given by $B\pi r^2$. Using the same procedure as in part (a), we find that

$$E(2\pi r) = -\pi r^2 \frac{dB}{dt} = \pi r^2 \mu_0 n I_{max} \omega \sin \omega t$$

$$(3) \qquad E = \frac{\mu_0 n I_{max} \omega}{2} r \sin \omega t \qquad (\text{for } r < R)$$

This shows that the amplitude of the electric field induced inside the solenoid by the changing magnetic flux through the solenoid increases linearly with r and varies sinusoidally with time.

Exercise Show that Equations (2) and (3) for the exterior and interior regions of the solenoid match at the boundary, $r = R$.

Exercise Would the electric field be different if the solenoid had an iron core?

Answer Yes, it could be much stronger because the maximum magnetic field (and thus the change in flux) through the solenoid could be thousands of times larger. (See Example 30.10.)

Optional Section

31.5 GENERATORS AND MOTORS

Turbines turn generators at a hydroelectric power plant. *(Luis Castaneda/The Image Bank)*

Electric generators are used to produce electrical energy. To understand how they work, let us consider the **alternating current** (ac) **generator,** a device that converts mechanical energy to electrical energy. In its simplest form, it consists of a loop of wire rotated by some external means in a magnetic field (Fig. 31.19a).

In commercial power plants, the energy required to rotate the loop can be derived from a variety of sources. For example, in a hydroelectric plant, falling water directed against the blades of a turbine produces the rotary motion; in a coal-fired plant, the energy released by burning coal is used to convert water to steam, and this steam is directed against the turbine blades. As a loop rotates in a magnetic field, the magnetic flux through the area enclosed by the loop changes with time; this induces an emf and a current in the loop according to Faraday's law. The ends of the loop are connected to slip rings that rotate with the loop. Connections from these slip rings, which act as output terminals of the generator, to the external circuit are made by stationary brushes in contact with the slip rings.

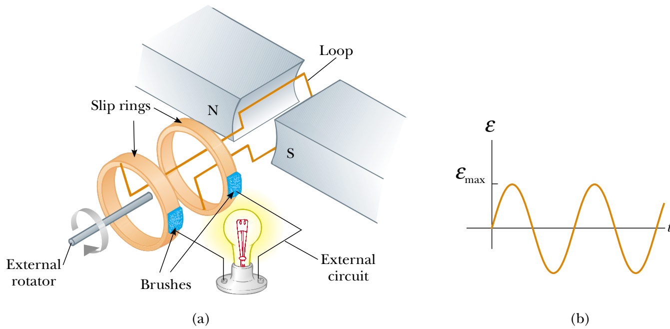


Figure 31.19 (a) Schematic diagram of an ac generator. An emf is induced in a loop that rotates in a magnetic field. (b) The alternating emf induced in the loop plotted as a function of time.

Suppose that, instead of a single turn, the loop has N turns (a more practical situation), all of the same area A, and rotates in a magnetic field with a constant angular speed ω. If θ is the angle between the magnetic field and the normal to the plane of the loop, as shown in Figure 31.20, then the magnetic flux through the loop at any time t is

$$\Phi_B = BA\cos\theta = BA\cos\omega t$$

where we have used the relationship $\theta = \omega t$ between angular displacement and angular speed (see Eq. 10.3). (We have set the clock so that $t = 0$ when $\theta = 0$.) Hence, the induced emf in the coil is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -NAB\frac{d}{dt}(\cos\omega t) = NAB\omega\sin\omega t \qquad \textbf{(31.10)}$$

This result shows that the emf varies sinusoidally with time, as was plotted in Figure 31.19b. From Equation 31.10 we see that the maximum emf has the value

$$\mathcal{E}_{\text{max}} = NAB\omega \qquad \textbf{(31.11)}$$

which occurs when $\omega t = 90°$ or $270°$. In other words, $\mathcal{E} = \mathcal{E}_{\text{max}}$ when the magnetic field is in the plane of the coil and the time rate of change of flux is a maximum. Furthermore, the emf is zero when $\omega t = 0$ or $180°$, that is, when **B** is perpendicular to the plane of the coil and the time rate of change of flux is zero.

The frequency for commercial generators in the United States and Canada is 60 Hz, whereas in some European countries it is 50 Hz. (Recall that $\omega = 2\pi f$, where f is the frequency in hertz.)

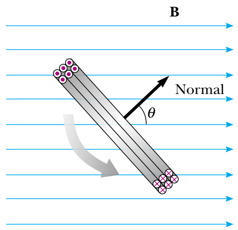


Figure 31.20 A loop enclosing an area A and containing N turns, rotating with constant angular speed ω in a magnetic field. The emf induced in the loop varies sinusoidally in time.

EXAMPLE 31.9 emf Induced in a Generator

An ac generator consists of 8 turns of wire, each of area $A = 0.090\,0\text{ m}^2$, and the total resistance of the wire is 12.0 Ω. The loop rotates in a 0.500-T magnetic field at a constant frequency of 60.0 Hz. (a) Find the maximum induced emf.

Solution First, we note that $\omega = 2\pi f = 2\pi(60.0\text{ Hz}) = 377\text{ s}^{-1}$. Thus, Equation 31.11 gives

$$\mathcal{E}_{\text{max}} = NAB\omega = 8(0.090\,0\text{ m}^2)(0.500\text{ T})(377\text{ s}^{-1}) = \boxed{136\text{ V}}$$

(b) What is the maximum induced current when the output terminals are connected to a low-resistance conductor?

Solution From Equation 27.8 and the results to part (a), we have

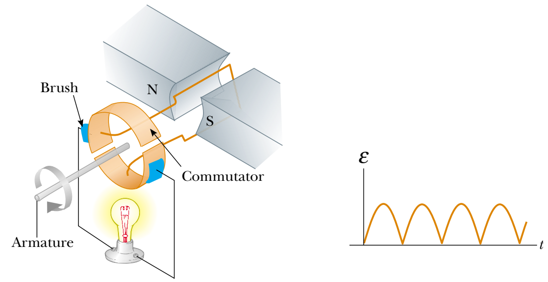
$$I_{\text{max}} = \frac{\mathcal{E}_{\text{max}}}{R} = \frac{136\text{ V}}{12.0\ \Omega} = \boxed{11.3\text{ A}}$$

Exercise Determine how the induced emf and induced current vary with time.

Answer $\mathcal{E} = \mathcal{E}_{\text{max}}\sin\omega t = (136\text{ V})\sin 377t$; $I = I_{\text{max}}\sin\omega t = (11.3\text{ A})\sin 377t$.

The **direct current** (dc) **generator** is illustrated in Figure 31.21a. Such generators are used, for instance, in older cars to charge the storage batteries used. The components are essentially the same as those of the ac generator except that the contacts to the rotating loop are made using a split ring called a *commutator.*

In this configuration, the output voltage always has the same polarity and pulsates with time, as shown in Figure 31.21b. We can understand the reason for this by noting that the contacts to the split ring reverse their roles every half cycle. At the same time, the polarity of the induced emf reverses; hence, the polarity of the



(a) (b)

Figure 31.21 (a) Schematic diagram of a dc generator. (b) The magnitude of the emf varies in time but the polarity never changes.

split ring (which is the same as the polarity of the output voltage) remains the same.

A pulsating dc current is not suitable for most applications. To obtain a more steady dc current, commercial dc generators use many coils and commutators distributed so that the sinusoidal pulses from the various coils are out of phase. When these pulses are superimposed, the dc output is almost free of fluctuations.

Motors are devices that convert electrical energy to mechanical energy. Essentially, a motor is a generator operating in reverse. Instead of generating a current by rotating a loop, a current is supplied to the loop by a battery, and the torque acting on the current-carrying loop causes it to rotate.

Useful mechanical work can be done by attaching the rotating armature to some external device. However, as the loop rotates in a magnetic field, the changing magnetic flux induces an emf in the loop; this induced emf always acts to reduce the current in the loop. If this were not the case, Lenz's law would be violated. The back emf increases in magnitude as the rotational speed of the armature increases. (The phrase *back emf* is used to indicate an emf that tends to reduce the supplied current.) Because the voltage available to supply current equals the difference between the supply voltage and the back emf, the current in the rotating coil is limited by the back emf.

When a motor is turned on, there is initially no back emf; thus, the current is very large because it is limited only by the resistance of the coils. As the coils begin to rotate, the induced back emf opposes the applied voltage, and the current in the coils is reduced. If the mechanical load increases, the motor slows down; this causes the back emf to decrease. This reduction in the back emf increases the current in the coils and therefore also increases the power needed from the external voltage source. For this reason, the power requirements for starting a motor and for running it are greater for heavy loads than for light ones. If the motor is allowed to run under no mechanical load, the back emf reduces the current to a value just large enough to overcome energy losses due to internal energy and friction. If a very heavy load jams the motor so that it cannot rotate, the lack of a back emf can lead to dangerously high current in the motor's wire. If the problem is not corrected, a fire could result.

EXAMPLE 31.10 The Induced Current in a Motor

Assume that a motor in which the coils have a total resistance of 10 Ω is supplied by a voltage of 120 V. When the motor is running at its maximum speed, the back emf is 70 V. Find the current in the coils (a) when the motor is turned on and (b) when it has reached maximum speed.

Solution (a) When the motor is turned on, the back emf is zero (because the coils are motionless). Thus, the current in the coils is a maximum and equal to

$$I = \frac{\mathcal{E}}{R} = \frac{120\ \text{V}}{10\ \Omega} = \boxed{12\ \text{A}}$$

(b) At the maximum speed, the back emf has its maximum value. Thus, the effective supply voltage is that of the external source minus the back emf. Hence, the current is reduced to

$$I = \frac{\mathcal{E} - \mathcal{E}_{\text{back}}}{R} = \frac{120\ \text{V} - 70\ \text{V}}{10\ \Omega} = \frac{50\ \text{V}}{10\ \Omega} = \boxed{5.0\ \text{A}}$$

Exercise If the current in the motor is 8.0 A at some instant, what is the back emf at this time?

Answer 40 V.

Optional Section

31.6 EDDY CURRENTS

As we have seen, an emf and a current are induced in a circuit by a changing magnetic flux. In the same manner, circulating currents called **eddy currents** are induced in bulk pieces of metal moving through a magnetic field. This can easily be demonstrated by allowing a flat copper or aluminum plate attached at the end of a rigid bar to swing back and forth through a magnetic field (Fig. 31.22). As the plate enters the field, the changing magnetic flux induces an emf in the plate, which in turn causes the free electrons in the plate to move, producing the swirling eddy currents. According to Lenz's law, the direction of the eddy currents must oppose the change that causes them. For this reason, the eddy currents must produce effective magnetic poles on the plate, which are repelled by the poles of the magnet; this gives rise to a repulsive force that opposes the motion of the plate. (If the opposite were true, the plate would accelerate and its energy would increase after each swing, in violation of the law of conservation of energy.) As you may have noticed while carrying out the QuickLab on page 997, you can "feel" the retarding force by pulling a copper or aluminum sheet through the field of a strong magnet.

QuickLab

Hang a strong magnet from two strings so that it swings back and forth in a plane. Start it oscillating and determine approximately how much time passes before it stops swinging. Start it oscillating again and quickly bring the flat surface of an aluminum cooking sheet up to within a millimeter of the plane of oscillation, taking care not to touch the magnet. How long does it take the oscillating magnet to stop now?

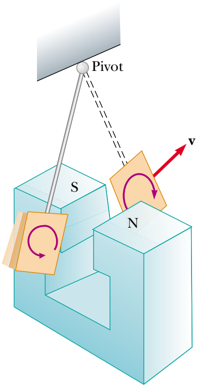


Figure 31.22 Formation of eddy currents in a conducting plate moving through a magnetic field. As the plate enters or leaves the field, the changing magnetic flux induces an emf, which causes eddy currents in the plate.

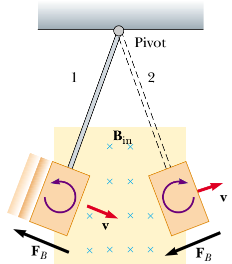


Figure 31.23 As the conducting plate enters the field (position 1), the eddy currents are counterclockwise. As the plate leaves the field (position 2), the currents are clockwise. In either case, the force on the plate is opposite the velocity, and eventually the plate comes to rest.

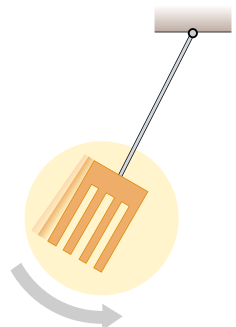


Figure 31.24 When slots are cut in the conducting plate, the eddy currents are reduced and the plate swings more freely through the magnetic field.

As indicated in Figure 31.23, with **B** directed into the page, the induced eddy current is counterclockwise as the swinging plate enters the field at position 1. This is because the external magnetic flux into the page through the plate is increasing, and hence by Lenz's law the induced current must provide a magnetic flux out of the page. The opposite is true as the plate leaves the field at position 2, where the current is clockwise. Because the induced eddy current always produces a magnetic retarding force $\mathbf{F}_B$ when the plate enters or leaves the field, the swinging plate eventually comes to rest.

If slots are cut in the plate, as shown in Figure 31.24, the eddy currents and the corresponding retarding force are greatly reduced. We can understand this by realizing that the cuts in the plate prevent the formation of any large current loops.

The braking systems on many subway and rapid-transit cars make use of electromagnetic induction and eddy currents. An electromagnet attached to the train is positioned near the steel rails. (An electromagnet is essentially a solenoid with an iron core.) The braking action occurs when a large current is passed through the electromagnet. The relative motion of the magnet and rails induces eddy currents in the rails, and the direction of these currents produces a drag force on the moving train. The loss in mechanical energy of the train is transformed to internal energy in the rails and wheels. Because the eddy currents decrease steadily in magnitude as the train slows down, the braking effect is quite smooth. Eddy-current brakes are also used in some mechanical balances and in various machines. Some power tools use eddy currents to stop rapidly spinning blades once the device is turned off.

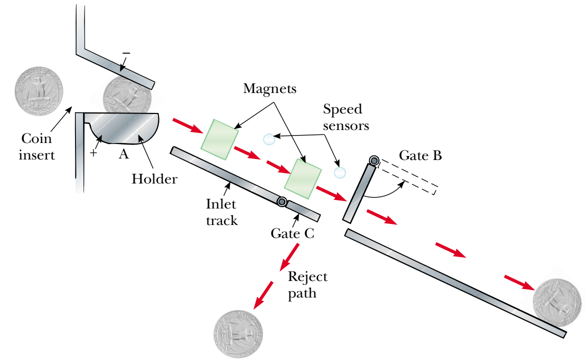


Figure 31.25 As the coin enters the vending machine, a potential difference is applied across the coin at A, and its resistance is measured. If the resistance is acceptable, the holder drops down, releasing the coin and allowing it to roll along the inlet track. Two magnets induce eddy currents in the coin, and magnetic forces control its speed. If the speed sensors indicate that the coin has the correct speed, gate B swings up to allow the coin to be accepted. If the coin is not moving at the correct speed, gate C opens to allow the coin to follow the reject path.

Eddy currents are often undesirable because they represent a transformation of mechanical energy to internal energy. To reduce this energy loss, moving conducting parts are often laminated—that is, they are built up in thin layers separated by a nonconducting material such as lacquer or a metal oxide. This layered structure increases the resistance of the possible paths of the eddy currents and effectively confines the currents to individual layers. Such a laminated structure is used in transformer cores and motors to minimize eddy currents and thereby increase the efficiency of these devices.

Even a task as simple as buying a candy bar from a vending machine involves eddy currents, as shown in Figure 31.25. After entering the slot, a coin is stopped momentarily while its electrical resistance is checked. If its resistance falls within an acceptable range, the coin is allowed to continue down a ramp and through a magnetic field. As it moves through the field, eddy currents are produced in the coin, and magnetic forces slow it down slightly. How much it is slowed down depends on its metallic composition. Sensors measure the coin's speed after it moves past the magnets, and this speed is compared with expected values. If the coin is legal and passes these tests, a gate is opened and the coin is accepted; otherwise, a second gate moves it into the reject path.

31.7 MAXWELL'S WONDERFUL EQUATIONS

12.10 We conclude this chapter by presenting four equations that are regarded as the basis of all electrical and magnetic phenomena. These equations, developed by James Clerk Maxwell, are as fundamental to electromagnetic phenomena as Newton's laws are to mechanical phenomena. In fact, the theory that Maxwell developed was more far-reaching than even he imagined because it turned out to be in agreement with the special theory of relativity, as Einstein showed in 1905.

Maxwell's equations represent the laws of electricity and magnetism that we have already discussed, but they have additional important consequences. In Chapter 34 we shall show that these equations predict the existence of electromagnetic waves (traveling patterns of electric and magnetic fields), which travel with a speed $c = 1/\sqrt{\mu_0 \epsilon_0} = 3.00 \times 10^8$ m/s, the speed of light. Furthermore, the theory shows that such waves are radiated by accelerating charges.

For simplicity, we present **Maxwell's equations** as applied to free space, that is, in the absence of any dielectric or magnetic material. The four equations are

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \qquad \textbf{(31.12)}$$

Gauss's law

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \qquad \textbf{(31.13)}$$

Gauss's law in magnetism

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \qquad \textbf{(31.14)}$$

Faraday's law

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \qquad \textbf{(31.15)}$$

Ampère–Maxwell law

Equation 31.12 is Gauss's law: **The total electric flux through any closed surface equals the net charge inside that surface divided by ϵ_0.** This law relates an electric field to the charge distribution that creates it.

Equation 31.13, which can be considered Gauss's law in magnetism, states that **the net magnetic flux through a closed surface is zero.** That is, the number of magnetic field lines that enter a closed volume must equal the number that leave that volume. This implies that magnetic field lines cannot begin or end at any point. If they did, it would mean that isolated magnetic monopoles existed at those points. The fact that isolated magnetic monopoles have not been observed in nature can be taken as a confirmation of Equation 31.13.

Equation 31.14 is Faraday's law of induction, which describes the creation of an electric field by a changing magnetic flux. This law states that **the emf, which is the line integral of the electric field around any closed path, equals the rate of change of magnetic flux through any surface area bounded by that path.** One consequence of Faraday's law is the current induced in a conducting loop placed in a time-varying magnetic field.

Equation 31.15, usually called the Ampère–Maxwell law, is the generalized form of Ampère's law, which describes the creation of a magnetic field by an electric field and electric currents: **The line integral of the magnetic field around any closed path is the sum of μ_0 times the net current through that path and $\epsilon_0\mu_0$ times the rate of change of electric flux through any surface bounded by that path.**

Once the electric and magnetic fields are known at some point in space, the force acting on a particle of charge q can be calculated from the expression

Lorentz force law

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \qquad \textbf{(31.16)}$$

This relationship is called the **Lorentz force law.** (We saw this relationship earlier as Equation 29.16.) Maxwell's equations, together with this force law, completely describe all classical electromagnetic interactions.

It is interesting to note the symmetry of Maxwell's equations. Equations 31.12 and 31.13 are symmetric, apart from the absence of the term for magnetic monopoles in Equation 31.13. Furthermore, Equations 31.14 and 31.15 are symmetric in that the line integrals of **E** and **B** around a closed path are related to the rate of change of magnetic flux and electric flux, respectively. "Maxwell's wonderful equations," as they were called by John R. Pierce,[3] are of fundamental importance not only to electromagnetism but to all of science. Heinrich Hertz once wrote, "One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we put into them."

SUMMARY

Faraday's law of induction states that the emf induced in a circuit is directly proportional to the time rate of change of magnetic flux through the circuit:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \qquad \textbf{(31.1)}$$

where $\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$ is the magnetic flux.

[3] John R. Pierce, *Electrons and Waves*, New York, Doubleday Science Study Series, 1964. Chapter 6 of this interesting book is recommended as supplemental reading.

When a conducting bar of length ℓ moves at a velocity $\mathbf{v}$ through a magnetic field $\mathbf{B}$, where $\mathbf{B}$ is perpendicular to the bar and to $\mathbf{v}$, the **motional emf** induced in the bar is

$$\mathcal{E} = -B\ell v \quad \textbf{(31.5)}$$

Lenz's law states that the induced current and induced emf in a conductor are in such a direction as to oppose the change that produced them.

A general form of **Faraday's law of induction** is

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \quad \textbf{(31.9)}$$

where $\mathbf{E}$ is the nonconservative electric field that is produced by the changing magnetic flux.

When used with the Lorentz force law, $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$, **Maxwell's equations** describe all electromagnetic phenomena:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \quad \textbf{(31.12)}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \quad \textbf{(31.13)}$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \quad \textbf{(31.14)}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \quad \textbf{(31.15)}$$

The Ampère–Maxwell law (Eq. 31.15) describes how a magnetic field can be produced by both a conduction current and a changing electric flux.

QUESTIONS

1. A loop of wire is placed in a uniform magnetic field. For what orientation of the loop is the magnetic flux a maximum? For what orientation is the flux zero? Draw pictures of these two situations.
2. As the conducting bar shown in Figure Q31.2 moves to the right, an electric field directed downward is set up in the bar. Explain why the electric field would be upward if the bar were to move to the left.
3. As the bar shown in Figure Q31.2 moves in a direction perpendicular to the field, is an applied force required to keep it moving with constant speed? Explain.
4. The bar shown in Figure Q31.4 moves on rails to the right with a velocity $\mathbf{v}$, and the uniform, constant magnetic field is directed out of the page. Why is the induced current clockwise? If the bar were moving to the left, what would be the direction of the induced current?
5. Explain why an applied force is necessary to keep the bar shown in Figure Q31.4 moving with a constant speed.
6. A large circular loop of wire lies in the horizontal plane. A bar magnet is dropped through the loop. If the axis of the magnet remains horizontal as it falls, describe the emf induced in the loop. How is the situation altered if the axis of the magnet remains vertical as it falls?

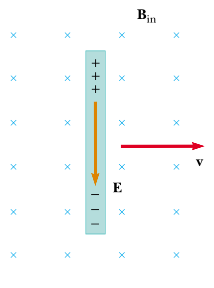


Figure Q31.2 (Questions 2 and 3).

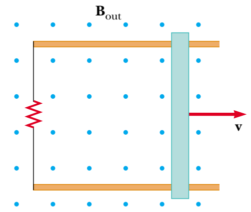


Figure Q31.4 (Questions 4 and 5).

7. When a small magnet is moved toward a solenoid, an emf is induced in the coil. However, if the magnet is moved around inside a toroid, no emf is induced. Explain.
8. Will dropping a magnet down a long copper tube produce a current in the walls of the tube? Explain.
9. How is electrical energy produced in dams (that is, how is the energy of motion of the water converted to alternating current electricity)?
10. In a beam–balance scale, an aluminum plate is sometimes used to slow the oscillations of the beam near equilibrium. The plate is mounted at the end of the beam and moves between the poles of a small horseshoe magnet attached to the frame. Why are the oscillations strongly damped near equilibrium?
11. What happens when the rotational speed of a generator coil is increased?
12. Could a current be induced in a coil by the rotation of a magnet inside the coil? If so, how?
13. When the switch shown in Figure Q31.13a is closed, a current is set up in the coil, and the metal ring springs upward (Fig. Q31.13b). Explain this behavior.

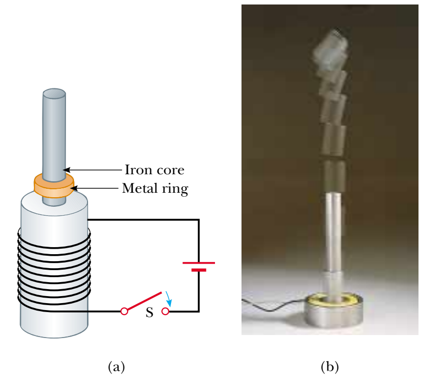


Figure Q31.13 (Questions 13 and 14). *(Photo courtesy of Central Scientific Company)*

14. Assume that the battery shown in Figure Q31.13a is replaced by an alternating current source and that the switch is held closed. If held down, the metal ring on top of the solenoid becomes hot. Why?
15. Do Maxwell's equations allow for the existence of magnetic monopoles? Explain.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging   ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/**   = Computer useful in solving problem   = Interactive Physics
 = paired numerical/symbolic problems

Section 31.1 Faraday's Law of Induction
Section 31.2 Motional emf
Section 31.3 Lenz's Law

1. A 50-turn rectangular coil of dimensions 5.00 cm × 10.0 cm is allowed to fall from a position where $B = 0$ to a new position where $B = 0.500$ T and is directed perpendicular to the plane of the coil. Calculate the magnitude of the average emf induced in the coil if the displacement occurs in 0.250 s.
2. A flat loop of wire consisting of a single turn of cross-sectional area 8.00 cm² is perpendicular to a magnetic field that increases uniformly in magnitude from 0.500 T to 2.50 T in 1.00 s. What is the resulting induced current if the loop has a resistance of 2.00 Ω?
3. A 25-turn circular coil of wire has a diameter of 1.00 m. It is placed with its axis along the direction of the Earth's magnetic field of 50.0 μT, and then in 0.200 s it is flipped 180°. An average emf of what magnitude is generated in the coil?
4. A rectangular loop of area A is placed in a region where the magnetic field is perpendicular to the plane of the loop. The magnitude of the field is allowed to vary in time according to the expression $B = B_{max}e^{-t/\tau}$, where B_{max} and τ are constants. The field has the constant value B_{max} for $t < 0$. (a) Use Faraday's law to show that the emf induced in the loop is given by

$$\mathcal{E} = (AB_{max}/\tau)e^{-t/\tau}$$

(b) Obtain a numerical value for $\mathcal{E}$ at $t = 4.00$ s when

$A = 0.160\ m^2$, $B_{max} = 0.350$ T, and $\tau = 2.00$ s. (c) For the values of A, B_{max}, and τ given in part (b), what is the maximum value of $\mathcal{E}$?

WEB **5.** A strong electromagnet produces a uniform field of 1.60 T over a cross-sectional area of 0.200 m^2. A coil having 200 turns and a total resistance of 20.0 Ω is placed around the electromagnet. The current in the electromagnet is then smoothly decreased until it reaches zero in 20.0 ms. What is the current induced in the coil?

6. A magnetic field of 0.200 T exists within a solenoid of 500 turns and a diameter of 10.0 cm. How rapidly (that is, within what period of time) must the field be reduced to zero if the average induced emf within the coil during this time interval is to be 10.0 kV?

WEB **7.** An aluminum ring with a radius of 5.00 cm and a resistance of 3.00×10^{-4} Ω is placed on top of a long air-core solenoid with 1 000 turns per meter and a radius of 3.00 cm, as shown in Figure P31.7. Assume that the axial component of the field produced by the solenoid over the area of the end of the solenoid is one-half as strong as at the center of the solenoid. Assume that the solenoid produces negligible field outside its cross-sectional area. (a) If the current in the solenoid is increasing at a rate of 270 A/s, what is the induced current in the ring? (b) At the center of the ring, what is the magnetic field produced by the induced current in the ring? (c) What is the direction of this field?

8. An aluminum ring of radius r_1 and resistance R is placed on top of a long air-core solenoid with n turns per meter and smaller radius r_2, as shown in Figure P31.7. Assume that the axial component of the field produced by the solenoid over the area of the end of the solenoid is one-half as strong as at the center of the solenoid. Assume that the solenoid produces negligible field outside its cross-sectional area. (a) If the current in the solenoid is increasing at a rate of $\Delta I/\Delta t$, what is the induced current in the ring? (b) At the center of the ring, what is the magnetic field produced by the induced current in the ring? (c) What is the direction of this field?

9. A loop of wire in the shape of a rectangle of width w and length L and a long, straight wire carrying a current I lie on a tabletop as shown in Figure P31.9. (a) Determine the magnetic flux through the loop due to the current I. (b) Suppose that the current is changing with time according to $I = a + bt$, where a and b are constants. Determine the induced emf in the loop if $b = 10.0$ A/s, $h = 1.00$ cm, $w = 10.0$ cm, and $L = 100$ cm. What is the direction of the induced current in the rectangle?

10. A coil of 15 turns and radius 10.0 cm surrounds a long solenoid of radius 2.00 cm and 1.00×10^3 turns per meter (Fig. P31.10). If the current in the solenoid changes as $I = (5.00\text{ A}) \sin(120t)$, find the induced emf in the 15-turn coil as a function of time.

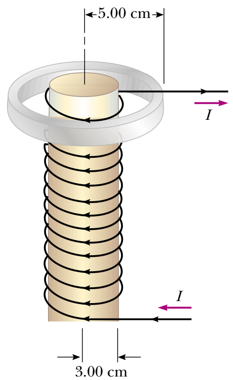


Figure P31.7 Problems 7 and 8.

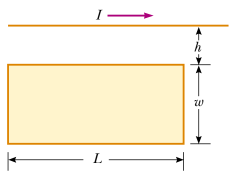


Figure P31.9 Problems 9 and 73.

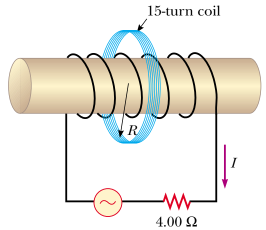


Figure P31.10

11. Find the current through section PQ of length $a = 65.0$ cm shown in Figure P31.11. The circuit is located in a magnetic field whose magnitude varies with time according to the expression $B = (1.00 \times 10^{-3}\text{ T/s})t$. Assume that the resistance per length of the wire is 0.100 Ω/m.

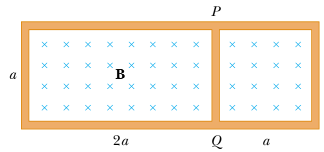


Figure P31.11

12. A 30-turn circular coil of radius 4.00 cm and resistance 1.00 Ω is placed in a magnetic field directed perpendicular to the plane of the coil. The magnitude of the magnetic field varies in time according to the expression $B = 0.010\,0t + 0.040\,0t^2$, where t is in seconds and B is in tesla. Calculate the induced emf in the coil at $t = 5.00$ s.

13. A long solenoid has 400 turns per meter and carries a current $I = (30.0\text{ A})(1 - e^{-1.60t})$. Inside the solenoid and coaxial with it is a coil that has a radius of 6.00 cm and consists of a total of 250 turns of fine wire (Fig. P31.13). What emf is induced in the coil by the changing current?

14. A long solenoid has n turns per meter and carries a current $I = I_{max}(1 - e^{-\alpha t})$. Inside the solenoid and coaxial with it is a coil that has a radius R and consists of a total of N turns of fine wire (see Fig. P31.13). What emf is induced in the coil by the changing current?

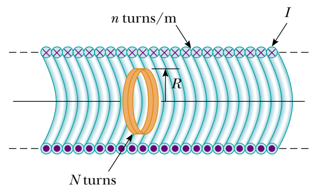


Figure P31.13 Problems 13 and 14.

15. A coil formed by wrapping 50 turns of wire in the shape of a square is positioned in a magnetic field so that the normal to the plane of the coil makes an angle of 30.0° with the direction of the field. When the magnetic field is increased uniformly from 200 μT to 600 μT in 0.400 s, an emf of magnitude 80.0 mV is induced in the coil. What is the total length of the wire?

16. A closed loop of wire is given the shape of a circle with a radius of 0.500 m. It lies in a plane perpendicular to a uniform magnetic field of magnitude 0.400 T. If in 0.100 s the wire loop is reshaped into a square but remains in the same plane, what is the magnitude of the average induced emf in the wire during this time?

17. A toroid having a rectangular cross-section ($a = 2.00$ cm by $b = 3.00$ cm) and inner radius $R = 4.00$ cm consists of 500 turns of wire that carries a current $I = I_{max} \sin \omega t$, with $I_{max} = 50.0$ A and a frequency $f = \omega/2\pi = 60.0$ Hz. A coil that consists of 20 turns of wire links with the toroid, as shown in Figure P31.17. Determine the emf induced in the coil as a function of time.

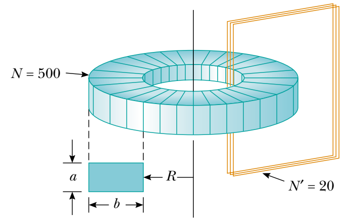


Figure P31.17

18. A single-turn, circular loop of radius R is coaxial with a long solenoid of radius r and length ℓ and having N turns (Fig. P31.18). The variable resistor is changed so that the solenoid current decreases linearly from I_1 to I_2 in an interval Δt. Find the induced emf in the loop.

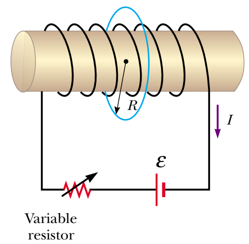


Figure P31.18

19. A circular coil enclosing an area of 100 cm^2 is made of 200 turns of copper wire, as shown in Figure P31.19. Ini-

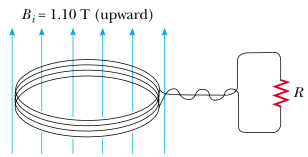


Figure P31.19

tially, a 1.10-T uniform magnetic field points in a perpendicular direction upward through the plane of the coil. The direction of the field then reverses. During the time the field is changing its direction, how much charge flows through the coil if $R = 5.00\ \Omega$?

20. Consider the arrangement shown in Figure P31.20. Assume that $R = 6.00\ \Omega$, $\ell = 1.20$ m, and a uniform 2.50-T magnetic field is directed into the page. At what speed should the bar be moved to produce a current of 0.500 A in the resistor?

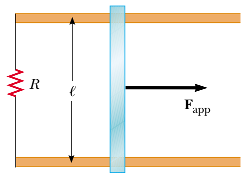


Figure P31.20 Problems 20, 21, and 22.

21. Figure P31.20 shows a top view of a bar that can slide without friction. The resistor is 6.00 Ω and a 2.50-T magnetic field is directed perpendicularly downward, into the paper. Let $\ell = 1.20$ m. (a) Calculate the applied force required to move the bar to the right at a constant speed of 2.00 m/s. (b) At what rate is energy delivered to the resistor?

22. A conducting rod of length ℓ moves on two horizontal, frictionless rails, as shown in Figure P31.20. If a constant force of 1.00 N moves the bar at 2.00 m/s through a magnetic field **B** that is directed into the page, (a) what is the current through an 8.00-Ω resistor R? (b) What is the rate at which energy is delivered to the resistor? (c) What is the mechanical power delivered by the force $\mathbf{F}_{app}$?

23. A Boeing-747 jet with a wing span of 60.0 m is flying horizontally at a speed of 300 m/s over Phoenix, Arizona, at a location where the Earth's magnetic field is 50.0 μT at 58.0° below the horizontal. What voltage is generated between the wingtips?

24. The square loop in Figure P31.24 is made of wires with total series resistance 10.0 Ω. It is placed in a uniform 0.100-T magnetic field directed perpendicular into the plane of the paper. The loop, which is hinged at each corner, is pulled as shown until the separation between points A and B is 3.00 m. If this process takes 0.100 s, what is the average current generated in the loop? What is the direction of the current?

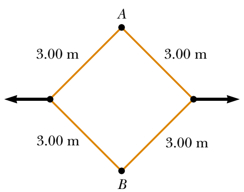


Figure P31.24

25. A helicopter has blades with a length of 3.00 m extending outward from a central hub and rotating at 2.00 rev/s. If the vertical component of the Earth's magnetic field is 50.0 μT, what is the emf induced between the blade tip and the center hub?

26. Use Lenz's law to answer the following questions concerning the direction of induced currents: (a) What is the direction of the induced current in resistor R shown in Figure P31.26a when the bar magnet is moved to the left? (b) What is the direction of the current induced in the resistor R right after the switch S in Figure P31.26b is closed? (c) What is the direction of the induced current in R when the current I in Figure P31.26c decreases rapidly to zero? (d) A copper bar is moved to the right while its axis is maintained in a direction perpendicular to a magnetic field, as shown in Figure P31.26d. If the top of the bar becomes positive relative to the bottom, what is the direction of the magnetic field?

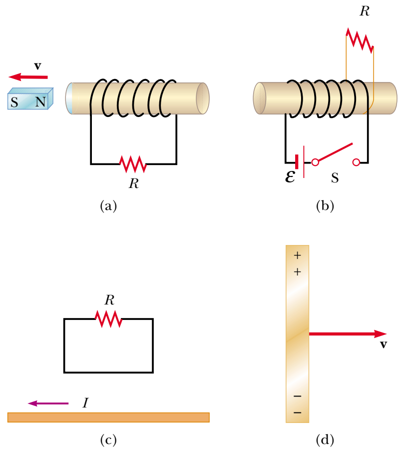


Figure P31.26

27. A rectangular coil with resistance R has N turns, each of length ℓ and width w as shown in Figure P31.27. The coil moves into a uniform magnetic field **B** with a velocity **v**. What are the magnitude and direction of the resultant force on the coil (a) as it enters the magnetic field, (b) as it moves within the field, and (c) as it leaves the field?

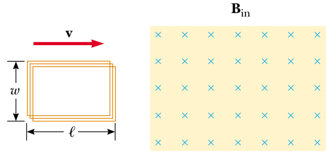


Figure P31.27

28. In 1832 Faraday proposed that the apparatus shown in Figure P31.28 could be used to generate electric current from the water flowing in the Thames River.[4] Two conducting plates of lengths a and widths b are placed facing each other on opposite sides of the river, a distance w apart, and are immersed entirely. The flow velocity of the river is **v** and the vertical component of the Earth's magnetic field is B. (a) Show that the current in the load resistor R is

$$I = \frac{abvB}{\rho + abR/w}$$

where ρ is the electrical resistivity of the water. (b) Calculate the short-circuit current ($R = 0$) if $a = 100$ m, $b = 5.00$ m, $v = 3.00$ m/s, $B = 50.0\ \mu$T, and $\rho = 100\ \Omega \cdot$m.

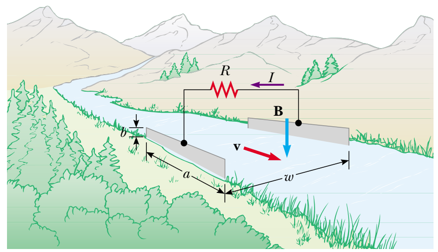


Figure P31.28

29. In Figure P31.29, the bar magnet is moved toward the loop. Is $V_a - V_b$ positive, negative, or zero? Explain.

30. A metal bar spins at a constant rate in the magnetic field of the Earth as in Figure 31.10. The rotation occurs in a region where the component of the Earth's magnetic field perpendicular to the plane of rotation is 3.30×10^{-5} T. If the bar is 1.00 m in length and its angular speed is 5.00 π rad/s, what potential difference is developed between its ends?

[4] The idea for this problem and Figure P31.28 is from Oleg D. Jefimenko, *Electricity and Magnetism: An Introduction to the Theory of Electric and Magnetic Fields.* Star City, WV, Electret Scientific Co., 1989.

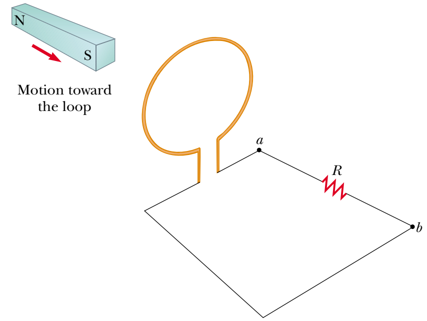


Figure P31.29

31. Two parallel rails with negligible resistance are 10.0 cm apart and are connected by a 5.00-Ω resistor. The circuit also contains two metal rods having resistances of 10.0 Ω and 15.0 Ω sliding along the rails (Fig. P31.31). The rods are pulled away from the resistor at constant speeds 4.00 m/s and 2.00 m/s, respectively. A uniform magnetic field of magnitude 0.010 0 T is applied perpendicular to the plane of the rails. Determine the current in the 5.00-Ω resistor.

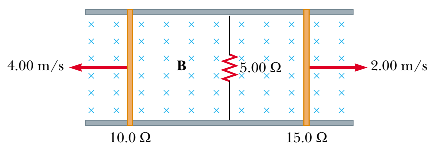


Figure P31.31

Section 31.4 Induced emf and Electric Fields

32. For the situation described in Figure P31.32, the magnetic field changes with time according to the expression $B = (2.00t^3 - 4.00t^2 + 0.800)$ T, and $r_2 = 2R = 5.00$ cm. (a) Calculate the magnitude and direction of

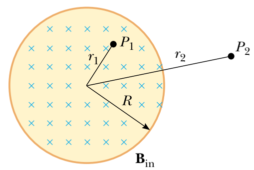


Figure P31.32 Problems 32 and 33.

the force exerted on an electron located at point P_2 when $t = 2.00$ s. (b) At what time is this force equal to zero?

33. A magnetic field directed into the page changes with time according to $B = (0.030\,0t^2 + 1.40)$ T, where t is in seconds. The field has a circular cross-section of radius $R = 2.50$ cm (see Fig. P31.32). What are the magnitude and direction of the electric field at point P_1 when $t = 3.00$ s and $r_1 = 0.020\,0$ m?

34. A solenoid has a radius of 2.00 cm and 1 000 turns per meter. Over a certain time interval the current varies with time according to the expression $I = 3e^{0.2t}$, where I is in amperes and t is in seconds. Calculate the electric field 5.00 cm from the axis of the solenoid at $t = 10.0$ s.

35. A long solenoid with 1 000 turns per meter and radius 2.00 cm carries an oscillating current $I = (5.00\text{ A})\sin(100\pi t)$. (a) What is the electric field induced at a radius $r = 1.00$ cm from the axis of the solenoid? (b) What is the direction of this electric field when the current is increasing counterclockwise in the coil?

(Optional)

Section 31.5 Generators and Motors

36. In a 250-turn automobile alternator, the magnetic flux in each turn is $\Phi_B = (2.50 \times 10^{-4}\text{ T}\cdot\text{m}^2)\cos(\omega t)$, where ω is the angular speed of the alternator. The alternator is geared to rotate three times for each engine revolution. When the engine is running at an angular speed of 1 000 rev/min, determine (a) the induced emf in the alternator as a function of time and (b) the maximum emf in the alternator.

WEB **37.** A coil of area 0.100 m^2 is rotating at 60.0 rev/s with the axis of rotation perpendicular to a 0.200-T magnetic field. (a) If there are 1 000 turns on the coil, what is the maximum voltage induced in it? (b) What is the orientation of the coil with respect to the magnetic field when the maximum induced voltage occurs?

38. A square coil (20.0 cm × 20.0 cm) that consists of 100 turns of wire rotates about a vertical axis at 1 500 rev/min, as indicated in Figure P31.38. The horizontal component of the Earth's magnetic field at the location of the coil is 2.00×10^{-5} T. Calculate the maximum emf induced in the coil by this field.

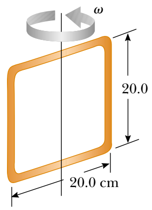


Figure P31.38

39. A long solenoid, with its axis along the x axis, consists of 200 turns per meter of wire that carries a steady current of 15.0 A. A coil is formed by wrapping 30 turns of thin wire around a circular frame that has a radius of 8.00 cm. The coil is placed inside the solenoid and mounted on an axis that is a diameter of the coil and coincides with the y axis. The coil is then rotated with an angular speed of 4.00π rad/s. (The plane of the coil is in the yz plane at $t = 0$.) Determine the emf developed in the coil as a function of time.

40. A bar magnet is spun at constant angular speed ω around an axis, as shown in Figure P31.40. A flat rectangular conducting loop surrounds the magnet, and at $t = 0$, the magnet is oriented as shown. Make a qualitative graph of the induced current in the loop as a function of time, plotting counterclockwise currents as positive and clockwise currents as negative.

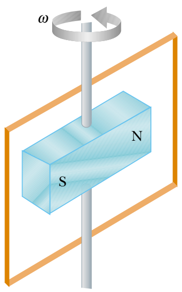


Figure P31.40

41. (a) What is the maximum torque delivered by an electric motor if it has 80 turns of wire wrapped on a rectangular coil of dimensions 2.50 cm by 4.00 cm? Assume that the motor uses 10.0 A of current and that a uniform 0.800-T magnetic field exists within the motor. (b) If the motor rotates at 3 600 rev/min, what is the peak power produced by the motor?

42. A semicircular conductor of radius $R = 0.250$ m is rotated about the axis AC at a constant rate of 120 rev/min (Fig. P31.42). A uniform magnetic field in all of the lower half of the figure is directed out of the plane of rotation and has a magnitude of 1.30 T. (a) Calculate the maximum value of the emf induced in the conductor. (b) What is the value of the average induced emf for each complete rotation? (c) How would the answers to parts (a) and (b) change if **B** were allowed to extend a distance R above the axis of rotation? Sketch the emf versus time (d) when the field is as drawn in Figure P31.42 and (e) when the field is extended as described in part (c).

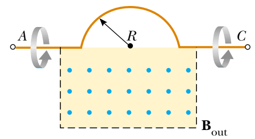


Figure P31.42

43. The rotating loop in an ac generator is a square 10.0 cm on a side. It is rotated at 60.0 Hz in a uniform field of 0.800 T. Calculate (a) the flux through the loop as a function of time, (b) the emf induced in the loop, (c) the current induced in the loop for a loop resistance of 1.00 Ω, (d) the power in the resistance of the loop, and (e) the torque that must be exerted to rotate the loop.

(Optional)

Section 31.6 Eddy Currents

44. A 0.150-kg wire in the shape of a closed rectangle 1.00 m wide and 1.50 m long has a total resistance of 0.750 Ω. The rectangle is allowed to fall through a magnetic field directed perpendicular to the direction of motion of the rectangle (Fig. P31.44). The rectangle accelerates downward as it approaches a terminal speed of 2.00 m/s, with its top not yet in the region of the field. Calculate the magnitude of **B**.

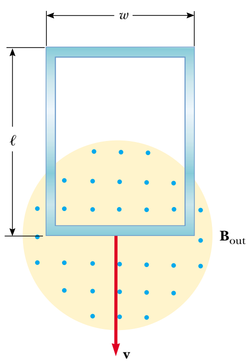


Figure P31.44 Problems 44 and 45.

WEB **45.** A conducting rectangular loop of mass M, resistance R, and dimensions w by ℓ falls from rest into a magnetic field **B** as in Figure P31.44. The loop approaches terminal speed v_t. (a) Show that

$$v_t = \frac{MgR}{B^2w^2}$$

(b) Why is v_t proportional to R? (c) Why is it inversely proportional to B^2?

46. Figure P31.46 represents an electromagnetic brake that utilizes eddy currents. An electromagnet hangs from a railroad car near one rail. To stop the car, a large steady current is sent through the coils of the electromagnet. The moving electromagnet induces eddy currents in the rails, whose fields oppose the change in the field of the electromagnet. The magnetic fields of the eddy currents exert force on the current in the electromagnet, thereby slowing the car. The direction of the car's motion and the direction of the current in the electromagnet are shown correctly in the picture. Determine which of the eddy currents shown on the rails is correct. Explain your answer.

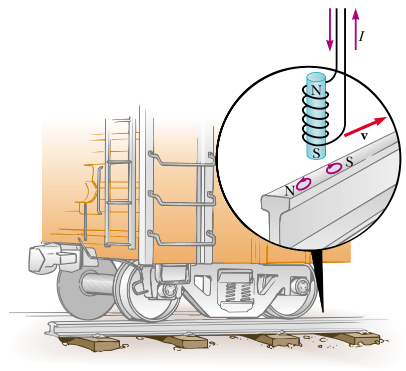


Figure P31.46

Section 31.7 Maxwell's Wonderful Equations

47. A proton moves through a uniform electric field $\mathbf{E} = 50.0\mathbf{j}$ V/m and a uniform magnetic field $\mathbf{B} = (0.200\mathbf{i} + 0.300\mathbf{j} + 0.400\mathbf{k})$ T. Determine the acceleration of the proton when it has a velocity $\mathbf{v} = 200\mathbf{i}$ m/s.

48. An electron moves through a uniform electric field $\mathbf{E} = (2.50\mathbf{i} + 5.00\mathbf{j})$ V/m and a uniform magnetic field $\mathbf{B} = 0.400\mathbf{k}$ T. Determine the acceleration of the electron when it has a velocity $\mathbf{v} = 10.0\mathbf{i}$ m/s.

ADDITIONAL PROBLEMS

49. A steel guitar string vibrates (see Fig. 31.5). The component of the magnetic field perpendicular to the area of a pickup coil nearby is given by

$$B = 50.0 \text{ mT} + (3.20 \text{ mT}) \sin(2\pi 523\ t/\text{s})$$

The circular pickup coil has 30 turns and radius 2.70 mm. Find the emf induced in the coil as a function of time.

50. Figure P31.50 is a graph of the induced emf versus time for a coil of N turns rotating with angular velocity ω in a uniform magnetic field directed perpendicular to the axis of rotation of the coil. Copy this graph (on a larger scale), and on the same set of axes show the graph of emf versus t (a) if the number of turns in the coil is doubled, (b) if instead the angular velocity is doubled, and (c) if the angular velocity is doubled while the number of turns in the coil is halved.

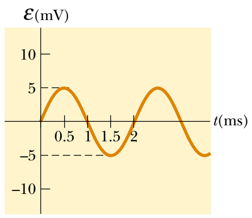


Figure P31.50

51. A technician wearing a brass bracelet enclosing an area of 0.005 00 m^2 places her hand in a solenoid whose magnetic field is 5.00 T directed perpendicular to the plane of the bracelet. The electrical resistance around the circumference of the bracelet is 0.020 0 Ω. An unexpected power failure causes the field to drop to 1.50 T in a time of 20.0 ms. Find (a) the current induced in the bracelet and (b) the power delivered to the resistance of the bracelet. (*Note:* As this problem implies, you should not wear any metallic objects when working in regions of strong magnetic fields.)

52. Two infinitely long solenoids (seen in cross-section) thread a circuit as shown in Figure P31.52. The magnitude of **B** inside each is the same and is increasing at the rate of 100 T/s. What is the current in each resistor?

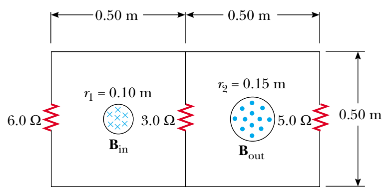


Figure P31.52

53. A conducting rod of length $\ell = 35.0$ cm is free to slide on two parallel conducting bars, as shown in Figure P31.53. Two resistors $R_1 = 2.00\ \Omega$ and $R_2 = 5.00\ \Omega$ are connected across the ends of the bars to form a loop. A constant magnetic field $B = 2.50$ T is directed perpendicular into the page. An external agent pulls the rod to the left with a constant speed of $v = 8.00$ m/s. Find (a) the currents in both resistors, (b) the total power delivered to the resistance of the circuit, and (c) the magnitude of the applied force that is needed to move the rod with this constant velocity.

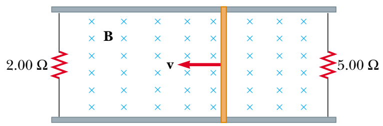


Figure P31.53

54. Suppose you wrap wire onto the core from a roll of cellophane tape to make a coil. Describe how you can use a bar magnet to produce an induced voltage in the coil. What is the order of magnitude of the emf you generate? State the quantities you take as data and their values.

55. A bar of mass m, length d, and resistance R slides without friction on parallel rails, as shown in Figure P31.55. A battery that maintains a constant emf $\mathcal{E}$ is connected between the rails, and a constant magnetic field **B** is directed perpendicular to the plane of the page. If the bar starts from rest, show that at time t it moves with a speed

$$v = \frac{\mathcal{E}}{Bd}\left(1 - e^{-B^2d^2t/mR}\right)$$

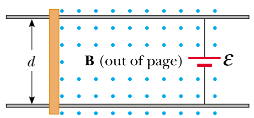


Figure P31.55

56. An automobile has a vertical radio antenna 1.20 m long. The automobile travels at 65.0 km/h on a horizontal road where the Earth's magnetic field is 50.0 μT directed toward the north and downward at an angle of 65.0° below the horizontal. (a) Specify the direction that the automobile should move to generate the maximum motional emf in the antenna, with the top of the antenna positive relative to the bottom. (b) Calculate the magnitude of this induced emf.

57. The plane of a square loop of wire with edge length $a = 0.200$ m is perpendicular to the Earth's magnetic field at a point where $B = 15.0\ \mu$T, as shown in Figure P31.57. The total resistance of the loop and the wires connecting it to the galvanometer is 0.500 Ω. If the loop is suddenly collapsed by horizontal forces as shown, what total charge passes through the galvanometer?

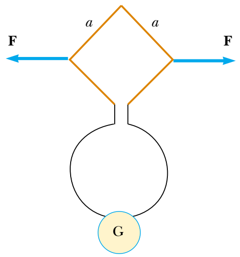


Figure P31.57

58. Magnetic field values are often determined by using a device known as a *search coil.* This technique depends on the measurement of the total charge passing through a coil in a time interval during which the magnetic flux linking the windings changes either because of the motion of the coil or because of a change in the value of B. (a) Show that as the flux through the coil changes from Φ_1 to Φ_2, the charge transferred through the coil will be given by $Q = N(\Phi_2 - \Phi_1)/R$, where R is the resistance of the coil and associated circuitry (galvanometer) and N is the number of turns. (b) As a specific example, calculate B when a 100-turn coil of resistance 200 Ω and cross-sectional area 40.0 cm^2 produces the following results. A total charge of 5.00×10^{-4} C passes through the coil when it is rotated in a uniform field from a position where the plane of the coil is perpendicular to the field to a position where the coil's plane is parallel to the field.

59. In Figure P31.59, the rolling axle, 1.50 m long, is pushed along horizontal rails at a constant speed $v = 3.00$ m/s. A resistor $R = 0.400\ \Omega$ is connected to the rails at points a and b, directly opposite each other. (The wheels make good electrical contact with the rails, and so the axle, rails, and R form a closed-loop circuit. The only significant resistance in the circuit is R.) There is a uniform magnetic field $B = 0.080$ 0 T vertically downward. (a) Find the induced current I in the resistor. (b) What horizontal force F is required to keep the axle rolling at constant speed? (c) Which end of the resistor, a or b, is at the higher electric potential? (d) After the axle rolls past the resistor, does the current in R reverse direction? Explain your answer.

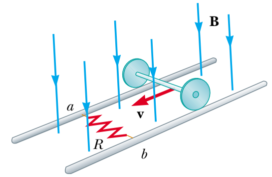


Figure P31.59

60. A conducting rod moves with a constant velocity **v** perpendicular to a long, straight wire carrying a current I as shown in Figure P31.60. Show that the magnitude of the emf generated between the ends of the rod is

$$|\mathcal{E}| = \frac{\mu_0 v I}{2\pi r}\,\ell$$

In this case, note that the emf decreases with increasing r, as you might expect.

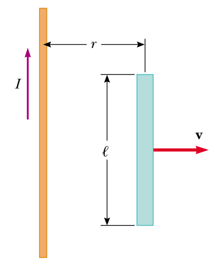


Figure P31.60

61. A circular loop of wire of radius r is in a uniform magnetic field, with the plane of the loop perpendicular to the direction of the field (Fig. P31.61). The magnetic field varies with time according to $B(t) = a + bt$, where a and b are constants. (a) Calculate the magnetic flux through the loop at $t = 0$. (b) Calculate the emf induced in the loop. (c) If the resistance of the loop is R, what is the induced current? (d) At what rate is electrical energy being delivered to the resistance of the loop?

62. In Figure P31.62, a uniform magnetic field decreases at a constant rate $dB/dt = -K$, where K is a positive constant. A circular loop of wire of radius a containing a re-

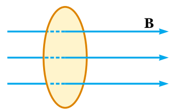


Figure P31.61

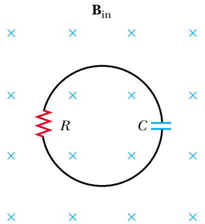


Figure P31.62

sistance R and a capacitance C is placed with its plane normal to the field. (a) Find the charge Q on the capacitor when it is fully charged. (b) Which plate is at the higher potential? (c) Discuss the force that causes the separation of charges.

63. A rectangular coil of 60 turns, dimensions 0.100 m by 0.200 m and total resistance 10.0 Ω, rotates with angular speed 30.0 rad/s about the y axis in a region where a 1.00-T magnetic field is directed along the x axis. The rotation is initiated so that the plane of the coil is perpendicular to the direction of **B** at $t = 0$. Calculate (a) the maximum induced emf in the coil, (b) the maximum rate of change of magnetic flux through the coil, (c) the induced emf at $t = 0.050\ 0$ s, and (d) the torque exerted on the coil by the magnetic field at the instant when the emf is a maximum.

64. A small circular washer of radius 0.500 cm is held directly below a long, straight wire carrying a current of 10.0 A. The washer is located 0.500 m above the top of the table (Fig. P31.64). (a) If the washer is dropped from rest, what is the magnitude of the average induced

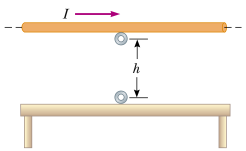


Figure P31.64

emf in the washer from the time it is released to the moment it hits the tabletop? Assume that the magnetic field is nearly constant over the area of the washer and equal to the magnetic field at the center of the washer. (b) What is the direction of the induced current in the washer?

65. To monitor the breathing of a hospital patient, a thin belt is wrapped around the patient's chest. The belt is a 200-turn coil. When the patient inhales, the area encircled by the coil increases by 39.0 cm^2. The magnitude of the Earth's magnetic field is 50.0 μT and makes an angle of 28.0° with the plane of the coil. If a patient takes 1.80 s to inhale, find the average induced emf in the coil during this time.

66. A conducting rod of length ℓ moves with velocity **v** parallel to a long wire carrying a steady current I. The axis of the rod is maintained perpendicular to the wire with the near end a distance r away, as shown in Figure P31.66. Show that the magnitude of the emf induced in the rod is

$$|\mathcal{E}| = \frac{\mu_0 I}{2\pi} v \ln\left(1 + \frac{\ell}{r}\right)$$

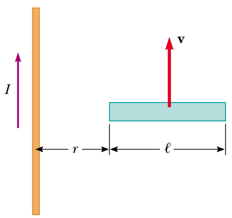


Figure P31.66

67. A rectangular loop of dimensions ℓ and w moves with a constant velocity **v** away from a long wire that carries a current I in the plane of the loop (Fig. P31.67). The to-

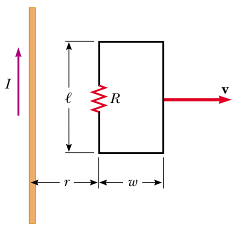


Figure P31.67

tal resistance of the loop is R. Derive an expression that gives the current in the loop at the instant the near side is a distance r from the wire.

68. A horizontal wire is free to slide on the vertical rails of a conducting frame, as shown in Figure P31.68. The wire has mass m and length ℓ, and the resistance of the circuit is R. If a uniform magnetic field is directed perpendicular to the frame, what is the terminal speed of the wire as it falls under the force of gravity?

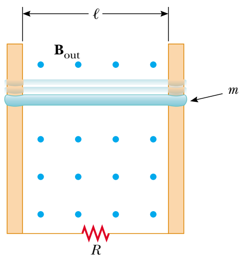


Figure P31.68

69. The magnetic flux threading a metal ring varies with time t according to $\Phi_B = 3(at^3 - bt^2)$ T·m^2, with $a = 2.00\ s^{-3}$ and $b = 6.00\ s^{-2}$. The resistance of the ring is 3.00 Ω. Determine the maximum current induced in the ring during the interval from $t = 0$ to $t = 2.00$ s.

70. **Review Problem.** The bar of mass m shown in Figure P31.70 is pulled horizontally across parallel rails by a massless string that passes over an ideal pulley and is attached to a suspended mass M. The uniform magnetic field has a magnitude B, and the distance between the rails is ℓ. The rails are connected at one end by a load resistor R. Derive an expression that gives the horizon-

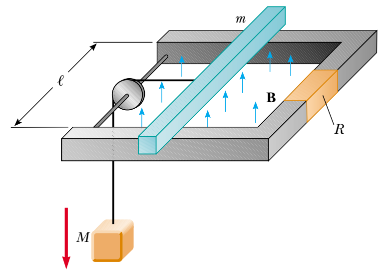


Figure P31.70

tal speed of the bar as a function of time, assuming that the suspended mass is released with the bar at rest at $t = 0$. Assume no friction between rails and bar.

71. A solenoid wound with 2 000 turns/m is supplied with current that varies in time according to $I = 4\sin(120\pi t)$, where I is in A and t is in s. A small coaxial circular coil of 40 turns and radius $r = 5.00$ cm is located inside the solenoid near its center. (a) Derive an expression that describes the manner in which the emf in the small coil varies in time. (b) At what average rate is energy transformed into internal energy in the small coil if the windings have a total resistance of 8.00 Ω?

72. A wire 30.0 cm long is held parallel to and 80.0 cm above a long wire carrying 200 A and resting on the floor (Fig. P31.72). The 30.0-cm wire is released and falls, remaining parallel with the current-carrying wire as it falls. Assume that the falling wire accelerates at 9.80 m/s^2 and derive an equation for the emf induced in it. Express your result as a function of the time t after the wire is dropped. What is the induced emf 0.300 s after the wire is released?

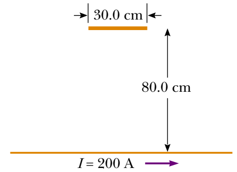


Figure P31.72

WEB **73.** A long, straight wire carries a current $I = I_{max}\sin(\omega t + \phi)$ and lies in the plane of a rectangular coil of N turns of wire, as shown in Figure P31.9. The quantities I_{max}, ω, and ϕ are all constants. Determine the emf induced in the coil by the magnetic field created by the current in the straight wire. Assume $I_{max} = 50.0$ A, $\omega = 200\pi\ s^{-1}$, $N = 100$, $h = w = 5.00$ cm, and $L = 20.0$ cm.

74. A dime is suspended from a thread and hung between the poles of a strong horseshoe magnet as shown in Figure P31.74. The dime rotates at constant angular speed ω about a vertical axis. Letting θ represent the angle between the direction of **B** and the normal to the face of the dime, sketch a graph of the torque due to induced currents as a function of θ for $0 < \theta < 2\pi$.

75. The wire shown in Figure P31.75 is bent in the shape of a tent, with $\theta = 60.0°$ and $L = 1.50$ m, and is placed in a uniform magnetic field of magnitude 0.300 T perpendicular to the tabletop. The wire is rigid but hinged at points a and b. If the "tent" is flattened out on the table in 0.100 s, what is the average induced emf in the wire during this time?

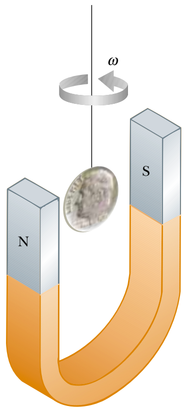


Figure P31.74

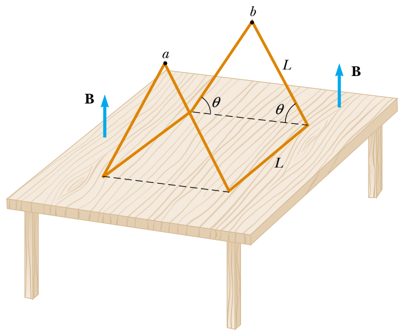


Figure P31.75

Answers to Quick Quizzes

31.1 Because the magnetic field now points in the opposite direction, you must replace θ with $\theta + \pi$. Because $\cos(\theta + \pi) = -\cos\theta$, the sign of the induced emf is reversed.

31.2 The one on the west side of the plane. As we saw in Section 30.9, the Earth's magnetic field has a downward component in the northern hemisphere. As the plane flies north, the right-hand rule illustrated in Figure 29.4 indicates that positive charge experiences a force directed toward the west. Thus, the left wingtip becomes positively charged and the right wingtip negatively charged.

31.3 Inserting. Because the south pole of the magnet is nearest the solenoid, the field lines created by the magnet point upward in Figure 31.14. Because the current induced in the solenoid is clockwise when viewed from above, the magnetic field lines produced by this current point downward in Figure 31.14. If the magnet were being withdrawn, it would create a decreasing upward flux. The induced current would counteract this decrease by producing its own upward flux. This would require a counterclockwise current in the solenoid, contrary to what is observed.

P U Z Z L E R

The marks in the pavement are part of a sensor that controls the traffic lights at this intersection. What are these marks, and how do they detect when a car is waiting at the light? *(© David R. Frazier)*

chapter

32 Inductance

Chapter Outline

In Chapter 31, we saw that emfs and currents are induced in a circuit when the magnetic flux through the area enclosed by the circuit changes with time. This electromagnetic induction has some practical consequences, which we describe in this chapter. First, we describe an effect known as *self-induction,* in which a time-varying current in a circuit produces in the circuit an induced emf that opposes the emf that initially set up the time-varying current. Self-induction is the basis of the *inductor,* an electrical element that has an important role in circuits that use time-varying currents. We discuss the energy stored in the magnetic field of an inductor and the energy density associated with the magnetic field.

Next, we study how an emf is induced in a circuit as a result of a changing magnetic flux produced by a second circuit; this is the basic principle of *mutual induction.* Finally, we examine the characteristics of circuits that contain inductors, resistors, and capacitors in various combinations.

32.1 SELF-INDUCTANCE

In this chapter, we need to distinguish carefully between emfs and currents that are caused by batteries or other sources and those that are induced by changing magnetic fields. We use the adjective *source* (as in the terms *source emf* and *source current*) to describe the parameters associated with a physical source, and we use the adjective *induced* to describe those emfs and currents caused by a changing magnetic field.

Consider a circuit consisting of a switch, a resistor, and a source of emf, as shown in Figure 32.1. When the switch is thrown to its closed position, the source current does not immediately jump from zero to its maximum value $\mathcal{E}/R$. Faraday's law of electromagnetic induction (Eq. 31.1) can be used to describe this effect as follows: As the source current increases with time, the magnetic flux through the circuit loop due to this current also increases with time. This increasing flux creates an induced emf in the circuit. The direction of the induced emf is such that it would cause an induced current in the loop (if a current were not already flowing in the loop), which would establish a magnetic field that would oppose the change in the source magnetic field. Thus, the direction of the induced emf is opposite the direction of the source emf; this results in a gradual rather than instantaneous increase in the source current to its final equilibrium value. This effect is called *self-induction* because the changing flux through the circuit and the resultant induced emf arise from the circuit itself. The emf $\mathcal{E}_L$ set up in this case is called a **self-induced emf.** It is also often called a **back emf.**

As a second example of self-induction, consider Figure 32.2, which shows a coil wound on a cylindrical iron core. (A practical device would have several hun-

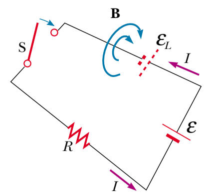


Figure 32.1 After the switch is thrown closed, the current produces a magnetic flux through the area enclosed by the loop. As the current increases toward its equilibrium value, this magnetic flux changes in time and induces an emf in the loop. The battery symbol drawn with dashed lines represents the self-induced emf.

Joseph Henry (1797–1878) Henry, an American physicist, became the first director of the Smithsonian Institution and first president of the Academy of Natural Science. He improved the design of the electromagnet and constructed one of the first motors. He also discovered the phenomenon of self-induction but failed to publish his findings. The unit of inductance, the henry, is named in his honor. *(North Wind Picture Archives)*

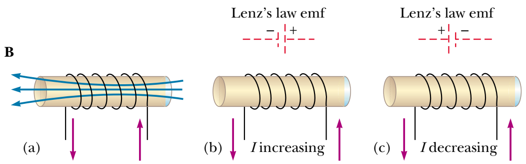


Figure 32.2 (a) A current in the coil produces a magnetic field directed to the left. (b) If the current increases, the increasing magnetic flux creates an induced emf having the polarity shown by the dashed battery. (c) The polarity of the induced emf reverses if the current decreases.

dred turns.) Assume that the source current in the coil either increases or decreases with time. When the source current is in the direction shown, a magnetic field directed from right to left is set up inside the coil, as seen in Figure 32.2a. As the source current changes with time, the magnetic flux through the coil also changes and induces an emf in the coil. From Lenz's law, the polarity of this induced emf must be such that it opposes the change in the magnetic field from the source current. If the source current is increasing, the polarity of the induced emf is as pictured in Figure 32.2b, and if the source current is decreasing, the polarity of the induced emf is as shown in Figure 32.2c.

To obtain a quantitative description of self-induction, we recall from Faraday's law that the induced emf is equal to the negative time rate of change of the magnetic flux. The magnetic flux is proportional to the magnetic field due to the source current, which in turn is proportional to the source current in the circuit. Therefore, **a self-induced emf $\mathcal{E}_L$ is always proportional to the time rate of change of the source current.** For a closely spaced coil of N turns (a toroid or an ideal solenoid) carrying a source current I, we find that

Self-induced emf

$$\mathcal{E}_L = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt} \qquad (32.1)$$

where L is a proportionality constant—called the **inductance** of the coil—that depends on the geometry of the circuit and other physical characteristics. From this expression, we see that the inductance of a coil containing N turns is

Inductance of an N-turn coil

$$L = \frac{N\Phi_B}{I} \qquad (32.2)$$

where it is assumed that the same flux passes through each turn. Later, we shall use this equation to calculate the inductance of some special circuit geometries.

From Equation 32.1, we can also write the inductance as the ratio

Inductance

$$L = -\frac{\mathcal{E}_L}{dI/dt} \qquad (32.3)$$

Just as resistance is a measure of the opposition to current ($R = \Delta V/I$), inductance is a measure of the opposition to a *change* in current.

The SI unit of inductance is the **henry** (H), which, as we can see from Equation 32.3, is 1 volt-second per ampere:

$$1\text{ H} = 1\,\frac{\text{V}\cdot\text{s}}{\text{A}}$$

That the inductance of a device depends on its geometry is analogous to the capacitance of a capacitor depending on the geometry of its plates, as we found in Chapter 26. Inductance calculations can be quite difficult to perform for complicated geometries; however, the following examples involve simple situations for which inductances are easily evaluated.

EXAMPLE 32.1 Inductance of a Solenoid

Find the inductance of a uniformly wound solenoid having N turns and length ℓ. Assume that ℓ is much longer than the radius of the windings and that the core of the solenoid is air.

Solution We can assume that the interior magnetic field due to the source current is uniform and given by Equation 30.17:

$$B = \mu_0 n I = \mu_0 \frac{N}{\ell} I$$

where $n = N/\ell$ is the number of turns per unit length. The magnetic flux through each turn is

$$\Phi_B = BA = \mu_0 \frac{NA}{\ell} I$$

where A is the cross-sectional area of the solenoid. Using this expression and Equation 32.2, we find that

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 A}{\ell} \qquad (32.4)$$

This result shows that L depends on geometry and is proportional to the square of the number of turns. Because $N = n\ell$, we can also express the result in the form

$$L = \mu_0 \frac{(n\ell)^2}{\ell} A = \mu_0 n^2 A\ell = \mu_0 n^2 V \qquad (32.5)$$

where $V = A\ell$ is the volume of the solenoid.

Exercise What would happen to the inductance if a ferromagnetic material were placed inside the solenoid?

13.6

Answer The inductance would increase. For a given current, the magnetic flux is now much greater because of the increase in the field originating from the magnetization of the ferromagnetic material. For example, if the material has a magnetic permeability of $500\mu_0$, the inductance would increase by a factor of 500.

The fact that various materials in the vicinity of a coil can substantially alter the coil's inductance is used to great advantage by traffic engineers. A flat, horizontal coil made of numerous loops of wire is placed in a shallow groove cut into the pavement of the lane approaching an intersection. (See the photograph at the beginning of this chapter.) These loops are attached to circuitry that measures inductance. When an automobile passes over the loops, the change in inductance caused by the large amount of iron passing over the loops is used to control the lights at the intersection.

EXAMPLE 32.2 Calculating Inductance and emf

(a) Calculate the inductance of an air-core solenoid containing 300 turns if the length of the solenoid is 25.0 cm and its cross-sectional area is 4.00 cm^2.

Solution Using Equation 32.4, we obtain

$$L = \frac{\mu_0 N^2 A}{\ell}$$

$$= (4\pi \times 10^{-7}\text{ T}\cdot\text{m/A})\,\frac{(300)^2(4.00 \times 10^{-4}\text{ m}^2)}{25.0 \times 10^{-2}\text{ m}}$$

$$= 1.81 \times 10^{-4}\text{ T}\cdot\text{m}^2/\text{A} = 0.181\text{ mH}$$

(b) Calculate the self-induced emf in the solenoid if the current through it is decreasing at the rate of 50.0 A/s.

Solution Using Equation 32.1 and given that $dI/dt = -50.0$ A/s, we obtain

$$\mathcal{E}_L = -L\frac{dI}{dt} = -(1.81 \times 10^{-4}\text{ H})(-50.0\text{ A/s})$$

$$= 9.05\text{ mV}$$

32.2 *RL* CIRCUITS

13.6

If a circuit contains a coil, such as a solenoid, the self-inductance of the coil prevents the current in the circuit from increasing or decreasing instantaneously. A circuit element that has a large self-inductance is called an **inductor** and has the circuit symbol . We always assume that the self-inductance of the remainder of a circuit is negligible compared with that of the inductor. Keep in mind, however, that even a circuit without a coil has some self-inductance that can affect the behavior of the circuit.

Because the inductance of the inductor results in a back emf, **an inductor in a circuit opposes changes in the current through that circuit.** If the battery voltage in the circuit is increased so that the current rises, the inductor opposes this change, and the rise is not instantaneous. If the battery voltage is decreased, the presence of the inductor results in a slow drop in the current rather than an immediate drop. Thus, the inductor causes the circuit to be "sluggish" as it reacts to changes in the voltage.

Quick Quiz 32.1

A switch controls the current in a circuit that has a large inductance. Is a spark more likely to be produced at the switch when the switch is being closed or when it is being opened, or doesn't it matter?

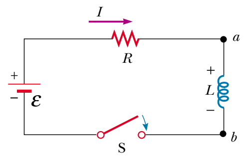


Figure 32.3 A series RL circuit. As the current increases toward its maximum value, an emf that opposes the increasing current is induced in the inductor.

Consider the circuit shown in Figure 32.3, in which the battery has negligible internal resistance. This is an ***RL* circuit** because the elements connected to the battery are a resistor and an inductor. Suppose that the switch S is thrown closed at $t = 0$. The current in the circuit begins to increase, and a back emf that opposes the increasing current is induced in the inductor. The back emf is, from Equation 32.1,

$$\mathcal{E}_L = -L\frac{dI}{dt}$$

Because the current is increasing, dI/dt is positive; thus, $\mathcal{E}_L$ is negative. This negative value reflects the decrease in electric potential that occurs in going from a to b across the inductor, as indicated by the positive and negative signs in Figure 32.3.

With this in mind, we can apply Kirchhoff's loop rule to this circuit, traversing the circuit in the clockwise direction:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \qquad \textbf{(32.6)}$$

where IR is the voltage drop across the resistor. (We developed Kirchhoff's rules for circuits with steady currents, but we can apply them to a circuit in which the current is changing if we imagine them to represent the circuit at one *instant* of time.) We must now look for a solution to this differential equation, which is similar to that for the RC circuit (see Section 28.4).

A mathematical solution of Equation 32.6 represents the current in the circuit as a function of time. To find this solution, we change variables for convenience, letting $x = \frac{\mathcal{E}}{R} - I$, so that $dx = -dI$. With these substitutions, we can write Equation 32.6 as

$$x + \frac{L}{R}\frac{dx}{dt} = 0$$

$$\frac{dx}{x} = -\frac{R}{L}\,dt$$

Integrating this last expression, we have

$$\ln\frac{x}{x_0} = -\frac{R}{L}t$$

where we take the integrating constant to be $-\ln x_0$ and x_0 is the value of x at time $t = 0$. Taking the antilogarithm of this result, we obtain

$$x = x_0 e^{-Rt/L}$$

Because $I = 0$ at $t = 0$, we note from the definition of x that $x_0 = \mathcal{E}/R$. Hence, this last expression is equivalent to

$$\frac{\mathcal{E}}{R} - I = \frac{\mathcal{E}}{R}e^{-Rt/L}$$

$$I = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$

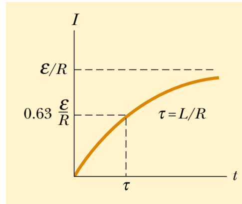


Figure 32.4 Plot of the current versus time for the RL circuit shown in Figure 32.3. The switch is thrown closed at $t = 0$, and the current increases toward its maximum value $\mathcal{E}/R$. The time constant τ is the time it takes I to reach 63% of its maximum value.

This expression shows the effect of the inductor. The current does not increase instantly to its final equilibrium value when the switch is closed but instead increases according to an exponential function. If we remove the inductance in the circuit, which we can do by letting L approach zero, the exponential term becomes zero and we see that there is no time dependence of the current in this case—the current increases instantaneously to its final equilibrium value in the absence of the inductance.

We can also write this expression as

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) \qquad \textbf{(32.7)}$$

where the constant τ is the **time constant** of the RL circuit:

Time constant of an RL circuit

$$\tau = L/R \qquad \textbf{(32.8)}$$

Physically, τ is the time it takes the current in the circuit to reach $(1 - e^{-1}) = 0.63$ of its final value $\mathcal{E}/R$. The time constant is a useful parameter for comparing the time responses of various circuits.

Figure 32.4 shows a graph of the current versus time in the RL circuit. Note that the equilibrium value of the current, which occurs as t approaches infinity, is $\mathcal{E}/R$. We can see this by setting dI/dt equal to zero in Equation 32.6 and solving for the current I. (At equilibrium, the change in the current is zero.) Thus, we see that the current initially increases very rapidly and then gradually approaches the equilibrium value $\mathcal{E}/R$ as t approaches infinity.

Let us also investigate the time rate of change of the current in the circuit. Taking the first time derivative of Equation 32.7, we have

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L}e^{-t/\tau} \qquad \textbf{(32.9)}$$

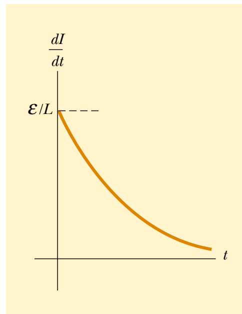


Figure 32.5 Plot of dI/dt versus time for the RL circuit shown in Figure 32.3. The time rate of change of current is a maximum at $t = 0$, which is the instant at which the switch is thrown closed. The rate decreases exponentially with time as I increases toward its maximum value.

From this result, we see that the time rate of change of the current is a maximum (equal to $\mathcal{E}/L$) at $t = 0$ and falls off exponentially to zero as t approaches infinity (Fig. 32.5).

Now let us consider the RL circuit shown in Figure 32.6. The circuit contains two switches that operate such that when one is closed, the other is opened. Suppose that S_1 has been closed for a length of time sufficient to allow the current to reach its equilibrium value $\mathcal{E}/R$. In this situation, the circuit is described completely by the outer loop in Figure 32.6. If S_2 is closed at the instant at which S_1 is opened, the circuit changes so that it is described completely by just the upper loop in Figure 32.6. The lower loop no longer influences the behavior of the circuit. Thus, we have a circuit with no battery ($\mathcal{E} = 0$). If we apply Kirchhoff's loop rule to the upper loop at the instant the switches are thrown, we obtain

$$IR + L\frac{dI}{dt} = 0$$

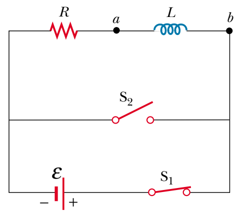


Figure 32.6 An RL circuit containing two switches. When S_1 is closed and S_2 open as shown, the battery is in the circuit. At the instant S_2 is closed, S_1 is opened, and the battery is no longer part of the circuit.

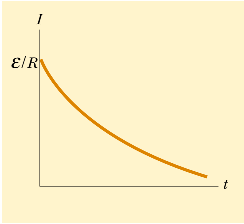


Figure 32.7 Current versus time for the upper loop of the circuit shown in Figure 32.6. For $t < 0$, S_1 is closed and S_2 is open. At $t = 0$, S_2 is closed as S_1 is opened, and the current has its maximum value $\mathcal{E}/R$.

It is left as a problem (Problem 18) to show that the solution of this differential equation is

$$I = \frac{\mathcal{E}}{R} e^{-t/\tau} = I_0 e^{-t/\tau} \quad \textbf{(32.10)}$$

where $\mathcal{E}$ is the emf of the battery and $I_0 = \mathcal{E}/R$ is the current at $t = 0$, the instant at which S_2 is closed as S_1 is opened.

If no inductor were present in the circuit, the current would immediately decrease to zero if the battery were removed. When the inductor is present, it acts to oppose the decrease in the current and to maintain the current. A graph of the current in the circuit versus time (Fig. 32.7) shows that the current is continuously decreasing with time. Note that the slope dI/dt is always negative and has its maximum value at $t = 0$. The negative slope signifies that $\mathcal{E}_L = -L\,(dI/dt)$ is now positive; that is, point a in Figure 32.6 is at a lower electric potential than point b.

Quick Quiz 32.2

Two circuits like the one shown in Figure 32.6 are identical except for the value of L. In circuit A the inductance of the inductor is L_A, and in circuit B it is L_B. Switch S_1 is thrown closed at $t = 0$, while switch S_2 remains open. At $t = 10$ s, switch S_1 is opened and switch S_2 is closed. The resulting time rates of change for the two currents are as graphed in Figure 32.8. If we assume that the time constant of each circuit is much less than 10 s, which of the following is true? (a) $L_A > L_B$; (b) $L_A < L_B$; (c) not enough information to tell.

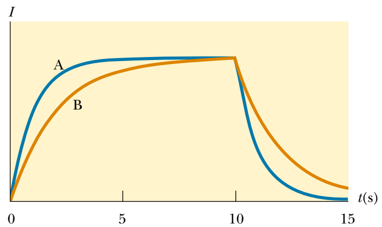


Figure 32.8

EXAMPLE 32.3 Time Constant of an RL Circuit

The switch in Figure 32.9a is thrown closed at $t = 0$. (a) Find the time constant of the circuit.

Solution The time constant is given by Equation 32.8:

$$\tau = \frac{L}{R} = \frac{30.0 \times 10^{-3}\ \text{H}}{6.00\ \Omega} = \boxed{5.00\ \text{ms}}$$

(b) Calculate the current in the circuit at $t = 2.00$ ms.

Solution Using Equation 32.7 for the current as a function of time (with t and τ in milliseconds), we find that at $t = 2.00$ ms

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) = \frac{12.0\ \text{V}}{6.00\ \Omega}(1 - e^{-0.400}) = \boxed{0.659\ \text{A}}$$

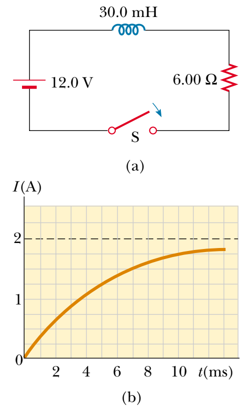


Figure 32.9 (a) The switch in this RL circuit is thrown closed at $t = 0$. (b) A graph of the current versus time for the circuit in part (a).

A plot of Equation 32.7 for this circuit is given in Figure 32.9b.

(c) Compare the potential difference across the resistor with that across the inductor.

Solution At the instant the switch is closed, there is no current and thus no potential difference across the resistor. At this instant, the battery voltage appears entirely across the inductor in the form of a back emf of 12.0 V as the inductor tries to maintain the zero-current condition. (The left end of the inductor is at a higher electric potential than the right end.) As time passes, the emf across the inductor decreases and the current through the resistor (and hence the potential difference across it) increases. The sum of the two potential differences at all times is 12.0 V, as shown in Figure 32.10.

Exercise Calculate the current in the circuit and the voltage across the resistor after a time interval equal to one time constant has elapsed.

Answer 1.26 A, 7.56 V.

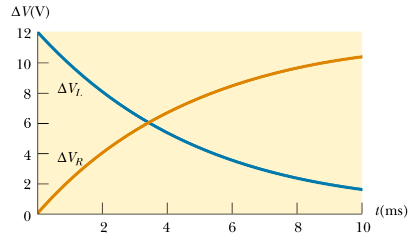


Figure 32.10 The sum of the potential differences across the resistor and inductor in Figure 32.9a is 12.0 V (the battery emf) at all times.

32.3 ENERGY IN A MAGNETIC FIELD

13.6 Because the emf induced in an inductor prevents a battery from establishing an instantaneous current, the battery must do work against the inductor to create a current. Part of the energy supplied by the battery appears as internal energy in the resistor, while the remaining energy is stored in the magnetic field of the inductor. If we multiply each term in Equation 32.6 by I and rearrange the expression, we have

$$I\mathcal{E} = I^2R + LI\frac{dI}{dt} \quad \textbf{(32.11)}$$

This expression indicates that the rate at which energy is supplied by the battery ($I\boldsymbol{\mathcal{E}}$) equals the sum of the rate at which energy is delivered to the resistor, I^2R, and the rate at which energy is stored in the inductor, $LI(dI/dt)$. Thus, Equation 32.11 is simply an expression of energy conservation. If we let U denote the energy stored in the inductor at any time, then we can write the rate dU/dt at which energy is stored as

$$\frac{dU}{dt} = LI\frac{dI}{dt}$$

To find the total energy stored in the inductor, we can rewrite this expression as $dU = LI\,dI$ and integrate:

$$U = \int dU = \int_0^I LI\,dI = L\int_0^I I\,dI$$

Energy stored in an inductor

$$U = \tfrac{1}{2}LI^2 \qquad \textbf{(32.12)}$$

where L is constant and has been removed from the integral. This expression represents the energy stored in the magnetic field of the inductor when the current is I. Note that this equation is similar in form to Equation 26.11 for the energy stored in the electric field of a capacitor, $U = Q^2/2C$. In either case, we see that energy is required to establish a field.

We can also determine the energy density of a magnetic field. For simplicity, consider a solenoid whose inductance is given by Equation 32.5:

$$L = \mu_0 n^2 A\ell$$

The magnetic field of a solenoid is given by Equation 30.17:

$$B = \mu_0 nI$$

Substituting the expression for L and $I = B/\mu_0 n$ into Equation 32.12 gives

$$U = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\mu_0 n^2 A\ell\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}A\ell \qquad \textbf{(32.13)}$$

Because $A\ell$ is the volume of the solenoid, the energy stored per unit volume in the magnetic field surrounding the inductor is

Magnetic energy density

$$u_B = \frac{U}{A\ell} = \frac{B^2}{2\mu_0} \qquad \textbf{(32.14)}$$

Although this expression was derived for the special case of a solenoid, it is valid for any region of space in which a magnetic field exists. Note that Equation 32.14 is similar in form to Equation 26.13 for the energy per unit volume stored in an electric field, $u_E = \tfrac{1}{2}\epsilon_0 E^2$. In both cases, the energy density is proportional to the square of the magnitude of the field.

EXAMPLE 32.4 What Happens to the Energy in the Inductor?

Consider once again the RL circuit shown in Figure 32.6, in which switch S_2 is closed at the instant S_1 is opened (at $t = 0$). Recall that the current in the upper loop decays exponentially with time according to the expression $I = I_0e^{-t/\tau}$, where $I_0 = \boldsymbol{\mathcal{E}}/R$ is the initial current in the circuit and $\tau = L/R$ is the time constant. Show that all the energy initially stored in the magnetic field of the inductor appears as internal energy in the resistor as the current decays to zero.

Solution The rate dU/dt at which energy is delivered to the resistor (which is the power) is equal to I^2R, where I is the instantaneous current:

$$\frac{dU}{dt} = I^2R = (I_0e^{-Rt/L})^2R = I_0{}^2Re^{-2Rt/L}$$

To find the total energy delivered to the resistor, we solve for dU and integrate this expression over the limits $t = 0$ to $t \rightarrow \infty$ (the upper limit is infinity because it takes an infinite amount of time for the current to reach zero):

$$(1) \qquad U = \int_0^\infty I_0{}^2Re^{-2Rt/L}dt = I_0{}^2R\int_0^\infty e^{-2Rt/L}dt$$

The value of the definite integral is $L/2R$ (this is left for the student to show in the exercise at the end of this example), and so U becomes

$$U = I_0{}^2R\left(\frac{L}{2R}\right) = \frac{1}{2}LI_0{}^2$$

Note that this is equal to the initial energy stored in the magnetic field of the inductor, given by Equation 32.13, as we set out to prove.

Exercise Show that the integral on the right-hand side of Equation (1) has the value $L/2R$.

EXAMPLE 32.5 The Coaxial Cable

Coaxial cables are often used to connect electrical devices, such as your stereo system and a loudspeaker. Model a long coaxial cable as consisting of two thin concentric cylindrical conducting shells of radii a and b and length ℓ, as shown in Figure 32.11. The conducting shells carry the same current I in opposite directions. Imagine that the inner conductor carries current to a device and that the outer one acts as a return path carrying the current back to the source. (a) Calculate the self-inductance L of this cable.

Solution To obtain L, we must know the magnetic flux through any cross-section in the region between the two shells, such as the light blue rectangle in Figure 32.11. Ampère's law (see Section 30.3) tells us that the magnetic field in the region between the shells is $B = \mu_0 I/2\pi r$, where r is measured from the common center of the shells. The magnetic field is zero outside the outer shell ($r > b$) because the net current through the area enclosed by a circular path surrounding the cable is zero, and hence from Ampère's law, $\oint \mathbf{B}\cdot d\mathbf{s} = 0$. The magnetic field is zero inside the inner shell because the shell is hollow and no current is present within a radius $r < a$.

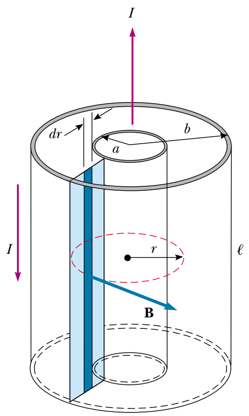


Figure 32.11 Section of a long coaxial cable. The inner and outer conductors carry equal currents in opposite directions.

The magnetic field is perpendicular to the light blue rectangle of length ℓ and width $b - a$, the cross-section of interest. Because the magnetic field varies with radial position across this rectangle, we must use calculus to find the total magnetic flux. Dividing this rectangle into strips of width dr, such as the dark blue strip in Figure 32.11, we see that the area of each strip is $\ell\,dr$ and that the flux through each strip is $B\,dA = B\ell\,dr$. Hence, we find the total flux through the entire cross-section by integrating:

$$\Phi_B = \int B\,dA = \int_a^b \frac{\mu_0 I}{2\pi r}\,\ell\,dr = \frac{\mu_0 I\ell}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 I\ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

Using this result, we find that the self-inductance of the cable is

$$L = \frac{\Phi_B}{I} = \frac{\mu_0\ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

(b) Calculate the total energy stored in the magnetic field of the cable.

Solution Using Equation 32.12 and the results to part (a) gives

$$U = \tfrac{1}{2}LI^2 = \frac{\mu_0\ell I^2}{4\pi}\ln\left(\frac{b}{a}\right)$$

32.4 MUTUAL INDUCTANCE

Very often, the magnetic flux through the area enclosed by a circuit varies with time because of time-varying currents in nearby circuits. This condition induces an emf through a process known as *mutual induction,* so called because it depends on the interaction of two circuits.

Consider the two closely wound coils of wire shown in cross-sectional view in Figure 32.12. The current I_1 in coil 1, which has N_1 turns, creates magnetic field lines, some of which pass through coil 2, which has N_2 turns. The magnetic flux caused by the current in coil 1 and passing through coil 2 is represented by Φ_{12}. In analogy to Equation 32.2, we define the **mutual inductance** M_{12} of coil 2 with respect to coil 1:

Definition of mutual inductance

$$M_{12} \equiv \frac{N_2\Phi_{12}}{I_1} \qquad (32.15)$$

Quick Quiz 32.3

Referring to Figure 32.12, tell what happens to M_{12} (a) if coil 1 is brought closer to coil 2 and (b) if coil 1 is rotated so that it lies in the plane of the page.

Quick Quiz 32.3 demonstrates that mutual inductance depends on the geometry of both circuits and on their orientation with respect to each other. As the circuit separation distance increases, the mutual inductance decreases because the flux linking the circuits decreases.

If the current I_1 varies with time, we see from Faraday's law and Equation 32.15 that the emf induced by coil 1 in coil 2 is

$$\mathcal{E}_2 = -N_2\frac{d\Phi_{12}}{dt} = -N_2\frac{d}{dt}\left(\frac{M_{12}I_1}{N_2}\right) = -M_{12}\frac{dI_1}{dt} \qquad (32.16)$$

In the preceding discussion, we assumed that the source current is in coil 1. We can also imagine a source current I_2 in coil 2. The preceding discussion can be repeated to show that there is a mutual inductance M_{21}. If the current I_2 varies with time, the emf induced by coil 2 in coil 1 is

$$\mathcal{E}_1 = -M_{21}\frac{dI_2}{dt} \qquad (32.17)$$

In mutual induction, the emf induced in one coil is always proportional to the rate at which the current in the other coil is changing. Although the

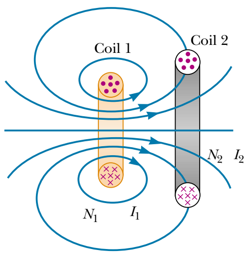


Figure 32.12 A cross-sectional view of two adjacent coils. A current in coil 1 sets up a magnetic flux, part of which passes through coil 2.

proportionality constants M_{12} and M_{21} appear to have different values, it can be shown that they are equal. Thus, with $M_{12} = M_{21} = M$, Equations 32.16 and 32.17 become

$$\mathcal{E}_2 = -M\frac{dI_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M\frac{dI_2}{dt}$$

These two equations are similar in form to Equation 32.1 for the self-induced emf $\mathcal{E} = -L(dI/dt)$. The unit of mutual inductance is the henry.

QuickLab

Tune in a relatively weak station on a radio. Now slowly rotate the radio about a vertical axis through its center. What happens to the reception? Can you explain this in terms of the mutual induction of the station's broadcast antenna and your radio's antenna?

Quick Quiz 32.4

(a) Can you have mutual inductance without self-inductance? (b) How about self-inductance without mutual inductance?

EXAMPLE 32.6 "Wireless" Battery Charger

An electric toothbrush has a base designed to hold the toothbrush handle when not in use. As shown in Figure 32.13a, the handle has a cylindrical hole that fits loosely over a matching cylinder on the base. When the handle is placed on the base, a changing current in a solenoid inside the base cylinder induces a current in a coil inside the handle. This induced current charges the battery in the handle.

We can model the base as a solenoid of length ℓ with N_B turns (Fig. 32.13b), carrying a source current I, and having a cross-sectional area A. The handle coil contains N_H turns. Find the mutual inductance of the system.

Solution Because the base solenoid carries a source current I, the magnetic field in its interior is

$$B = \frac{\mu_0 N_B I}{\ell}$$

Because the magnetic flux Φ_{BH} through the handle's coil caused by the magnetic field of the base coil is BA, the mutual inductance is

$$M = \frac{N_H\Phi_{BH}}{I} = \frac{N_H BA}{I} = \mu_0\frac{N_H N_B A}{\ell}$$

Exercise Calculate the mutual inductance of two solenoids with $N_B = 1\,500$ turns, $A = 1.0 \times 10^{-4}$ m^2, $\ell = 0.02$ m, and $N_H = 800$ turns.

Answer 7.5 mH.

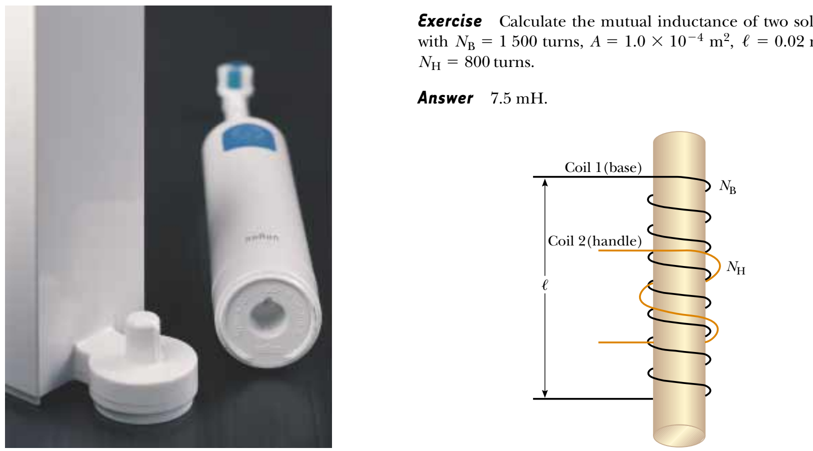


Figure 32.13 (a) This electric toothbrush uses the mutual induction of solenoids as part of its battery-charging system. (b) A coil of N_H turns wrapped around the center of a solenoid of N_B turns.

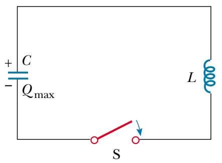


Figure 32.14 A simple LC circuit. The capacitor has an initial charge Q_{max}, and the switch is thrown closed at $t = 0$.

32.5 OSCILLATIONS IN AN *LC* CIRCUIT

13.7 When a capacitor is connected to an inductor as illustrated in Figure 32.14, the combination is an ***LC* circuit.** If the capacitor is initially charged and the switch is then closed, both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values. If the resistance of the circuit is zero, no energy is transformed to internal energy. In the following analysis, we neglect the resistance in the circuit. We also assume an idealized situation in which energy is not radiated away from the circuit. We shall discuss this radiation in Chapter 34, but we neglect it for now. With these idealizations—zero resistance and no radiation—the oscillations in the circuit persist indefinitely.

Assume that the capacitor has an initial charge Q_{max} (the maximum charge) and that the switch is thrown closed at $t = 0$. Let us look at what happens from an energy viewpoint.

When the capacitor is fully charged, the energy U in the circuit is stored in the electric field of the capacitor and is equal to $Q_{max}^2/2C$ (Eq. 26.11). At this time, the current in the circuit is zero, and thus no energy is stored in the inductor. After the switch is thrown closed, the rate at which charges leave or enter the capacitor plates (which is also the rate at which the charge on the capacitor changes) is equal to the current in the circuit. As the capacitor begins to discharge after the switch is closed, the energy stored in its electric field decreases. The discharge of the capacitor represents a current in the circuit, and hence some energy is now stored in the magnetic field of the inductor. Thus, energy is transferred from the electric field of the capacitor to the magnetic field of the inductor. When the capacitor is fully discharged, it stores no energy. At this time, the current reaches its maximum value, and all of the energy is stored in the inductor. The current continues in the same direction, decreasing in magnitude, with the capacitor eventually becoming fully charged again but with the polarity of its plates now opposite the initial polarity. This is followed by another discharge until the circuit returns to its original state of maximum charge Q_{max} and the plate polarity shown in Figure 32.14. The energy continues to oscillate between inductor and capacitor.

The oscillations of the LC circuit are an electromagnetic analog to the mechanical oscillations of a block–spring system, which we studied in Chapter 13. Much of what we discussed is applicable to LC oscillations. For example, we investigated the effect of driving a mechanical oscillator with an external force, which leads to the phenomenon of *resonance.* We observe the same phenomenon in the LC circuit. For example, a radio tuner has an LC circuit with a natural frequency, which we determine as follows: When the circuit is driven by the electromagnetic oscillations of a radio signal detected by the antenna, the tuner circuit responds with a large amplitude of electrical oscillation only for the station frequency that matches the natural frequency. Thus, only the signal from one station is passed on to the amplifier, even though signals from all stations are driving the circuit at the same time. When you turn the knob on the radio tuner to change the station, you are changing the natural frequency of the circuit so that it will exhibit a resonance response to a different driving frequency.

A graphical description of the energy transfer between the inductor and the capacitor in an LC circuit is shown in Figure 32.15. The right side of the figure shows the analogous energy transfer in the oscillating block–spring system studied in Chapter 13. In each case, the situation is shown at intervals of one-fourth the period of oscillation T. The potential energy $\frac{1}{2}kx^2$ stored in a stretched spring is analogous to the electric potential energy $Q_{max}^2/2C$ stored in the capacitor. The kinetic energy $\frac{1}{2}mv^2$ of the moving block is analogous to the magnetic energy $\frac{1}{2}LI^2$

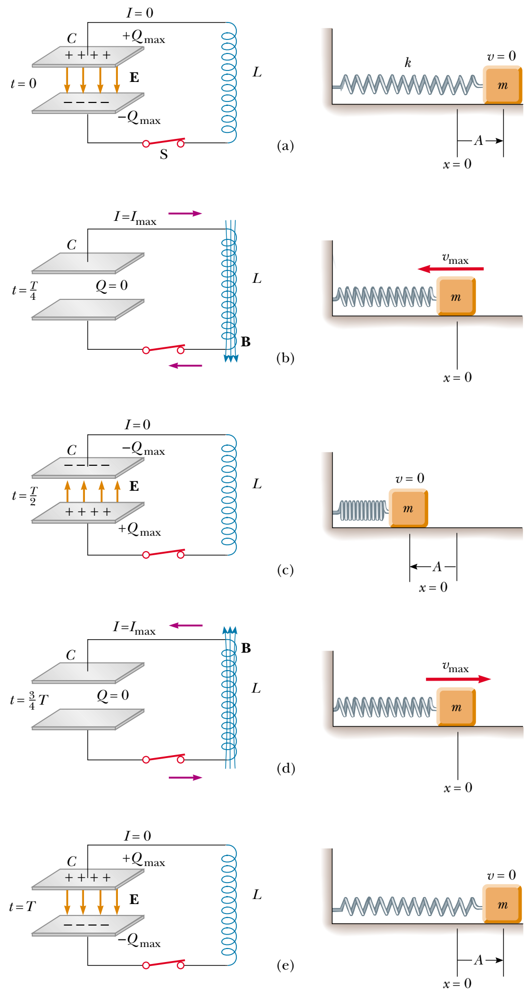


Figure 32.15 Energy transfer in a resistanceless, non-radiating LC circuit. The capacitor has a charge Q_{max} at $t = 0$, the instant at which the switch is thrown closed. The mechanical analog of this circuit is a block–spring system.

stored in the inductor, which requires the presence of moving charges. In Figure 32.15a, all of the energy is stored as electric potential energy in the capacitor at $t = 0$. In Figure 32.15b, which is one fourth of a period later, all of the energy is stored as magnetic energy $\frac{1}{2}LI_{\max}^2$ in the inductor, where $I_{\max}$ is the maximum current in the circuit. In Figure 32.15c, the energy in the LC circuit is stored completely in the capacitor, with the polarity of the plates now opposite what it was in Figure 32.15a. In parts d and e the system returns to the initial configuration over the second half of the cycle. At times other than those shown in the figure, part of the energy is stored in the electric field of the capacitor and part is stored in the magnetic field of the inductor. In the analogous mechanical oscillation, part of the energy is potential energy in the spring and part is kinetic energy of the block.

Let us consider some arbitrary time t after the switch is closed, so that the capacitor has a charge $Q < Q_{\max}$ and the current is $I < I_{\max}$. At this time, both elements store energy, but the sum of the two energies must equal the total initial energy U stored in the fully charged capacitor at $t = 0$:

Total energy stored in an LC circuit

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2 \qquad (32.18)$$

Because we have assumed the circuit resistance to be zero, no energy is transformed to internal energy, and hence *the total energy must remain constant in time.* This means that $dU/dt = 0$. Therefore, by differentiating Equation 32.18 with respect to time while noting that Q and I vary with time, we obtain

The total energy in an ideal LC circuit remains constant; $dU/dt = 0$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0 \qquad (32.19)$$

We can reduce this to a differential equation in one variable by remembering that the current in the circuit is equal to the rate at which the charge on the capacitor changes: $I = dQ/dt$. From this, it follows that $dI/dt = d^2Q/dt^2$. Substitution of these relationships into Equation 32.19 gives

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q \qquad (32.20)$$

We can solve for Q by noting that this expression is of the same form as the analogous Equations 13.16 and 13.17 for a block–spring system:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

where k is the spring constant, m is the mass of the block, and $\omega = \sqrt{k/m}$. The solution of this equation has the general form

$$x = A\cos(\omega t + \phi)$$

where ω is the angular frequency of the simple harmonic motion, A is the amplitude of motion (the maximum value of x), and ϕ is the phase constant; the values of A and ϕ depend on the initial conditions. Because it is of the same form as the differential equation of the simple harmonic oscillator, we see that Equation 32.20 has the solution

Charge versus time for an ideal LC circuit

$$Q = Q_{\max}\cos(\omega t + \phi) \qquad (32.21)$$

where $Q_{\max}$ is the maximum charge of the capacitor and the angular frequency ω is

Angular frequency of oscillation

$$\omega = \frac{1}{\sqrt{LC}} \qquad (32.22)$$

Note that the angular frequency of the oscillations depends solely on the inductance and capacitance of the circuit. This is the *natural frequency* of oscillation of the LC circuit.

Because Q varies sinusoidally, the current in the circuit also varies sinusoidally. We can easily show this by differentiating Equation 32.21 with respect to time:

Current versus time for an ideal LC current

$$I = \frac{dQ}{dt} = -\omega Q_{\max}\sin(\omega t + \phi) \qquad (32.23)$$

To determine the value of the phase angle ϕ, we examine the initial conditions, which in our situation require that at $t = 0$, $I = 0$ and $Q = Q_{\max}$. Setting $I = 0$ at $t = 0$ in Equation 32.23, we have

$$0 = -\omega Q_{\max}\sin\phi$$

which shows that $\phi = 0$. This value for ϕ also is consistent with Equation 32.21 and with the condition that $Q = Q_{\max}$ at $t = 0$. Therefore, in our case, the expressions for Q and I are

$$Q = Q_{\max}\cos\omega t \qquad (32.24)$$

$$I = -\omega Q_{\max}\sin\omega t = -I_{\max}\sin\omega t \qquad (32.25)$$

Graphs of Q versus t and I versus t are shown in Figure 32.16. Note that the charge on the capacitor oscillates between the extreme values $Q_{\max}$ and $-Q_{\max}$, and that the current oscillates between $I_{\max}$ and $-I_{\max}$. Furthermore, the current is 90° out of phase with the charge. That is, when the charge is a maximum, the current is zero, and when the charge is zero, the current has its maximum value.

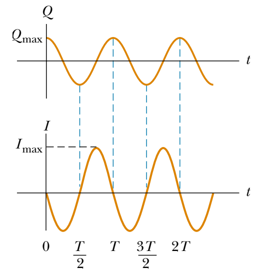


Figure 32.16 Graphs of charge versus time and current versus time for a resistanceless, nonradiating LC circuit. Note that Q and I are 90° out of phase with each other.

Quick Quiz 32.5

What is the relationship between the amplitudes of the two curves in Figure 32.16?

Let us return to the energy discussion of the LC circuit. Substituting Equations 32.24 and 32.25 in Equation 32.18, we find that the total energy is

$$U = U_C + U_L = \frac{Q_{\max}^2}{2C}\cos^2\omega t + \frac{LI_{\max}^2}{2}\sin^2\omega t \qquad (32.26)$$

This expression contains all of the features described qualitatively at the beginning of this section. It shows that the energy of the LC circuit continuously oscillates between energy stored in the electric field of the capacitor and energy stored in the magnetic field of the inductor. When the energy stored in the capacitor has its maximum value $Q_{\max}^2/2C$, the energy stored in the inductor is zero. When the energy stored in the inductor has its maximum value $\frac{1}{2}LI_{\max}^2$, the energy stored in the capacitor is zero.

Plots of the time variations of U_C and U_L are shown in Figure 32.17. The sum $U_C + U_L$ is a constant and equal to the total energy $Q_{\max}^2/2C$ or $LI_{\max}^2/2$. Analytical verification of this is straightforward. The amplitudes of the two graphs in Figure 32.17 must be equal because the maximum energy stored in the capacitor

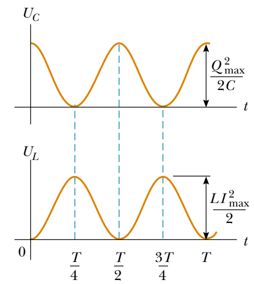


Figure 32.17 Plots of U_C versus t and U_L versus t for a resistanceless, nonradiating LC circuit. The sum of the two curves is a constant and equal to the total energy stored in the circuit.

(when $I = 0$) must equal the maximum energy stored in the inductor (when $Q = 0$). This is mathematically expressed as

$$\frac{Q_{\max}^2}{2C} = \frac{LI_{\max}^2}{2}$$

Using this expression in Equation 32.26 for the total energy gives

$$U = \frac{Q_{\max}^2}{2C}(\cos^2 \omega t + \sin^2 \omega t) = \frac{Q_{\max}^2}{2C} \qquad \textbf{(32.27)}$$

because $\cos^2 \omega t + \sin^2 \omega t = 1$.

In our idealized situation, the oscillations in the circuit persist indefinitely; however, we remember that the total energy U of the circuit remains constant only if energy transfers and transformations are neglected. In actual circuits, there is always some resistance, and hence energy is transformed to internal energy. We mentioned at the beginning of this section that we are also ignoring radiation from the circuit. In reality, radiation is inevitable in this type of circuit, and the total energy in the circuit continuously decreases as a result of this process.

EXAMPLE 32.7 An Oscillatory *LC* Circuit

In Figure 32.18, the capacitor is initially charged when switch S_1 is open and S_2 is closed. Switch S_1 is then thrown closed at the same instant that S_2 is opened, so that the capacitor is connected directly across the inductor. (a) Find the frequency of oscillation of the circuit.

Solution Using Equation 32.22 gives for the frequency

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi[(2.81 \times 10^{-3}\text{ H})(9.00 \times 10^{-12}\text{ F})]^{1/2}}$$

$$= \boxed{1.00 \times 10^6\text{ Hz}}$$

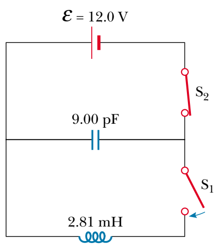


Figure 32.18 First the capacitor is fully charged with the switch S_1 open and S_2 closed. Then, S_1 is thrown closed at the same time that S_2 is thrown open.

(b) What are the maximum values of charge on the capacitor and current in the circuit?

Solution The initial charge on the capacitor equals the maximum charge, and because $C = Q/\mathcal{E}$, we have

$$Q_{\max} = C\mathcal{E} = (9.00 \times 10^{-12}\text{ F})(12.0\text{ V}) = \boxed{1.08 \times 10^{-10}\text{ C}}$$

From Equation 32.25, we can see how the maximum current is related to the maximum charge:

$$I_{\max} = \omega Q_{\max} = 2\pi f Q_{\max}$$

$$= (2\pi \times 10^6\text{ s}^{-1})(1.08 \times 10^{-10}\text{ C})$$

$$= \boxed{6.79 \times 10^{-4}\text{ A}}$$

(c) Determine the charge and current as functions of time.

Solution Equations 32.24 and 32.25 give the following expressions for the time variation of Q and I:

$$Q = Q_{\max} \cos \omega t$$

$$= \boxed{(1.08 \times 10^{-10}\text{ C}) \cos[(2\pi \times 10^6\text{ rad/s})t]}$$

$$I = -I_{\max} \sin \omega t$$

$$= \boxed{(-6.79 \times 10^{-4}\text{ A}) \sin[(2\pi \times 10^6\text{ rad/s})t]}$$

Exercise What is the total energy stored in the circuit?

Answer 6.48×10^{-10} J.

Optional Section

32.6 THE *RLC* CIRCUIT

13.7 We now turn our attention to a more realistic circuit consisting of an inductor, a capacitor, and a resistor connected in series, as shown in Figure 32.19. We let the resistance of the resistor represent all of the resistance in the circuit. We assume that the capacitor has an initial charge $Q_{\max}$ before the switch is closed. Once the switch is thrown closed and a current is established, the total energy stored in the capacitor and inductor at any time is given, as before, by Equation 32.18. However, the total energy is no longer constant, as it was in the LC circuit, because the resistor causes transformation to internal energy. Because the rate of energy transformation to internal energy within a resistor is I^2R, we have

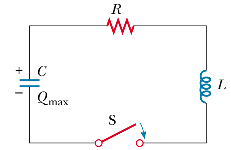


Figure 32.19 A series RLC circuit. The capacitor has a charge $Q_{\max}$ at $t = 0$, the instant at which the switch is thrown closed.

$$\frac{dU}{dt} = -I^2R$$

where the negative sign signifies that the energy U of the circuit is decreasing in time. Substituting this result into Equation 32.19 gives

$$LI\frac{dI}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -I^2R \qquad \textbf{(32.28)}$$

To convert this equation into a form that allows us to compare the electrical oscillations with their mechanical analog, we first use the fact that $I = dQ/dt$ and move all terms to the left-hand side to obtain

$$LI\frac{d^2Q}{dt^2} + \frac{Q}{C}I + I^2R = 0$$

Now we divide through by I:

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} + IR = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \qquad \textbf{(32.29)}$$

The RLC circuit is analogous to the damped harmonic oscillator discussed in Section 13.6 and illustrated in Figure 32.20. The equation of motion for this mechanical system is, from Equation 13.32,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \qquad \textbf{(32.30)}$$

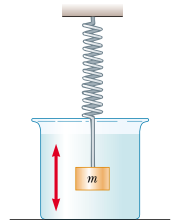


Figure 32.20 A block–spring system moving in a viscous medium with damped harmonic motion is analogous to an RLC circuit.

Comparing Equations 32.29 and 32.30, we see that Q corresponds to the position x of the block at any instant, L to the mass m of the block, R to the damping coefficient b, and C to $1/k$, where k is the force constant of the spring. These and other relationships are listed in Table 32.1.

Because the analytical solution of Equation 32.29 is cumbersome, we give only a qualitative description of the circuit behavior. In the simplest case, when $R = 0$, Equation 32.29 reduces to that of a simple LC circuit, as expected, and the charge and the current oscillate sinusoidally in time. This is equivalent to removal of all damping in the mechanical oscillator.

When R is small, a situation analogous to light damping in the mechanical oscillator, the solution of Equation 32.29 is

$$Q = Q_{\max}e^{-Rt/2L} \cos \omega_d t \qquad \textbf{(32.31)}$$

TABLE 32.1 Analogies Between Electrical and Mechanical Systems

Electric Circuit		One-Dimensional Mechanical System
Charge	$Q \leftrightarrow x$	Displacement
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving mass
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped mass on a spring

where

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2} \qquad \textbf{(32.32)}$$

is the angular frequency at which the circuit oscillates. That is, the value of the charge on the capacitor undergoes a damped harmonic oscillation in analogy with a mass–spring system moving in a viscous medium. From Equation 32.32, we see that, when $R \ll \sqrt{4L/C}$ (so that the second term in the brackets is much smaller than the first), the frequency ω_d of the damped oscillator is close to that of the undamped oscillator, $1/\sqrt{LC}$. Because $I = dQ/dt$, it follows that the current also undergoes damped harmonic oscillation. A plot of the charge versus time for the damped oscillator is shown in Figure 32.21a. Note that the maximum value of Q decreases after each oscillation, just as the amplitude of a damped block–spring system decreases in time.

Quick Quiz 32.6

Figure 32.21a has two dashed blue lines that form an "envelope" around the curve. What is the equation for the upper dashed line?

When we consider larger values of R, we find that the oscillations damp out more rapidly; in fact, there exists a critical resistance value $R_c = \sqrt{4L/C}$ above which no oscillations occur. A system with $R = R_c$ is said to be *critically damped.* When R exceeds R_c, the system is said to be *overdamped* (Fig. 32.22).

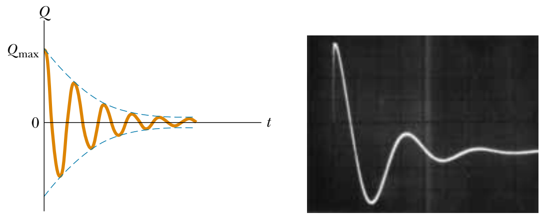


Figure 32.21 (a) Charge versus time for a damped *RLC* circuit. The charge decays in this way when $R \ll \sqrt{4L/C}$. The Q-versus-t curve represents a plot of Equation 32.31. (b) Oscilloscope pattern showing the decay in the oscillations of an *RLC* circuit. The parameters used were $R = 75\ \Omega$, $L = 10$ mH, and $C = 0.19\ \mu$F.

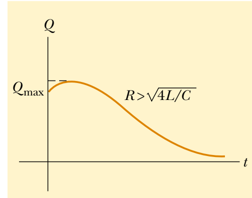


Figure 32.22 Plot of Q versus t for an overdamped *RLC* circuit, which occurs for values of $R > \sqrt{4L/C}$.

SUMMARY

When the current in a coil changes with time, an emf is induced in the coil according to Faraday's law. The **self-induced emf** is

$$\mathcal{E}_L = -L\frac{dI}{dt} \qquad \textbf{(32.1)}$$

where L is the **inductance** of the coil. Inductance is a measure of how much opposition an electrical device offers to a change in current passing through the device. Inductance has the SI unit of **henry** (H), where $1\ \text{H} = 1\ \text{V}\cdot\text{s/A}$.

The inductance of any coil is

$$L = \frac{N\Phi_B}{I} \qquad \textbf{(32.2)}$$

where Φ_B is the magnetic flux through the coil and N is the total number of turns. The inductance of a device depends on its geometry. For example, the inductance of an air-core solenoid is

$$L = \frac{\mu_0 N^2 A}{\ell} \qquad \textbf{(32.4)}$$

where A is the cross-sectional area, and ℓ is the length of the solenoid.

If a resistor and inductor are connected in series to a battery of emf $\mathcal{E}$, and if a switch in the circuit is thrown closed at $t = 0$, then the current in the circuit varies in time according to the expression

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) \qquad \textbf{(32.7)}$$

where $\tau = L/R$ is the time constant of the *RL* circuit. That is, the current increases to an equilibrium value of $\mathcal{E}/R$ after a time that is long compared with τ. If the battery in the circuit is replaced by a resistanceless wire, the current decays exponentially with time according to the expression

$$I = \frac{\mathcal{E}}{R}e^{-t/\tau} \qquad \textbf{(32.10)}$$

where $\mathcal{E}/R$ is the initial current in the circuit.

The energy stored in the magnetic field of an inductor carrying a current I is

$$U = \tfrac{1}{2}LI^2 \quad \textbf{(32.12)}$$

This energy is the magnetic counterpart to the energy stored in the electric field of a charged capacitor.

The energy density at a point where the magnetic field is B is

$$u_B = \frac{B^2}{2\mu_0} \quad \textbf{(32.14)}$$

The mutual inductance of a system of two coils is given by

$$M_{12} = \frac{N_2\Phi_{12}}{I_1} = M_{21} = \frac{N_1\Phi_{21}}{I_2} = M \quad \textbf{(32.15)}$$

This mutual inductance allows us to relate the induced emf in a coil to the changing source current in a nearby coil using the relationships

$$\mathcal{E}_2 = -M_{12}\frac{dI_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M_{21}\frac{dI_2}{dt} \quad \textbf{(32.16, 32.17)}$$

In an LC circuit that has zero resistance and does not radiate electromagnetically (an idealization), the values of the charge on the capacitor and the current in the circuit vary in time according to the expressions

$$Q = Q_{\text{max}}\cos(\omega t + \phi) \quad \textbf{(32.21)}$$

$$I = \frac{dQ}{dt} = -\omega Q_{\text{max}}\sin(\omega t + \phi) \quad \textbf{(32.23)}$$

where Q_{max} is the maximum charge on the capacitor, ϕ is a phase constant, and ω is the angular frequency of oscillation:

$$\omega = \frac{1}{\sqrt{LC}} \quad \textbf{(32.22)}$$

The energy in an LC circuit continuously transfers between energy stored in the capacitor and energy stored in the inductor. The total energy of the LC circuit at any time t is

$$U = U_C + U_L = \frac{Q_{\text{max}}^2}{2C}\cos^2\omega t + \frac{LI_{\text{max}}^2}{2}\sin^2\omega t \quad \textbf{(32.26)}$$

At $t = 0$, all of the energy is stored in the electric field of the capacitor ($U = Q_{\text{max}}^2/2C$). Eventually, all of this energy is transferred to the inductor ($U = LI_{\text{max}}^2/2$). However, the total energy remains constant because energy transformations are neglected in the ideal LC circuit.

QUESTIONS

1. Why is the induced emf that appears in an inductor called a "counter" or "back" emf?
2. The current in a circuit containing a coil, resistor, and battery reaches a constant value. Does the coil have an inductance? Does the coil affect the value of the current?
3. What parameters affect the inductance of a coil? Does the inductance of a coil depend on the current in the coil?
4. How can a long piece of wire be wound on a spool so that the wire has a negligible self-inductance?
5. A long, fine wire is wound as a solenoid with a self-inductance L. If it is connected across the terminals of a battery, how does the maximum current depend on L?
6. For the series RL circuit shown in Figure Q32.6, can the back emf ever be greater than the battery emf? Explain.

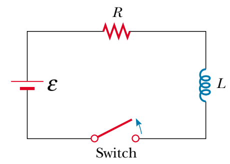


Figure Q32.6

7. Consider this thesis: "Joseph Henry, America's first professional physicist, changed the view of the Universe during a school vacation at the Albany Academy in 1830. Before that time, one could think of the Universe as consisting of just one thing: matter. In Henry's experiment, after a battery is removed from a coil, the energy that keeps the current flowing for a while does not belong to any piece of matter. This energy belongs to the magnetic field surrounding the coil. With Henry's discovery of self-induction, Nature forced us to admit that the Universe consists of fields as well as matter." What in your view constitutes the Universe? Argue for your answer.
8. Discuss the similarities and differences between the energy stored in the electric field of a charged capacitor and the energy stored in the magnetic field of a current-carrying coil.
9. What is the inductance of two inductors connected in series? Does it matter if they are solenoids or toroids?
10. The centers of two circular loops are separated by a fixed distance. For what relative orientation of the loops is their mutual inductance a maximum? a minimum? Explain.
11. Two solenoids are connected in series so that each carries the same current at any instant. Is mutual induction present? Explain.
12. In the LC circuit shown in Figure 32.15, the charge on the capacitor is sometimes zero, even though current is in the circuit. How is this possible?
13. If the resistance of the wires in an LC circuit were not zero, would the oscillations persist? Explain.
14. How can you tell whether an RLC circuit is overdamped or underdamped?
15. What is the significance of critical damping in an RLC circuit?
16. Can an object exert a force on itself? When a coil induces an emf in itself, does it exert a force on itself?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** ▭ = Computer useful in solving problem ◎ = Interactive Physics
▭ = paired numerical/symbolic problems

Section 32.1 Self-Inductance

1. A coil has an inductance of 3.00 mH, and the current through it changes from 0.200 A to 1.50 A in a time of 0.200 s. Find the magnitude of the average induced emf in the coil during this time.
2. A coiled telephone cord forms a spiral with 70 turns, a diameter of 1.30 cm, and an unstretched length of 60.0 cm. Determine the self-inductance of one conductor in the unstretched cord.
3. A 2.00-H inductor carries a steady current of 0.500 A. When the switch in the circuit is thrown open, the current is effectively zero in 10.0 ms. What is the average induced emf in the inductor during this time?
4. A small air-core solenoid has a length of 4.00 cm and a radius of 0.250 cm. If the inductance is to be 0.060 0 mH, how many turns per centimeter are required?
5. Calculate the magnetic flux through the area enclosed by a 300-turn, 7.20-mH coil when the current in the coil is 10.0 mA.
6. The current in a solenoid is increasing at a rate of 10.0 A/s. The cross-sectional area of the solenoid is π cm², and there are 300 turns on its 15.0-cm length. What is the induced emf opposing the increasing current?

WEB 7. A 10.0-mH inductor carries a current $I = I_{\text{max}}\sin\omega t$, with $I_{\text{max}} = 5.00$ A and $\omega/2\pi = 60.0$ Hz. What is the back emf as a function of time?
8. An emf of 24.0 mV is induced in a 500-turn coil at an instant when the current is 4.00 A and is changing at the rate of 10.0 A/s. What is the magnetic flux through each turn of the coil?
9. An inductor in the form of a solenoid contains 420 turns, is 16.0 cm in length, and has a cross-sectional area of 3.00 cm². What uniform rate of decrease of current through the inductor induces an emf of 175 μV?
10. An inductor in the form of a solenoid contains N turns, has length ℓ, and has cross-sectional area A. What uniform rate of decrease of current through the inductor induces an emf $\mathcal{E}$?
11. The current in a 90.0-mH inductor changes with time as $I = t^2 - 6.00t$ (in SI units). Find the magnitude of the induced emf at (a) $t = 1.00$ s and (b) $t = 4.00$ s. (c) At what time is the emf zero?
12. A 40.0-mA current is carried by a uniformly wound air-core solenoid with 450 turns, a 15.0-mm diameter, and 12.0-cm length. Compute (a) the magnetic field inside the solenoid, (b) the magnetic flux through each turn,

and (c) the inductance of the solenoid. (d) Which of these quantities depends on the current?

13. A solenoid has 120 turns uniformly wrapped around a wooden core, which has a diameter of 10.0 mm and a length of 9.00 cm. (a) Calculate the inductance of the solenoid. (b) The wooden core is replaced with a soft iron rod that has the same dimensions but a magnetic permeability $\mu_m = 800\mu_0$. What is the new inductance?

14. A toroid has a major radius R and a minor radius r, and it is tightly wound with N turns of wire, as shown in Figure P32.14. If $R \gg r$, the magnetic field within the region of the torus, of cross-sectional area $A = \pi r^2$, is essentially that of a long solenoid that has been bent into a large circle of radius R. Using the uniform field of a long solenoid, show that the self-inductance of such a toroid is approximately

$$L \cong \mu_0 N^2 A / 2\pi R$$

(An exact expression for the inductance of a toroid with a rectangular cross-section is derived in Problem 64.)

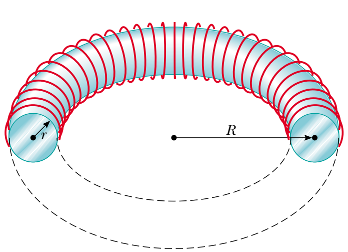


Figure P32.14

15. An emf self-induced in a solenoid of inductance L changes in time as $\mathcal{E} = \mathcal{E}_0 e^{-kt}$. Find the total charge that passes through the solenoid, if the charge is finite.

Section 32.2 *RL* Circuits

16. Calculate the resistance in an RL circuit in which $L = 2.50$ H and the current increases to 90.0% of its final value in 3.00 s.

17. A 12.0-V battery is connected into a series circuit containing a 10.0-Ω resistor and a 2.00-H inductor. How long will it take the current to reach (a) 50.0% and (b) 90.0% of its final value?

18. Show that $I = I_0 e^{-t/\tau}$ is a solution of the differential equation

$$IR + L\frac{dI}{dt} = 0$$

where $\tau = L/R$ and I_0 is the current at $t = 0$.

19. Consider the circuit in Figure P32.19, taking $\mathcal{E} = 6.00$ V, $L = 8.00$ mH, and $R = 4.00$ Ω. (a) What is the inductive time constant of the circuit? (b) Calculate the current in the circuit 250 μs after the switch is closed. (c) What is the value of the final steady-state current? (d) How long does it take the current to reach 80.0% of its maximum value?

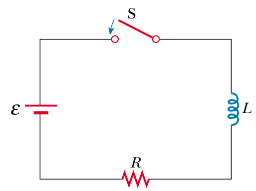


Figure P32.19 Problems 19, 20, 21, and 24.

20. In the circuit shown in Figure P32.19, let $L = 7.00$ H, $R = 9.00$ Ω, and $\mathcal{E} = 120$ V. What is the self-induced emf 0.200 s after the switch is closed?

WEB **21.** For the RL circuit shown in Figure P32.19, let $L = 3.00$ H, $R = 8.00$ Ω, and $\mathcal{E} = 36.0$ V. (a) Calculate the ratio of the potential difference across the resistor to that across the inductor when $I = 2.00$ A. (b) Calculate the voltage across the inductor when $I = 4.50$ A.

22. A 12.0-V battery is connected in series with a resistor and an inductor. The circuit has a time constant of 500 μs, and the maximum current is 200 mA. What is the value of the inductance?

23. An inductor that has an inductance of 15.0 H and a resistance of 30.0 Ω is connected across a 100-V battery. What is the rate of increase of the current (a) at $t = 0$ and (b) at $t = 1.50$ s?

24. When the switch in Figure P32.19 is thrown closed, the current takes 3.00 ms to reach 98.0% of its final value. If $R = 10.0$ Ω, what is the inductance?

25. The switch in Figure P32.25 is closed at time $t = 0$. Find the current in the inductor and the current through the switch as functions of time thereafter.

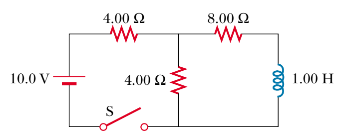


Figure P32.25

26. A series RL circuit with $L = 3.00$ H and a series RC circuit with $C = 3.00$ μF have equal time constants. If the two circuits contain the same resistance R, (a) what is the value of R and (b) what is the time constant?

27. A current pulse is fed to the partial circuit shown in Figure P32.27. The current begins at zero, then becomes 10.0 A between $t = 0$ and $t = 200$ μs, and then is zero once again. Determine the current in the inductor as a function of time.

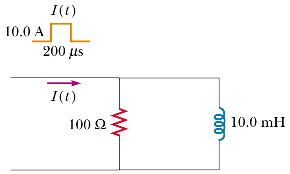


Figure P32.27

28. One application of an RL circuit is the generation of time-varying high voltage from a low-voltage source, as shown in Figure P32.28. (a) What is the current in the circuit a long time after the switch has been in position A? (b) Now the switch is thrown quickly from A to B. Compute the initial voltage across each resistor and the inductor. (c) How much time elapses before the voltage across the inductor drops to 12.0 V?

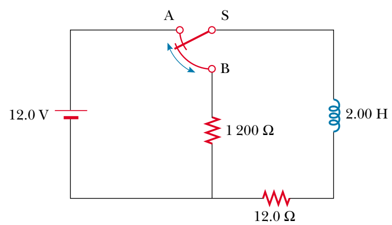


Figure P32.28

WEB **29.** A 140-mH inductor and a 4.90-Ω resistor are connected with a switch to a 6.00-V battery, as shown in Figure P32.29. (a) If the switch is thrown to the left (connecting the battery), how much time elapses before the current reaches 220 mA? (b) What is the current in the inductor 10.0 s after the switch is closed? (c) Now the switch is quickly thrown from A to B. How much time elapses before the current falls to 160 mA?

30. Consider two ideal inductors, L_1 and L_2, that have *zero* internal resistance and are far apart, so that their magnetic fields do not influence each other. (a) If these inductors are connected in series, show that they are equivalent to a single ideal inductor having $L_{eq} = L_1 + L_2$. (b) If these same two inductors are connected in parallel, show that they are equivalent to a single ideal inductor having $1/L_{eq} = 1/L_1 + 1/L_2$. (c) Now consider two inductors L_1 and L_2 that have *nonzero* internal resistances R_1 and R_2, respectively. Assume that they are still far apart so that their magnetic fields do not influence each other. If these inductors are connected in series, show that they are equivalent to a single inductor having $L_{eq} = L_1 + L_2$ and $R_{eq} = R_1 + R_2$. (d) If these same inductors are now connected in parallel, is it necessarily true that they are equivalent to a single ideal inductor having $1/L_{eq} = 1/L_1 + 1/L_2$ and $1/R_{eq} = 1/R_1 + 1/R_2$? Explain your answer.

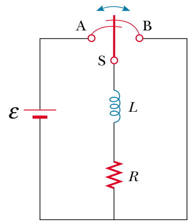


Figure P32.29

Section 32.3 Energy in a Magnetic Field

31. Calculate the energy associated with the magnetic field of a 200-turn solenoid in which a current of 1.75 A produces a flux of 3.70×10^{-4} T·m^2 in each turn.

32. The magnetic field inside a superconducting solenoid is 4.50 T. The solenoid has an inner diameter of 6.20 cm and a length of 26.0 cm. Determine (a) the magnetic energy density in the field and (b) the energy stored in the magnetic field within the solenoid.

33. An air-core solenoid with 68 turns is 8.00 cm long and has a diameter of 1.20 cm. How much energy is stored in its magnetic field when it carries a current of 0.770 A?

34. At $t = 0$, an emf of 500 V is applied to a coil that has an inductance of 0.800 H and a resistance of 30.0 Ω. (a) Find the energy stored in the magnetic field when the current reaches half its maximum value. (b) After the emf is connected, how long does it take the current to reach this value?

WEB **35.** On a clear day there is a 100-V/m vertical electric field near the Earth's surface. At the same place, the Earth's magnetic field has a magnitude of 0.500×10^{-4} T. Compute the energy densities of the two fields.

36. An RL circuit in which $L = 4.00$ H and $R = 5.00$ Ω is connected to a 22.0-V battery at $t = 0$. (a) What energy is stored in the inductor when the current is 0.500 A? (b) At what rate is energy being stored in the inductor when $I = 1.00$ A? (c) What power is being delivered to the circuit by the battery when $I = 0.500$ A?

37. A 10.0-V battery, a 5.00-Ω resistor, and a 10.0-H inductor are connected in series. After the current in the circuit has reached its maximum value, calculate (a) the power being supplied by the battery, (b) the power being delivered to the resistor, (c) the power being delivered to the inductor, and (d) the energy stored in the magnetic field of the inductor.

38. A uniform electric field with a magnitude of 680 kV/m throughout a cylindrical volume results in a total energy of 3.40 μJ. What magnetic field over this same region stores the same total energy?

39. Assume that the magnitude of the magnetic field outside a sphere of radius R is $B = B_0(R/r)^2$, where B_0 is a constant. Determine the total energy stored in the magnetic field outside the sphere and evaluate your result for $B_0 = 5.00 \times 10^{-5}$ T and $R = 6.00 \times 10^6$ m, values appropriate for the Earth's magnetic field.

Section 32.4 Mutual Inductance

40. Two coils are close to each other. The first coil carries a time-varying current given by $I(t) = (5.00 \text{ A})\, e^{-0.025\,0t} \sin(377t)$. At $t = 0.800$ s, the voltage measured across the second coil is -3.20 V. What is the mutual inductance of the coils?

41. Two coils, held in fixed positions, have a mutual inductance of 100 μH. What is the peak voltage in one when a sinusoidal current given by $I(t) = (10.0 \text{ A}) \sin(1\,000t)$ flows in the other?

42. An emf of 96.0 mV is induced in the windings of a coil when the current in a nearby coil is increasing at the rate of 1.20 A/s. What is the mutual inductance of the two coils?

43. Two solenoids A and B, spaced close to each other and sharing the same cylindrical axis, have 400 and 700 turns, respectively. A current of 3.50 A in coil A produces an average flux of 300 $\mu\text{T}\cdot\text{m}^2$ through each turn of A and a flux of 90.0 $\mu\text{T}\cdot\text{m}^2$ through each turn of B. (a) Calculate the mutual inductance of the two solenoids. (b) What is the self-inductance of A? (c) What emf is induced in B when the current in A increases at the rate of 0.500 A/s?

44. A 70-turn solenoid is 5.00 cm long and 1.00 cm in diameter and carries a 2.00-A current. A single loop of wire, 3.00 cm in diameter, is held so that the plane of the loop is perpendicular to the long axis of the solenoid, as illustrated in Figure P31.18 (page 1004). What is the mutual inductance of the two if the plane of the loop passes through the solenoid 2.50 cm from one end?

45. Two single-turn circular loops of wire have radii R and r, with $R \gg r$. The loops lie in the same plane and are concentric. (a) Show that the mutual inductance of the pair is $M = \mu_0 \pi r^2 / 2R$. (*Hint:* Assume that the larger loop carries a current I and compute the resulting flux through the smaller loop.) (b) Evaluate M for $r = 2.00$ cm and $R = 20.0$ cm.

46. On a printed circuit board, a relatively long straight conductor and a conducting rectangular loop lie in the same plane, as shown in Figure P31.9 (page 1003). If $h = 0.400$ mm, $w = 1.30$ mm, and $L = 2.70$ mm, what is their mutual inductance?

47. Two inductors having self-inductances L_1 and L_2 are connected in parallel, as shown in Figure P32.47a. The mutual inductance between the two inductors is M. Determine the equivalent self-inductance L_{eq} for the system (Fig. P32.47b).

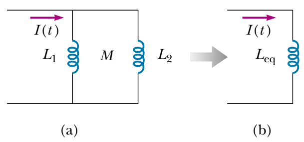


Figure P32.47

Section 32.5 Oscillations in an *LC* Circuit

48. A 1.00-μF capacitor is charged by a 40.0-V power supply. The fully-charged capacitor is then discharged through a 10.0-mH inductor. Find the maximum current in the resulting oscillations.

49. An LC circuit consists of a 20.0-mH inductor and a 0.500-μF capacitor. If the maximum instantaneous current is 0.100 A, what is the greatest potential difference across the capacitor?

50. In the circuit shown in Figure P32.50, $\mathcal{E} = 50.0$ V, $R = 250\ \Omega$, and $C = 0.500\ \mu$F. The switch S is closed for a long time, and no voltage is measured across the capacitor. After the switch is opened, the voltage across the capacitor reaches a maximum value of 150 V. What is the inductance L?

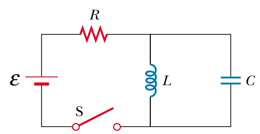


Figure P32.50

51. A fixed inductance $L = 1.05\ \mu$H is used in series with a variable capacitor in the tuning section of a radio. What capacitance tunes the circuit to the signal from a station broadcasting at 6.30 MHz?

52. Calculate the inductance of an LC circuit that oscillates at 120 Hz when the capacitance is 8.00 μF.

53. An LC circuit like the one shown in Figure 32.14 contains an 82.0-mH inductor and a 17.0-μF capacitor that initially carries a 180-μC charge. The switch is thrown closed at $t = 0$. (a) Find the frequency (in hertz) of the resulting oscillations. At $t = 1.00$ ms, find (b) the charge on the capacitor and (c) the current in the circuit.

54. The switch in Figure P32.54 is connected to point a for a long time. After the switch is thrown to point b, what are (a) the frequency of oscillation of the LC circuit, (b) the maximum charge that appears on the capacitor, (c) the maximum current in the inductor, and (d) the total energy the circuit possesses at $t = 3.00$ s?

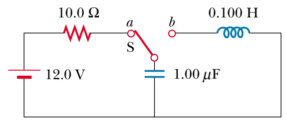


Figure P32.54

WEB **55.** An LC circuit like that illustrated in Figure 32.14 consists of a 3.30-H inductor and an 840-pF capacitor, initially carrying a 105-μC charge. At $t = 0$ the switch is thrown closed. Compute the following quantities at $t = 2.00$ ms: (a) the energy stored in the capacitor; (b) the energy stored in the inductor; (c) the total energy in the circuit.

(Optional)
Section 32.6 The *RLC* Circuit

56. In Figure 32.19, let $R = 7.60\ \Omega$, $L = 2.20$ mH, and $C = 1.80\ \mu$F. (a) Calculate the frequency of the damped oscillation of the circuit. (b) What is the critical resistance?

57. Consider an LC circuit in which $L = 500$ mH and $C = 0.100\ \mu$F. (a) What is the resonant frequency ω_0? (b) If a resistance of 1.00 kΩ is introduced into this circuit, what is the frequency of the (damped) oscillations? (c) What is the percent difference between the two frequencies?

58. Show that Equation 32.29 in the text is Kirchhoff's loop rule as applied to Figure 32.19.

59. Electrical oscillations are initiated in a series circuit containing a capacitance C, inductance L, and resistance R. (a) If $R \ll \sqrt{4L/C}$ (weak damping), how much time elapses before the amplitude of the current oscillation falls off to 50.0% of its initial value? (b) How long does it take the energy to decrease to 50.0% of its initial value?

ADDITIONAL PROBLEMS

60. Initially, the capacitor in a series LC circuit is charged. A switch is closed, allowing the capacitor to discharge, and after time t the energy stored in the capacitor is one-fourth its initial value. Determine L if C is known.

61. A 1.00-mH inductor and a 1.00-μF capacitor are connected in series. The current in the circuit is described by $I = 20.0t$, where t is in seconds and I is in amperes. The capacitor initially has no charge. Determine (a) the voltage across the inductor as a function of time, (b) the voltage across the capacitor as a function of time, and (c) the time when the energy stored in the capacitor first exceeds that in the inductor.

62. An inductor having inductance L and a capacitor having capacitance C are connected in series. The current in the circuit increases linearly in time as described by $I = Kt$. The capacitor is initially uncharged. Determine (a) the voltage across the inductor as a function of time, (b) the voltage across the capacitor as a function of time, and (c) the time when the energy stored in the capacitor first exceeds that in the inductor.

63. A capacitor in a series LC circuit has an initial charge Q and is being discharged. Find, in terms of L and C, the flux through each of the N turns in the coil, when the charge on the capacitor is $Q/2$.

64. The toroid in Figure P32.64 consists of N turns and has a rectangular cross-section. Its inner and outer radii are a and b, respectively. (a) Show that

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$$

(b) Using this result, compute the self-inductance of a 500-turn toroid for which $a = 10.0$ cm, $b = 12.0$ cm, and $h = 1.00$ cm. (c) In Problem 14, an approximate formula for the inductance of a toroid with $R \gg r$ was derived. To get a feel for the accuracy of that result, use the expression in Problem 14 to compute the approximate inductance of the toroid described in part (b). Compare the result with the answer to part (b).

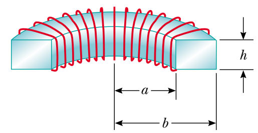


Figure P32.64

65. (a) A flat circular coil does not really produce a uniform magnetic field in the area it encloses, but estimate the self-inductance of a flat circular coil, with radius R and N turns, by supposing that the field at its center is uniform over its area. (b) A circuit on a laboratory table consists of a 1.5-V battery, a 270-Ω resistor, a switch, and three 30-cm-long cords connecting them. Suppose that the circuit is arranged to be circular. Think of it as a flat coil with one turn. Compute the order of magnitude of its self-inductance and (c) of the time constant describing how fast the current increases when you close the switch.

66. A soft iron rod ($\mu_m = 800\ \mu_0$) is used as the core of a solenoid. The rod has a diameter of 24.0 mm and is

10.0 cm long. A 10.0-m piece of 22-gauge copper wire (diameter = 0.644 mm) is wrapped around the rod in a single uniform layer, except for a 10.0-cm length at each end, which is to be used for connections. (a) How many turns of this wire can wrap around the rod? (*Hint:* The diameter of the wire adds to the diameter of the rod in determining the circumference of each turn. Also, the wire spirals diagonally along the surface of the rod.) (b) What is the resistance of this inductor? (c) What is its inductance?

67. A wire of nonmagnetic material with radius R carries current uniformly distributed over its cross-section. If the total current carried by the wire is I, show that the magnetic energy per unit length inside the wire is $\mu_0 I^2/16\pi$.

68. An 820-turn wire coil of resistance 24.0 Ω is placed around a 12 500-turn solenoid, 7.00 cm long, as shown in Figure P32.68. Both coil and solenoid have cross-sectional areas of 1.00×10^{-4} m^2. (a) How long does it take the solenoid current to reach 63.2 percent of its maximum value? Determine (b) the average back emf caused by the self-inductance of the solenoid during this interval, (c) the average rate of change in magnetic flux through the coil during this interval, and (d) the magnitude of the average induced current in the coil.

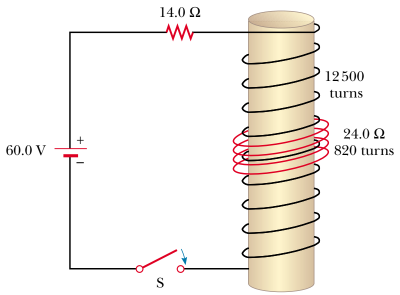


Figure P32.68

69. At $t = 0$, the switch in Figure P32.69 is thrown closed. Using Kirchhoff's laws for the instantaneous currents and voltages in this two-loop circuit, show that the current in the inductor is

$$I(t) = \frac{\mathcal{E}}{R_1}\left[1 - e^{-(R'/L)t}\right]$$

where $R' = R_1 R_2/(R_1 + R_2)$.

70. In Figure P32.69, take $\mathcal{E} = 6.00$ V, $R_1 = 5.00\ \Omega$, and $R_2 = 1.00\ \Omega$. The inductor has negligible resistance. When the switch is thrown open after having been closed for a long time, the current in the inductor drops to 0.250 A in 0.150 s. What is the inductance of the inductor?

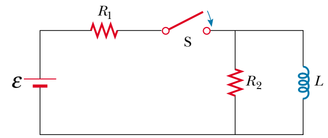


Figure P32.69 Problems 69 and 70.

71. In Figure P32.71, the switch is closed for $t < 0$, and steady-state conditions are established. The switch is thrown open at $t = 0$. (a) Find the initial voltage $\mathcal{E}_0$ across L just after $t = 0$. Which end of the coil is at the higher potential: a or b? (b) Make freehand graphs of the currents in R_1 and in R_2 as a function of time, treating the steady-state directions as positive. Show values before and after $t = 0$. (c) How long after $t = 0$ does the current in R_2 have the value 2.00 mA?

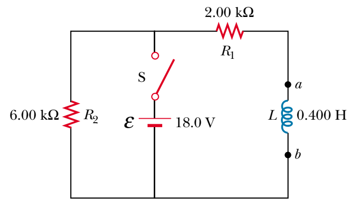


Figure P32.71

72. The switch in Figure P32.72 is thrown closed at $t = 0$. Before the switch is closed, the capacitor is uncharged, and all currents are zero. Determine the currents in L, C, and R and the potential differences across L, C, and R (a) the instant after the switch is closed and (b) long after it is closed.

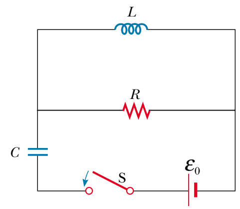


Figure P32.72

73. To prevent damage from arcing in an electric motor, a discharge resistor is sometimes placed in parallel with the armature. If the motor is suddenly unplugged while running, this resistor limits the voltage that appears across the armature coils. Consider a 12.0-V dc motor with an armature that has a resistance of 7.50 Ω and an inductance of 450 mH. Assume that the back emf in the armature coils is 10.0 V when the motor is running at normal speed. (The equivalent circuit for the armature is shown in Fig. P32.73.) Calculate the maximum resistance R that limits the voltage across the armature to 80.0 V when the motor is unplugged.

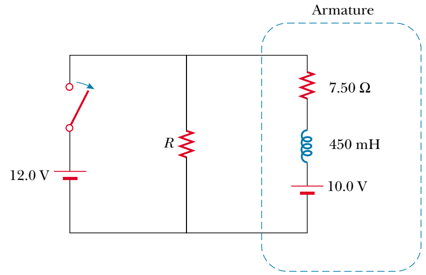


Figure P32.73

74. An air-core solenoid 0.500 m in length contains 1 000 turns and has a cross-sectional area of 1.00 cm^2. (a) If end effects are neglected, what is the self-inductance? (b) A secondary winding wrapped around the center of the solenoid has 100 turns. What is the mutual inductance? (c) The secondary winding carries a constant current of 1.00 A, and the solenoid is connected to a load of 1.00 kΩ. The constant current is suddenly stopped. How much charge flows through the load resistor?

75. The lead-in wires from a television antenna are often constructed in the form of two parallel wires (Fig. P32.75). (a) Why does this configuration of conductors have an inductance? (b) What constitutes the flux loop for this configuration? (c) Neglecting any magnetic flux inside the wires, show that the inductance of a length x of this type of lead-in is

$$L = \frac{\mu_0 x}{\pi} \ln\left(\frac{w - a}{a}\right)$$

where a is the radius of the wires and w is their center-to-center separation.

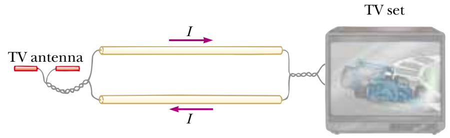


Figure P32.75

Note: Problems 76 through 79 require the application of ideas from this chapter and earlier chapters to some properties of superconductors, which were introduced in Section 27.5.

76. **Review Problem.** *The resistance of a superconductor.* In an experiment carried out by S. C. Collins between 1955 and 1958, a current was maintained in a superconducting lead ring for 2.50 yr with no observed loss. If the inductance of the ring was 3.14×10^{-8} H and the sensitivity of the experiment was 1 part in 10^9, what was the maximum resistance of the ring? (*Hint:* Treat this as a decaying current in an RL circuit, and recall that $e^{-x} \cong 1 - x$ for small x.)

77. **Review Problem.** A novel method of storing electrical energy has been proposed. A huge underground superconducting coil, 1.00 km in diameter, would be fabricated. It would carry a maximum current of 50.0 kA through each winding of a 150-turn Nb_3Sn solenoid. (a) If the inductance of this huge coil were 50.0 H, what would be the total energy stored? (b) What would be the compressive force per meter length acting between two adjacent windings 0.250 m apart?

78. **Review Problem.** *Superconducting Power Transmission.* The use of superconductors has been proposed for the manufacture of power transmission lines. A single coaxial cable (Fig. P32.78) could carry 1.00×10^3 MW (the output of a large power plant) at 200 kV, dc, over a distance of 1 000 km without loss. An inner wire with a radius of 2.00 cm, made from the superconductor Nb_3Sn, carries the current I in one direction. A surrounding superconducting cylinder, of radius 5.00 cm, would carry the return current I. In such a system, what is the magnetic field (a) at the surface of the inner conductor and (b) at the inner surface of the outer conductor? (c) How much energy would be stored in the space between the conductors in a 1 000-km superconducting line? (d) What is the pressure exerted on the outer conductor?

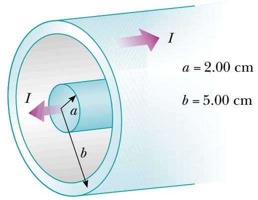


Figure P32.78

79. Review Problem. *The Meissner Effect.* Compare this problem with Problem 63 in Chapter 26 on the force attracting a perfect dielectric into a strong electric field. A fundamental property of a Type I superconducting material is *perfect diamagnetism,* or demonstration of the *Meissner effect,* illustrated in the photograph on page 855 and again in Figure 30.34, and described as follows: The superconducting material has $\mathbf{B} = 0$ everywhere inside it. If a sample of the material is placed into an externally produced magnetic field, or if it is cooled to become superconducting while it is in a magnetic field, electric currents appear on the surface of the sample. The currents have precisely the strength and orientation required to make the total magnetic field zero throughout the interior of the sample. The following problem will help you to understand the magnetic force that can then act on the superconducting sample.

Consider a vertical solenoid with a length of 120 cm and a diameter of 2.50 cm consisting of 1 400 turns of copper wire carrying a counterclockwise current of 2.00 A, as shown in Figure P32.79a. (a) Find the magnetic field in the vacuum inside the solenoid. (b) Find the energy density of the magnetic field, and note that the units J/m^3 of energy density are the same as the units $\text{N/m}^2 (= \text{Pa})$ of pressure. (c) A superconducting bar 2.20 cm in diameter is inserted partway into the solenoid. Its upper end is far outside the solenoid, where the magnetic field is small. The lower end of the bar is deep inside the solenoid. Identify the direction required for the current on the curved surface of the bar so that the total magnetic field is zero within the bar. The field created by the supercurrents is sketched in Figure P32.79b, and the total field is sketched in Figure P32.79c. (d) The field of the solenoid exerts a force on the current in the superconductor. Identify the direction of the force on the bar. (e) Calculate the magnitude of the force by multiplying the energy density of the solenoid field by the area of the bottom end of the superconducting bar.

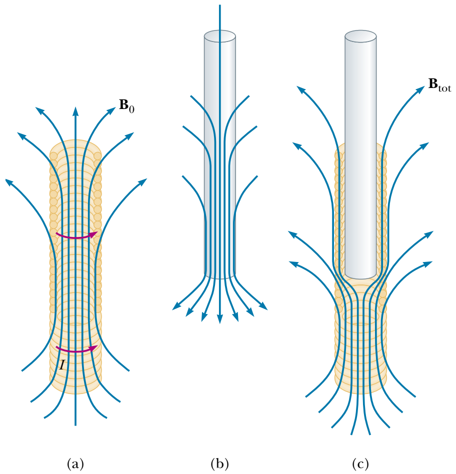


Figure P32.79

ANSWERS TO QUICK QUIZZES

32.1 When it is being opened. When the switch is initially open, there is no current in the circuit; when the switch is then closed, the inductor tends to maintain the no-current condition, and as a result there is very little chance of sparking. When the switch is initially closed, there is current in the circuit; when the switch is then opened, the current decreases. An induced emf is set up across the inductor, and this emf tends to maintain the original current. Sparking can occur as the current bridges the air gap between the poles of the switch.

32.2 (b). Figure 32.8 shows that circuit B has the greater time constant because in this circuit it takes longer for the current to reach its maximum value and then longer for this current to decrease to zero after switch S_2 is closed. Equation 32.8 indicates that, for equal resistances R_A and R_B, the condition $\tau_B > \tau_A$ means that $L_A < L_B$.

32.3 (a) M_{12} increases because the magnetic flux through coil 2 increases. (b) M_{12} decreases because rotation of coil 1 decreases its flux through coil 2.

32.4 (a) No. Mutual inductance requires a system of coils, and each coil has self-inductance. (b) Yes. A single coil has self-inductance but no mutual inductance because it does not interact with any other coils.

32.5 From Equation 32.25, $I_{\max} = \omega Q_{\max}$. Thus, the amplitude of the I-t graph is ω times the amplitude of the Q-t graph.

32.6 Equation 32.31 without the cosine factor. The dashed lines represent the positive and negative amplitudes (maximum values) for each oscillation period, and it is the $Q = Q_{\max}e^{-Rt/2L}$ part of Equation 32.31 that gives the value of the ever-decreasing amplitude.

PUZZLER

Small "black boxes" like this one are commonly used to supply power to electronic devices such as CD players and tape players. Whereas these devices need only about 12 V to operate, wall outlets provide an output of 120 V. What do the black boxes do, and how do they work? *(George Semple)*

chapter 33

Alternating-Current Circuits

Chapter Outline

In this chapter we describe alternating-current (ac) circuits. Every time we turn on a television set, a stereo, or any of a multitude of other electrical appliances, we are calling on alternating currents to provide the power to operate them. We begin our study by investigating the characteristics of simple series circuits that contain resistors, inductors, and capacitors and that are driven by a sinusoidal voltage. We shall find that the maximum alternating current in each element is proportional to the maximum alternating voltage across the element. We shall also find that when the applied voltage is sinusoidal, the current in each element is sinusoidal, too, but not necessarily in phase with the applied voltage. We conclude the chapter with two sections concerning transformers, power transmission, and *RC* filters.

33.1 AC SOURCES AND PHASORS

An ac circuit consists of circuit elements and a generator that provides the alternating current. As you recall from Section 31.5, the basic principle of the ac generator is a direct consequence of Faraday's law of induction. When a conducting loop is rotated in a magnetic field at constant angular frequency ω, a sinusoidal voltage (emf) is induced in the loop. This instantaneous voltage Δv is

$$\Delta v = \Delta V_{\max} \sin \omega t$$

where $\Delta V_{\max}$ is the maximum output voltage of the ac generator, or the **voltage amplitude.** From Equation 13.6, the angular frequency is

$$\omega = 2\pi f = \frac{2\pi}{T}$$

where f is the frequency of the generator (the voltage source) and T is the period. The generator determines the frequency of the current in any circuit connected to the generator. Because the output voltage of an ac generator varies sinusoidally with time, the voltage is positive during one half of the cycle and negative during the other half. Likewise, the current in any circuit driven by an ac generator is an alternating current that also varies sinusoidally with time. Commercial electric-power plants in the United States use a frequency of 60 Hz, which corresponds to an angular frequency of 377 rad/s.

The primary aim of this chapter can be summarized as follows: If an ac generator is connected to a series circuit containing resistors, inductors, and capacitors, we want to know the amplitude and time characteristics of the alternating current. To simplify our analysis of circuits containing two or more elements, we use graphical constructions called *phasor diagrams.* In these constructions, alternating (sinusoidal) quantities, such as current and voltage, are represented by rotating vectors called **phasors.** The length of the phasor represents the amplitude (maximum value) of the quantity, and the projection of the phasor onto the vertical axis represents the instantaneous value of the quantity. As we shall see, a phasor diagram greatly simplifies matters when we must combine several sinusoidally varying currents or voltages that have different phases.

33.2 RESISTORS IN AN AC CIRCUIT

Consider a simple ac circuit consisting of a resistor and an ac generator —(~)— , as shown in Figure 33.1. At any instant, the algebraic sum of the voltages around a closed loop in a circuit must be zero (Kirchhoff's loop rule). Therefore, $\Delta v - \Delta v_R = 0$, or[1]

$$\Delta v = \Delta v_R = \Delta V_{\max} \sin \omega t \qquad \textbf{(33.1)}$$

where Δv_R is the **instantaneous voltage across the resistor.** Therefore, the instantaneous current in the resistor is

$$i_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{\max}}{R} \sin \omega t = I_{\max} \sin \omega t \qquad \textbf{(33.2)}$$

where $I_{\max}$ is the maximum current:

Maximum current in a resistor

$$I_{\max} = \frac{\Delta V_{\max}}{R}$$

From Equations 33.1 and 33.2, we see that the instantaneous voltage across the resistor is

$$\Delta v_R = I_{\max} R \sin \omega t \qquad \textbf{(33.3)}$$

Let us discuss the current-versus-time curve shown in Figure 33.2a. At point *a*, the current has a maximum value in one direction, arbitrarily called the positive direction. Between points *a* and *b*, the current is decreasing in magnitude but is still in the positive direction. At *b*, the current is momentarily zero; it then begins to increase in the negative direction between points *b* and *c*. At *c*, the current has reached its maximum value in the negative direction.

The current in a resistor is in phase with the voltage

The current and voltage are in step with each other because they vary identically with time. Because i_R and Δv_R both vary as $\sin \omega t$ and reach their maximum values at the same time, as shown in Figure 33.2a, they are said to be **in phase.** Thus we can say that, for a sinusoidal applied voltage, the current in a resistor is always in phase with the voltage across the resistor.

A *phasor diagram* is used to represent current–voltage phase relationships. The lengths of the arrows correspond to $\Delta V_{\max}$ and $I_{\max}$. The projections of the phasor arrows onto the vertical axis give Δv_R and i_R values. As we showed in Section 13.5, if the phasor arrow is imagined to rotate steadily with angular speed ω, its vertical-axis component oscillates sinusoidally in time. In the case of the single-loop resistive circuit of Figure 33.1, the current and voltage phasors lie along the same line, as in Figure 33.2b, because i_R and Δv_R are in phase.

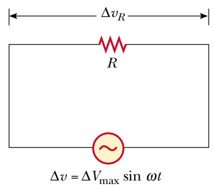


Figure 33.1 A circuit consisting of a resistor of resistance R connected to an ac generator, designated by the symbol  .

Note that **the average value of the current over one cycle is zero.** That is, the current is maintained in the positive direction for the same amount of time and at the same magnitude as it is maintained in the negative direction. However, the direction of the current has no effect on the behavior of the resistor. We can understand this by realizing that collisions between electrons and the fixed atoms of the resistor result in an increase in the temperature of the resistor. Although this temperature increase depends on the magnitude of the current, it is independent of the direction of the current.

We can make this discussion quantitative by recalling that the rate at which electrical energy is converted to internal energy in a resistor is the power $\mathcal{P} = i^2R$, where i is the instantaneous current in the resistor. Because this rate is proportional to the square of the current, it makes no difference whether the current is direct or alternating—that is, whether the sign associated with the current is positive or negative. However, the temperature increase produced by an alternating

[1] The lowercase symbols v and i are used to indicate the instantaneous values of the voltage and the current.

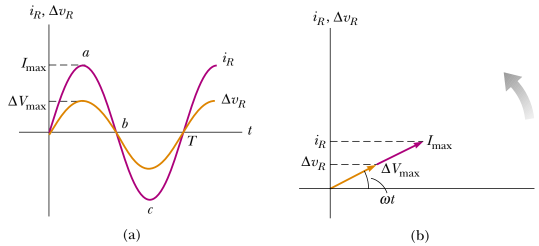


Figure 33.2 (a) Plots of the instantaneous current i_R and instantaneous voltage Δv_R across a resistor as functions of time. The current is in phase with the voltage, which means that the current is zero when the voltage is zero, maximum when the voltage is maximum, and minimum when the voltage is minimum. At time $t = T$, one cycle of the time-varying voltage and current has been completed. (b) Phasor diagram for the resistive circuit showing that the current is in phase with the voltage.

current having a maximum value $I_{\max}$ is not the same as that produced by a direct current equal to $I_{\max}$. This is because the alternating current is at this maximum value for only an instant during each cycle (Fig. 33.3a). What is of importance in an ac circuit is an average value of current, referred to as the **rms current.** As we learned in Section 21.1, the notation *rms* stands for *root mean square,* which in this case means the square root of the mean (average) value of the square of the current: $I_{\text{rms}} = \sqrt{\overline{i^2}}$. Because i^2 varies as $\sin^2 \omega t$ and because the average value of i^2 is $\frac{1}{2} I_{\max}^2$ (see Fig. 33.3b), the rms current is[2]

rms current

$$I_{\text{rms}} = \frac{I_{\max}}{\sqrt{2}} = 0.707 I_{\max} \qquad \textbf{(33.4)}$$

This equation states that an alternating current whose maximum value is 2.00 A delivers to a resistor the same power as a direct current that has a value of (0.707)(2.00 A) = 1.41 A. Thus, we can say that the average power delivered to a resistor that carries an alternating current is

Average power delivered to a resistor

$$\mathcal{P}_{\text{av}} = I_{\text{rms}}^2 R$$

[2] That the square root of the average value of i^2 is equal to $I_{\max}/\sqrt{2}$ can be shown as follows: The current in the circuit varies with time according to the expression $i = I_{\max} \sin \omega t$, so $i^2 = I_{\max}^2 \sin^2 \omega t$. Therefore, we can find the average value of i^2 by calculating the average value of $\sin^2 \omega t$. A graph of $\cos^2 \omega t$ versus time is identical to a graph of $\sin^2 \omega t$ versus time, except that the points are shifted on the time axis. Thus, the time average of $\sin^2 \omega t$ is equal to the time average of $\cos^2 \omega t$ when taken over one or more complete cycles. That is,

$$(\sin^2 \omega t)_{\text{av}} = (\cos^2 \omega t)_{\text{av}}$$

Using this fact and the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$, we obtain

$$(\sin^2 \omega t)_{\text{av}} + (\cos^2 \omega t)_{\text{av}} = 2(\sin^2 \omega t)_{\text{av}} = 1$$

$$(\sin^2 \omega t)_{\text{av}} = \tfrac{1}{2}$$

When we substitute this result in the expression $i^2 = I_{\max}^2 \sin^2 \omega t$, we obtain $(i^2)_{\text{av}} = \overline{i^2} = I_{\text{rms}}^2 = I_{\max}^2/2$, or $I_{\text{rms}} = I_{\max}/\sqrt{2}$. The factor $1/\sqrt{2}$ is valid only for sinusoidally varying currents. Other waveforms, such as sawtooth variations, have different factors.

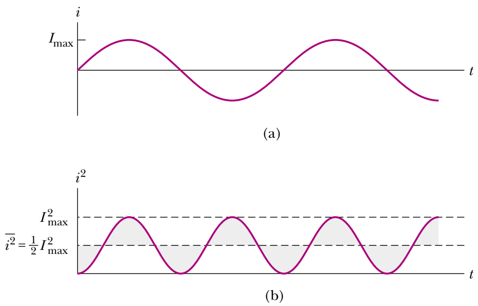


Figure 33.3 (a) Graph of the current in a resistor as a function of time. (b) Graph of the current squared in a resistor as a function of time. Notice that the gray shaded regions *under* the curve and *above* the dashed line for $I^2_{max}/2$ have the same area as the gray shaded regions *above* the curve and *below* the dashed line for $I^2_{max}/2$. Thus, the average value of i^2 is $I^2_{max}/2$.

Alternating voltage also is best discussed in terms of rms voltage, and the relationship is identical to that for current:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707\,\Delta V_{max} \qquad (33.5)$$

rms voltage

When we speak of measuring a 120-V alternating voltage from an electrical outlet, we are referring to an rms voltage of 120 V. A quick calculation using Equation 33.5 shows that such an alternating voltage has a maximum value of about 170 V. One reason we use rms values when discussing alternating currents and voltages in this chapter is that ac ammeters and voltmeters are designed to read rms values. Furthermore, with rms values, many of the equations we use have the same form as their direct-current counterparts.

Quick Quiz 33.1

Which of the following statements might be true for a resistor connected to an ac generator? (a) $\mathcal{P}_{av} = 0$ and $i_{av} = 0$; (b) $\mathcal{P}_{av} = 0$ and $i_{av} > 0$; (c) $\mathcal{P}_{av} > 0$ and $i_{av} = 0$; (d) $\mathcal{P}_{av} > 0$ and $i_{av} > 0$.

EXAMPLE 33.1 What Is the rms Current?

The voltage output of a generator is given by $\Delta v = (200\text{ V})\sin \omega t$. Find the rms current in the circuit when this generator is connected to a 100-Ω resistor.

Solution Comparing this expression for voltage output with the general form $\Delta v = \Delta V_{max} \sin \omega t$, we see that $\Delta V_{max} = 200$ V. Thus, the rms voltage is

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = \frac{200\text{ V}}{\sqrt{2}} = 141\text{ V}$$

Therefore,

$$I_{rms} = \frac{\Delta V_{rms}}{R} = \frac{141\text{ V}}{100\ \Omega} = \boxed{1.41\text{ A}}$$

Exercise Find the maximum current in the circuit.

Answer 2.00 A.

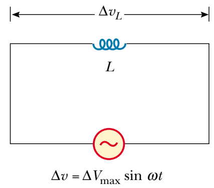


Figure 33.4 A circuit consisting of an inductor of inductance L connected to an ac generator.

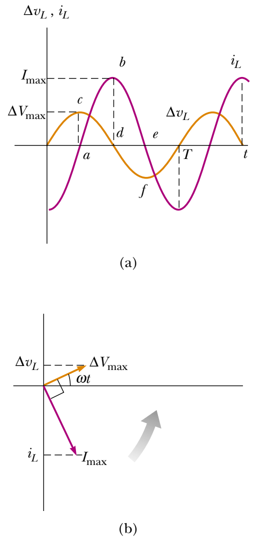


Figure 33.5 (a) Plots of the instantaneous current i_L and instantaneous voltage Δv_L across an inductor as functions of time. The current lags behind the voltage by 90°. (b) Phasor diagram for the inductive circuit, showing that the current lags behind the voltage by 90°.

The current in an inductor lags the voltage by 90°

33.3 INDUCTORS IN AN AC CIRCUIT

Now consider an ac circuit consisting only of an inductor connected to the terminals of an ac generator, as shown in Figure 33.4. If $\Delta v_L = \mathcal{E}_L = -L(di/dt)$ is the self-induced instantaneous voltage across the inductor (see Eq. 32.1), then Kirchhoff's loop rule applied to this circuit gives $\Delta v + \Delta v_L = 0$, or

$$\Delta v - L\frac{di}{dt} = 0$$

When we substitute $\Delta V_{max} \sin \omega t$ for Δv and rearrange, we obtain

$$L\frac{di}{dt} = \Delta V_{max} \sin \omega t \qquad (33.6)$$

Solving this equation for di, we find that

$$di = \frac{\Delta V_{max}}{L} \sin \omega t\, dt$$

Integrating this expression[3] gives the instantaneous current in the inductor as a function of time:

$$i_L = \frac{\Delta V_{max}}{L}\int \sin \omega t\, dt = -\frac{\Delta V_{max}}{\omega L}\cos \omega t \qquad (33.7)$$

When we use the trigonometric identity $\cos \omega t = -\sin(\omega t - \pi/2)$, we can express Equation 33.7 as

$$i_L = \frac{\Delta V_{max}}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) \qquad (33.8)$$

Comparing this result with Equation 33.6, we see that the instantaneous current i_L in the inductor and the instantaneous voltage Δv_L across the inductor are out of phase by $(\pi/2)$ rad = 90°.

In general, inductors in an ac circuit produce a current that is out of phase with the ac voltage. A plot of voltage and current versus time is provided in Figure 33.5a. At point *a*, the current begins to increase in the positive direction. At this instant the rate of change of current is at a maximum, and thus the voltage across the inductor is also at a maximum. As the current increases between points *a* and *b*, di/dt (the slope of the current curve) gradually decreases until it reaches zero at point *b*. As a result, the voltage across the inductor is decreasing during this same time interval, as the curve segment between *c* and *d* indicates. Immediately after point *b*, the current begins to decrease, although it still has the same direction it had during the previous quarter cycle (from *a* to *b*). As the current decreases to zero (from *b* to *e*), a voltage is again induced in the inductor (*d* to *f*), but the polarity of this voltage is opposite that of the voltage induced between *c* and *d* (because back emfs are always directed to oppose the change in the current). Note that the voltage reaches its maximum value one quarter of a period before the current reaches its maximum value. Thus, we see that

for a sinusoidal applied voltage, the current in an inductor always lags behind the voltage across the inductor by 90° (one-quarter cycle in time).

[3] We neglect the constant of integration here because it depends on the initial conditions, which are not important for this situation.

The phasor diagram for the inductive circuit of Figure 33.4 is shown in Figure 33.5b.

From Equation 33.7 we see that the current in an inductive circuit reaches its maximum value when $\cos \omega t = -1$:

$$I_{\max} = \frac{\Delta V_{\max}}{\omega L} = \frac{\Delta V_{\max}}{X_L} \quad \textbf{(33.9)}$$

Maximum current in an inductor

where the quantity X_L, called the **inductive reactance,** is

$$X_L = \omega L \quad \textbf{(33.10)}$$

Inductive reactance

Equation 33.9 indicates that, for a given applied voltage, the maximum current decreases as the inductive reactance increases. The expression for the rms current in an inductor is similar to Equation 33.9, with $I_{\max}$ replaced by I_{rms} and $\Delta V_{\max}$ replaced by ΔV_{rms}.

Inductive reactance, like resistance, has units of ohms. However, unlike resistance, reactance depends on frequency as well as on the characteristics of the inductor. Note that the reactance of an inductor in an ac circuit increases as the frequency of the current increases. This is because at higher frequencies, the instantaneous current must change more rapidly than it does at the lower frequencies; this causes an increase in the maximum induced emf associated with a given maximum current.

Using Equations 33.6 and 33.9, we find that the instantaneous voltage across the inductor is

$$\Delta v_L = -L\frac{di}{dt} = -\Delta V_{\max} \sin \omega t = -I_{\max} X_L \sin \omega t \quad \textbf{(33.11)}$$

CONCEPTUAL EXAMPLE 33.2

Figure 33.6 shows a circuit consisting of a series combination of an alternating voltage source, a switch, an inductor, and a lightbulb. The switch is thrown closed, and the circuit is allowed to come to equilibrium so that the lightbulb glows steadily. An iron rod is then inserted into the interior of the inductor. What happens to the brightness of the lightbulb, and why?

Solution The bulb gets dimmer. As the rod is inserted, the inductance increases because the magnetic field inside the inductor increases. According to Equation 33.10, this increase in L means that the inductive reactance of the inductor also increases. The voltage across the inductor increases while the voltage across the lightbulb decreases. With less voltage across it, the lightbulb glows more dimly. In theatrical productions of the early 20th century, this method was used to dim the lights in the theater gradually.

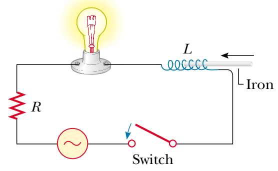


Figure 33.6

EXAMPLE 33.3 A Purely Inductive ac Circuit

In a purely inductive ac circuit (see Fig. 33.4), $L = 25.0$ mH and the rms voltage is 150 V. Calculate the inductive reactance and rms current in the circuit if the frequency is 60.0 Hz.

Solution Equation 33.10 gives

$$X_L = \omega L = 2\pi f L = 2\pi(60.0\ \text{Hz})(25.0 \times 10^{-3}\ \text{H}) = 9.42\ \Omega$$

From a modified version of Equation 33.9, the rms current is

$$I_{\text{rms}} = \frac{\Delta V_{L,\text{rms}}}{X_L} = \frac{150\ \text{V}}{9.42\ \Omega} = 15.9\ \text{A}$$

Exercise Calculate the inductive reactance and rms current in the circuit if the frequency is 6.00 kHz.

Answer 942 Ω, 0.159 A.

Exercise Show that inductive reactance has SI units of ohms.

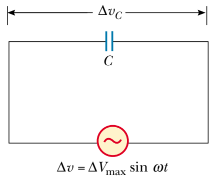


Figure 33.7 A circuit consisting of a capacitor of capacitance C connected to an ac generator.

33.4 CAPACITORS IN AN AC CIRCUIT

Figure 33.7 shows an ac circuit consisting of a capacitor connected across the terminals of an ac generator. Kirchhoff's loop rule applied to this circuit gives $\Delta v - \Delta v_C = 0$, or

$$\Delta v = \Delta v_C = \Delta V_{\max} \sin \omega t \quad \textbf{(33.12)}$$

where Δv_C is the instantaneous voltage across the capacitor. We know from the definition of capacitance that $C = q/\Delta v_C$; hence, Equation 33.12 gives

$$q = C\,\Delta V_{\max} \sin \omega t \quad \textbf{(33.13)}$$

where q is the instantaneous charge on the capacitor. Because $i = dq/dt$, differentiating Equation 33.13 gives the instantaneous current in the circuit:

$$i_C = \frac{dq}{dt} = \omega C\,\Delta V_{\max} \cos \omega t \quad \textbf{(33.14)}$$

Using the trigonometric identity

$$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$$

we can express Equation 33.14 in the alternative form

$$i_C = \omega C\,\Delta V_{\max} \sin\left(\omega t + \frac{\pi}{2}\right) \quad \textbf{(33.15)}$$

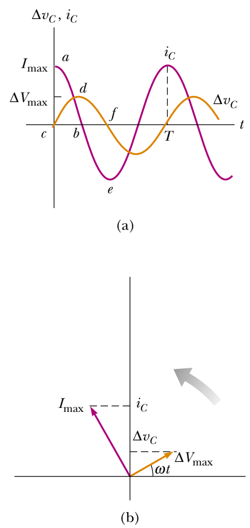


Figure 33.8 (a) Plots of the instantaneous current i_C and instantaneous voltage Δv_C across a capacitor as functions of time. The voltage lags behind the current by 90°. (b) Phasor diagram for the capacitive circuit, showing that the current leads the voltage by 90°.

Comparing this expression with Equation 33.12, we see that the current is $\pi/2$ rad = 90° out of phase with the voltage across the capacitor. A plot of current and voltage versus time (Fig. 33.8a) shows that the current reaches its maximum value one quarter of a cycle sooner than the voltage reaches its maximum value.

Looking more closely, we see that the segment of the current curve from a to b indicates that the current starts out at a relatively high value. We can understand this by recognizing that there is no charge on the capacitor at $t = 0$; as a consequence, nothing in the circuit except the resistance of the wires can hinder the flow of charge at this instant. However, the current decreases as the voltage across the capacitor increases (from c to d on the voltage curve), and the capacitor is charging. When the voltage is at point d, the current reverses and begins to increase in the opposite direction (from b to e on the current curve). During this time, the voltage across the capacitor decreases from d to f because the plates are now losing the charge they accumulated earlier. During the second half of the cycle, the current is initially at its maximum value in the opposite direction (point e) and then decreases as the voltage across the capacitor builds up. The phasor diagram in Figure 33.8b also shows that

for a sinusoidally applied voltage, the current in a capacitor always leads the voltage across the capacitor by 90°.

From Equation 33.14, we see that the current in the circuit reaches its maximum value when $\cos \omega t = 1$:

$$I_{max} = \omega C \Delta V_{max} = \frac{\Delta V_{max}}{X_C} \quad (33.16)$$

where X_C is called the **capacitive reactance:**

$$X_C = \frac{1}{\omega C} \quad (33.17)$$

Capacitive reactance

Note that capacitive reactance also has units of ohms.

The rms current is given by an expression similar to Equation 33.16, with I_{max} replaced by I_{rms} and ΔV_{max} replaced by ΔV_{rms}.

Combining Equations 33.12 and 33.16, we can express the instantaneous voltage across the capacitor as

$$\Delta v_C = \Delta V_{max} \sin \omega t = I_{max} X_C \sin \omega t \quad (33.18)$$

Equations 33.16 and 33.17 indicate that as the frequency of the voltage source increases, the capacitive reactance decreases and therefore the maximum current increases. Again, note that the frequency of the current is determined by the frequency of the voltage source driving the circuit. As the frequency approaches zero, the capacitive reactance approaches infinity, and hence the current approaches zero. This makes sense because the circuit approaches direct-current conditions as ω approaches 0.

EXAMPLE 33.4 A Purely Capacitive ac Circuit

An 8.00-μF capacitor is connected to the terminals of a 60.0-Hz ac generator whose rms voltage is 150 V. Find the capacitive reactance and the rms current in the circuit.

Solution Using Equation 33.17 and the fact that $\omega = 2\pi f = 377 \text{ s}^{-1}$ gives

$$X_C = \frac{1}{\omega C} = \frac{1}{(377 \text{ s}^{-1})(8.00 \times 10^{-6} \text{ F})} = 332 \ \Omega$$

Hence, from a modified Equation 33.16, the rms current is

$$I_{rms} = \frac{\Delta V_{rms}}{X_C} = \frac{150 \text{ V}}{332 \ \Omega} = 0.452 \text{ A}$$

Exercise If the frequency is doubled, what happens to the capacitive reactance and the current?

Answer X_C is halved, and I_{max} is doubled.

33.5 THE *RLC* SERIES CIRCUIT

13.7

Figure 33.9a shows a circuit that contains a resistor, an inductor, and a capacitor connected in series across an alternating-voltage source. As before, we assume that the applied voltage varies sinusoidally with time. It is convenient to assume that the instantaneous applied voltage is given by

$$\Delta v = \Delta V_{max} \sin \omega t$$

while the current varies as

$$i = I_{max} \sin(\omega t - \phi)$$

where ϕ is the **phase angle** between the current and the applied voltage. Our aim

Phase angle ϕ

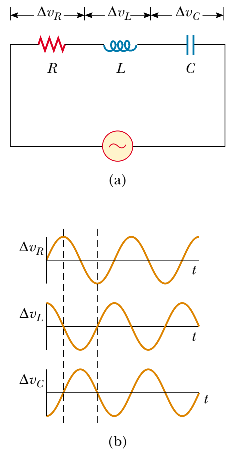


Figure 33.9 (a) A series circuit consisting of a resistor, an inductor, and a capacitor connected to an ac generator. (b) Phase relationships for instantaneous voltages in the series *RLC* circuit.

is to determine ϕ and I_{max}. Figure 33.9b shows the voltage versus time across each element in the circuit and their phase relationships.

To solve this problem, we must analyze the phasor diagram for this circuit. First, we note that because the elements are in series, the current everywhere in the circuit must be the same at any instant. That is, **the current at all points in a series ac circuit has the same amplitude and phase.** Therefore, as we found in the preceding sections, the voltage across each element has a different amplitude and phase, as summarized in Figure 33.10. In particular, the voltage across the resistor is in phase with the current, the voltage across the inductor leads the current by 90°, and the voltage across the capacitor lags behind the current by 90°. Using these phase relationships, we can express the instantaneous voltages across the three elements as

$$\Delta v_R = I_{max} R \sin \omega t = \Delta V_R \sin \omega t \quad (33.19)$$

$$\Delta v_L = I_{max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t \quad (33.20)$$

$$\Delta v_C = I_{max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t \quad (33.21)$$

where ΔV_R, ΔV_L, and ΔV_C are the maximum voltage values across the elements:

$$\Delta V_R = I_{max} R \qquad \Delta V_L = I_{max} X_L \qquad \Delta V_C = I_{max} X_C$$

At this point, we could proceed by noting that the instantaneous voltage Δv across the three elements equals the sum

$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$

Quick Quiz 33.2

For the circuit of Figure 33.9a, is the voltage of the ac source equal to (a) the sum of the maximum voltages across the elements, (b) the sum of the instantaneous voltages across the elements, or (c) the sum of the rms voltages across the elements?

Although this analytical approach is correct, it is simpler to obtain the sum by examining the phasor diagram. Because the current at any instant is the same in all

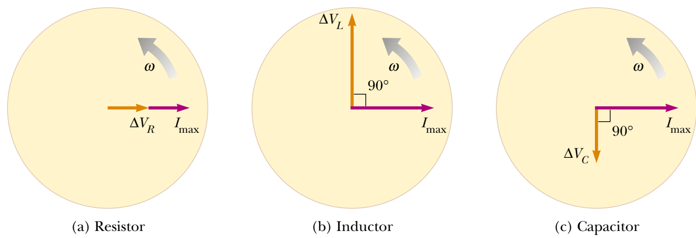


Figure 33.10 Phase relationships between the voltage and current phasors for (a) a resistor, (b) an inductor, and (c) a capacitor connected in series.

elements, we can obtain a phasor diagram for the circuit. We combine the three phasor pairs shown in Figure 33.10 to obtain Figure 33.11a, in which a single phasor I_{max} is used to represent the current in each element. To obtain the vector sum of the three voltage phasors in Figure 33.11a, we redraw the phasor diagram as in Figure 33.11b. From this diagram, we see that the vector sum of the voltage amplitudes ΔV_R, ΔV_L, and ΔV_C equals a phasor whose length is the maximum applied voltage ΔV_{max}, where the phasor ΔV_{max} makes an angle ϕ with the current phasor I_{max}. Note that the voltage phasors ΔV_L and ΔV_C are in opposite directions along the same line, and hence we can construct the difference phasor $\Delta V_L - \Delta V_C$, which is perpendicular to the phasor ΔV_R. From either one of the right triangles in Figure 33.11b, we see that

$$\Delta V_{max} = \sqrt{\Delta V_R^2 + (\Delta V_L - \Delta V_C)^2} = \sqrt{(I_{max}R)^2 + (I_{max}X_L - I_{max}X_C)^2}$$

$$\Delta V_{max} = I_{max}\sqrt{R^2 + (X_L - X_C)^2} \qquad \textbf{(33.22)}$$

Therefore, we can express the maximum current as

$$I_{max} = \frac{\Delta V_{max}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

The **impedance** Z of the circuit is defined as

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \qquad \textbf{(33.23)}$$

where impedance also has units of ohms. Therefore, we can write Equation 33.22 in the form

$$\Delta V_{max} = I_{max}Z \qquad \textbf{(33.24)}$$

We can regard Equation 33.24 as the ac equivalent of Equation 27.8, which defined *resistance* in a dc circuit as the ratio of the voltage across a conductor to the current in that conductor. Note that the impedance and therefore the current in an ac circuit depend upon the resistance, the inductance, the capacitance, and the frequency (because the reactances are frequency-dependent).

By removing the common factor I_{max} from each phasor in Figure 33.11a, we can construct the *impedance triangle* shown in Figure 33.12. From this phasor diagram we find that the phase angle ϕ between the current and the voltage is

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \qquad \textbf{(33.25)}$$

Also, from Figure 33.12, we see that $\cos\phi = R/Z$. When $X_L > X_C$ (which occurs at high frequencies), the phase angle is positive, signifying that the current lags behind the applied voltage, as in Figure 33.11a. When $X_L < X_C$, the phase angle is negative, signifying that the current leads the applied voltage. When $X_L = X_C$, the phase angle is zero. In this case, the impedance equals the resistance and the current has its maximum value, given by $\Delta V_{max}/R$. The frequency at which this occurs is called the *resonance frequency;* it is described further in Section 33.7.

Table 33.1 gives impedance values and phase angles for various series circuits containing different combinations of elements.

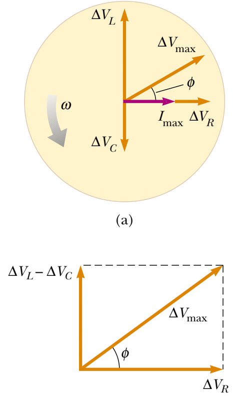


Figure 33.11 (a) Phasor diagram for the series *RLC* circuit shown in Figure 33.9a. The phasor ΔV_R is in phase with the current phasor I_{max}, the phasor ΔV_L leads I_{max} by 90°, and the phasor ΔV_C lags I_{max} by 90°. The total voltage ΔV_{max} makes an angle ϕ with I_{max}. (b) Simplified version of the phasor diagram shown in (a).

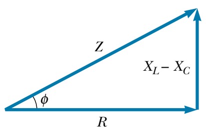


Figure 33.12 An impedance triangle for a series *RLC* circuit gives the relationship $Z = \sqrt{R^2 + (X_L - X_C)^2}$.

TABLE 33.1 Impedance Values and Phase Angles for Various Circuit-Element Combinations[a]

Circuit Elements	Impedance Z	Phase Angle ϕ
R	R	0°
C	X_C	−90°
L	X_L	+90°
R C	$\sqrt{R^2 + X_C^2}$	Negative, between −90° and 0°
R L	$\sqrt{R^2 + X_L^2}$	Positive, between 0° and 90°
R L C	$\sqrt{R^2 + (X_L - X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

[a] In each case, an ac voltage (not shown) is applied across the elements.

Quick Quiz 33.3

Label each part of Figure 33.13 as being $X_L > X_C$, $X_L = X_C$, or $X_L < X_C$.

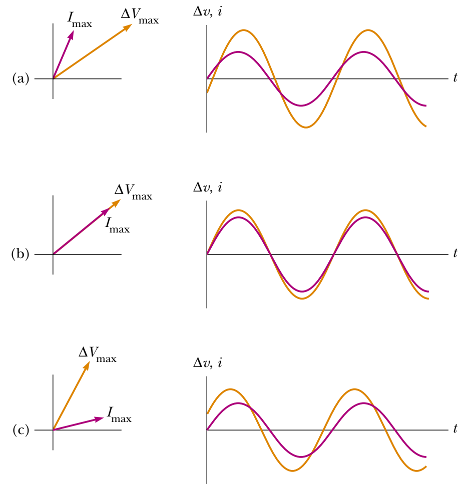


Figure 33.13

EXAMPLE 33.5 Finding *L* from a Phasor Diagram

In a series *RLC* circuit, the applied voltage has a maximum value of 120 V and oscillates at a frequency of 60.0 Hz. The circuit contains an inductor whose inductance can be varied, a 200-Ω resistor, and a 4.00-μF capacitor. What value of *L* should an engineer analyzing the circuit choose such that the voltage across the capacitor lags the applied voltage by 30.0°?

Solution The phase relationships for the drops in voltage across the elements are shown in Figure 33.14. From the figure we see that the phase angle is $\phi = -60.0°$. This is because the phasors representing I_{max} and ΔV_R are in the same direction (they are in phase). From Equation 33.25, we find that

$$X_L = X_C + R \tan \phi$$

Substituting Equations 33.10 and 33.17 (with $\omega = 2\pi f$) into this expression gives

$$2\pi f L = \frac{1}{2\pi f C} + R \tan \phi$$

$$L = \frac{1}{2\pi f}\left[\frac{1}{2\pi f C} + R \tan \phi\right]$$

Substituting the given values into the equation gives $L =$ 0.84 H.

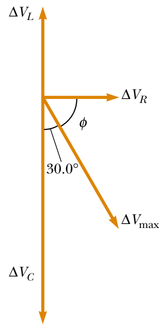


Figure 33.14

EXAMPLE 33.6 Analyzing a Series *RLC* Circuit

A series *RLC* ac circuit has $R = 425\ \Omega$, $L = 1.25$ H, $C = 3.50\ \mu$F, $\omega = 377\ s^{-1}$, and $\Delta V_{max} = 150$ V. (a) Determine the inductive reactance, the capacitive reactance, and the impedance of the circuit.

Solution The reactances are $X_L = \omega L =$ 471 Ω and $X_C = 1/\omega C =$ 758 Ω. The impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{(425\ \Omega)^2 + (471\ \Omega - 758\ \Omega)^2} = 513\ \Omega$$

(b) Find the maximum current in the circuit.

Solution

$$I_{max} = \frac{V_{max}}{Z} = \frac{150\ \text{V}}{513\ \Omega} = 0.292\ \text{A}$$

(c) Find the phase angle between the current and voltage.

Solution

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{471\ \Omega - 758\ \Omega}{425\ \Omega}\right)$$

$$= -34.0°$$

Because the circuit is more capacitive than inductive, ϕ is negative and the current leads the applied voltage.

(d) Find both the maximum voltage and the instantaneous voltage across each element.

Solution The maximum voltages are

$$\Delta V_R = I_{max} R = (0.292\ \text{A})(425\ \Omega) = 124\ \text{V}$$

$$\Delta V_L = I_{max} X_L = (0.292\ \text{A})(471\ \Omega) = 138\ \text{V}$$

$$\Delta V_C = I_{max} X_C = (0.292\ \text{A})(758\ \Omega) = 221\ \text{V}$$

Using Equations 33.19, 33.20, and 33.21, we find that we can write the instantaneous voltages across the three elements as

$$\Delta v_R = (124\ \text{V}) \sin 377t$$

$$\Delta v_L = (138\ \text{V}) \cos 377t$$

$$\Delta v_C = (-221\ \text{V}) \cos 377t$$

Comments The sum of the maximum voltages across the elements is $\Delta V_R + \Delta V_L + \Delta V_C = 483$ V. Note that this sum is much greater than the maximum voltage of the generator, 150 V. As we saw in Quick Quiz 33.2, the sum of the maximum voltages is a meaningless quantity because when sinusoidally varying quantities are added, *both their amplitudes and their phases* must be taken into account. We know that the maximum voltages across the various elements occur at different times. That is, the voltages must be added in a way that takes account of the different phases. When this is done, Equation 33.22 is satisfied. You should verify this result.

Exercise Construct a phasor diagram to scale, showing the voltages across the elements and the applied voltage. From your diagram, verify that the phase angle is −34.0°.

33.6 POWER IN AN AC CIRCUIT

No power losses are associated with pure capacitors and pure inductors in an ac circuit. To see why this is true, let us first analyze the power in an ac circuit containing only a generator and a capacitor.

When the current begins to increase in one direction in an ac circuit, charge begins to accumulate on the capacitor, and a voltage drop appears across it. When this voltage drop reaches its maximum value, the energy stored in the capacitor is $\frac{1}{2}C(\Delta V_{max})^2$. However, this energy storage is only momentary. The capacitor is charged and discharged twice during each cycle: Charge is delivered to the capacitor during two quarters of the cycle and is returned to the voltage source during the remaining two quarters. Therefore, **the average power supplied by the source is zero.** In other words, **no power losses occur in a capacitor in an ac circuit.**

Similarly, the voltage source must do work against the back emf of the inductor. When the current reaches its maximum value, the energy stored in the inductor is a maximum and is given by $\frac{1}{2}LI_{max}^2$. When the current begins to decrease in the circuit, this stored energy is returned to the source as the inductor attempts to maintain the current in the circuit.

In Example 28.1 we found that the power delivered by a battery to a dc circuit is equal to the product of the current and the emf of the battery. Likewise, the instantaneous power delivered by an ac generator to a circuit is the product of the generator current and the applied voltage. For the *RLC* circuit shown in Figure 33.9a, we can express the instantaneous power $\mathcal{P}$ as

$$\mathcal{P} = i\,\Delta v = I_{max} \sin(\omega t - \phi)\,\Delta V_{max} \sin \omega t$$

$$= I_{max}\,\Delta V_{max} \sin \omega t \sin(\omega t - \phi) \quad \textbf{(33.26)}$$

Clearly, this result is a complicated function of time and therefore is not very useful from a practical viewpoint. What is generally of interest is the average power over one or more cycles. Such an average can be computed by first using the trigonometric identity $\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$. Substituting this into Equation 33.26 gives

$$\mathcal{P} = I_{max}\,\Delta V_{max} \sin^2 \omega t \cos \phi - I_{max}\,\Delta V_{max} \sin \omega t \cos \omega t \sin \phi \quad \textbf{(33.27)}$$

We now take the time average of $\mathcal{P}$ over one or more cycles, noting that I_{max}, ΔV_{max}, ϕ, and ω are all constants. The time average of the first term on the right in Equation 33.27 involves the average value of $\sin^2 \omega t$, which is $\frac{1}{2}$ (as shown in footnote 2). The time average of the second term on the right is identically zero because $\sin \omega t \cos \omega t = \frac{1}{2} \sin 2\omega t$, and the average value of $\sin 2\omega t$ is zero. Therefore, we can express the **average power** $\mathcal{P}_{av}$ as

$$\mathcal{P}_{av} = \tfrac{1}{2} I_{max}\,\Delta V_{max} \cos \phi \quad \textbf{(33.28)}$$

It is convenient to express the average power in terms of the rms current and rms voltage defined by Equations 33.4 and 33.5:

$$\mathcal{P}_{av} = I_{rms}\,\Delta V_{rms} \cos \phi \quad \textbf{(33.29)}$$

where the quantity $\cos \phi$ is called the **power factor.** By inspecting Figure 33.11b, we see that the maximum voltage drop across the resistor is given by $\Delta V_R = \Delta V_{max} \cos \phi = I_{max} R$. Using Equation 33.5 and the fact that $\cos \phi = I_{max}R/\Delta V_{max}$, we find that we can express $\mathcal{P}_{av}$ as

$$\mathcal{P}_{av} = I_{rms}\,\Delta V_{rms} \cos \phi = I_{rms}\left(\frac{\Delta V_{max}}{\sqrt{2}}\right)\frac{I_{max}R}{\Delta V_{max}} = I_{rms}\,\frac{I_{max}R}{\sqrt{2}}$$

After making the substitution $I_{max} = \sqrt{2} I_{rms}$ from Equation 33.4, we have

$$\mathcal{P}_{av} = I_{rms}^2 R \qquad \textbf{(33.30)}$$

Average power delivered to an *RLC* circuit

In words, the **average power delivered by the generator is converted to internal energy in the resistor,** just as in the case of a dc circuit. **No power loss occurs in an ideal inductor or capacitor.** When the load is purely resistive, then $\phi = 0$, $\cos \phi = 1$, and from Equation 33.29 we see that

$$\mathcal{P}_{av} = I_{rms}\,\Delta V_{rms}$$

Equation 33.29 shows that the power delivered by an ac source to any circuit depends on the phase, and this result has many interesting applications. For example, a factory that uses large motors in machines, generators, or transformers has a large inductive load (because of all the windings). To deliver greater power to such devices in the factory without using excessively high voltages, technicians introduce capacitance in the circuits to shift the phase.

EXAMPLE 33.7 Average Power in an *RLC* Series Circuit

Calculate the average power delivered to the series *RLC* circuit described in Example 33.6.

Solution First, let us calculate the rms voltage and rms current, using the values of ΔV_{max} and I_{max} from Example 33.6:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = \frac{150 \text{ V}}{\sqrt{2}} = 106 \text{ V}$$

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = \frac{0.292 \text{ A}}{\sqrt{2}} = 0.206 \text{ A}$$

Because $\phi = -34.0°$, the power factor, $\cos \phi$, is 0.829; hence, the average power delivered is

$$\mathcal{P}_{av} = I_{rms}\,\Delta V_{rms} \cos \phi = (0.206 \text{ A})(106 \text{ V})(0.829)$$
$$= 18.1 \text{ W}$$

We can obtain the same result using Equation 33.30.

33.7 RESONANCE IN A SERIES *RLC* CIRCUIT

13.7 A series *RLC* circuit is said to be **in resonance** when the current has its maximum value. In general, the rms current can be written

$$I_{rms} = \frac{\Delta V_{rms}}{Z} \qquad \textbf{(33.31)}$$

where Z is the impedance. Substituting the expression for Z from Equation 33.23 into 33.31 gives

$$I_{rms} = \frac{\Delta V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \qquad \textbf{(33.32)}$$

Because the impedance depends on the frequency of the source, the current in the *RLC* circuit also depends on the frequency. The frequency ω_0 at which $X_L - X_C = 0$ is called the **resonance frequency** of the circuit. To find ω_0, we use the condition $X_L = X_C$, from which we obtain $\omega_0 L = 1/\omega_0 C$, or

Resonance frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \textbf{(33.33)}$$

Note that this frequency also corresponds to the natural frequency of oscillation of an *LC* circuit (see Section 32.5). Therefore, the current in a series *RLC* circuit reaches its maximum value when the frequency of the applied voltage matches the natural oscillator frequency—which depends only on L and C. Furthermore, at this frequency the current is in phase with the applied voltage.

Quick Quiz 33.4

What is the impedance of a series *RLC* circuit at resonance? What is the current in the circuit at resonance?

A plot of rms current versus frequency for a series *RLC* circuit is shown in Figure 33.15a. The data assume a constant $\Delta V_{rms} = 5.0$ mV, that $L = 5.0\ \mu$H, and that $C = 2.0$ nF. The three curves correspond to three values of R. Note that in each case the current reaches its maximum value at the resonance frequency ω_0. Furthermore, the curves become narrower and taller as the resistance decreases.

By inspecting Equation 33.32, we must conclude that, when $R = 0$, the current becomes infinite at resonance. Although the equation predicts this, real circuits always have some resistance, which limits the value of the current.

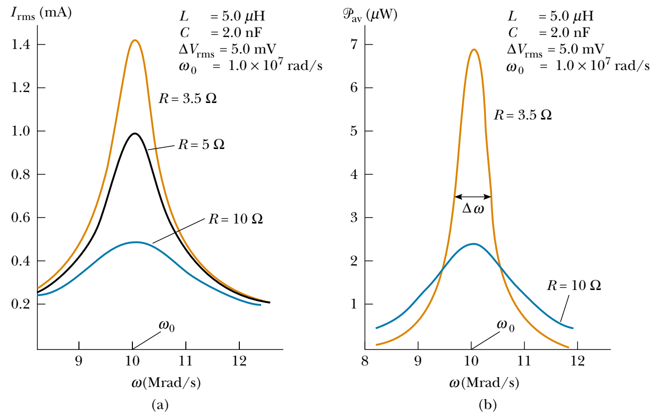


Figure 33.15 (a) The rms current versus frequency for a series *RLC* circuit, for three values of R. The current reaches its maximum value at the resonance frequency ω_0. (b) Average power versus frequency for the series *RLC* circuit, for two values of R.

It is also interesting to calculate the average power as a function of frequency for a series *RLC* circuit. Using Equations 33.30, 33.31, and 33.23, we find that

$$\mathcal{P}_{av} = I_{rms}^2 R = \frac{(\Delta V_{rms})^2}{Z^2} R = \frac{(\Delta V_{rms})^2 R}{R^2 + (X_L - X_C)^2} \quad \textbf{(33.34)}$$

Because $X_L = \omega L$, $X_C = 1/\omega C$, and $\omega_0{}^2 = 1/LC$, we can express the term $(X_L - X_C)^2$ as

$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 = \frac{L^2}{\omega^2}(\omega^2 - \omega_0{}^2)^2$$

Using this result in Equation 33.34 gives

Average power as a function of frequency in an *RLC* circuit

$$\mathcal{P}_{av} = \frac{(\Delta V_{rms})^2 R\omega^2}{R^2\omega^2 + L^2(\omega^2 - \omega_0{}^2)^2} \quad \textbf{(33.35)}$$

This expression shows that at resonance, when $\omega = \omega_0$, **the average power is a maximum** and has the value $(\Delta V_{rms})^2/R$. Figure 33.15b is a plot of average power versus frequency for two values of R in a series *RLC* circuit. As the resistance is made smaller, the curve becomes sharper in the vicinity of the resonance frequency. This curve sharpness is usually described by a dimensionless parameter known as the **quality factor,** denoted by Q:[4]

Quality factor

$$Q = \frac{\omega_0}{\Delta\omega}$$

where $\Delta\omega$ is the width of the curve measured between the two values of ω for which $\mathcal{P}_{av}$ has half its maximum value, called the *half-power points* (see Fig. 33.15b.) It is left as a problem (Problem 70) to show that the width at the half-power points has the value $\Delta\omega = R/L$, so

$$Q = \frac{\omega_0 L}{R} \quad \textbf{(33.36)}$$

The curves plotted in Figure 33.16 show that a high-Q circuit responds to only a very narrow range of frequencies, whereas a low-Q circuit can detect a much broader range of frequencies. Typical values of Q in electronic circuits range from 10 to 100.

The receiving circuit of a radio is an important application of a resonant circuit. One tunes the radio to a particular station (which transmits a specific electromagnetic wave or signal) by varying a capacitor, which changes the resonant frequency of the receiving circuit. When the resonance frequency of the circuit matches that of the incoming electromagnetic wave, the current in the receiving circuit increases. This signal caused by the incoming wave is then amplified and fed to a speaker. Because many signals are often present over a range of frequencies, it is important to design a high-Q circuit to eliminate unwanted signals. In this manner, stations whose frequencies are near but not equal to the resonance frequency give signals at the receiver that are negligibly small relative to the signal that matches the resonance frequency.

QuickLab

Tune a radio to your favorite station. Can you determine what the product of *LC* must be for the radio's tuning circuitry?

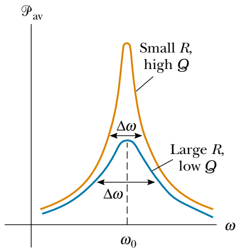


Figure 33.16 Average power versus frequency for a series *RLC* circuit. The width $\Delta\omega$ of each curve is measured between the two points where the power is half its maximum value. The power is a maximum at the resonance frequency ω_0.

[4] The quality factor is also defined as the ratio $2\pi E/\Delta E$, where E is the energy stored in the oscillating system and ΔE is the energy lost per cycle of oscillation. The quality factor for a mechanical system can also be defined, as noted in Section 13.7.

Quick Quiz 33.5

An airport metal detector (Fig. 33.17) is essentially a resonant circuit. The portal you step through is an inductor (a large loop of conducting wire) that is part of the circuit. The frequency of the circuit is tuned to the resonant frequency of the circuit when there is no metal in the inductor. Any metal on your body increases the effective inductance of the loop and changes the current in it. If you want the detector to be able to detect a small metallic object, should the circuit have a high quality factor or a low one?

Figure 33.17 When you pass through a metal detector, you become part of a resonant circuit. As you step through the detector, the inductance of the circuit changes, and thus the current in the circuit changes. *(Terry Qing/FPG International)*

EXAMPLE 33.8 A Resonating Series *RLC* Circuit

Consider a series *RLC* circuit for which $R = 150\ \Omega$, $L = 20.0$ mH, $\Delta V_{rms} = 20.0$ V, and $\omega = 5\,000\ s^{-1}$. Determine the value of the capacitance for which the current is a maximum.

Solution The current has its maximum value at the resonance frequency ω_0, which should be made to match the "driving" frequency of $5\,000\ s^{-1}$:

$$\omega_0 = 5.00 \times 10^3\ s^{-1} = \frac{1}{\sqrt{LC}}$$

$$C = \frac{1}{\omega_0{}^2 L} = \frac{1}{(25.0 \times 10^6\ s^{-2})(20.0 \times 10^{-3}\ H)}$$

$$= 2.00\ \mu F$$

Exercise Calculate the maximum value of the rms current in the circuit as the frequency is varied.

Answer 0.133 A.

33.8 THE TRANSFORMER AND POWER TRANSMISSION

When electric power is transmitted over great distances, it is economical to use a high voltage and a low current to minimize the I^2R loss in the transmission lines.

Consequently, 350-kV lines are common, and in many areas even higher-voltage (765-kV) lines are under construction. At the receiving end of such lines, the consumer requires power at a low voltage (for safety and for efficiency in design). Therefore, a device is required that can change the alternating voltage and current without causing appreciable changes in the power delivered. The ac transformer is that device.

In its simplest form, the **ac transformer** consists of two coils of wire wound around a core of iron, as illustrated in Figure 33.18. The coil on the left, which is connected to the input alternating voltage source and has N_1 turns, is called the *primary winding* (or the *primary*). The coil on the right, consisting of N_2 turns and connected to a load resistor R, is called the *secondary winding* (or the *secondary*). The purpose of the iron core is to increase the magnetic flux through the coil and to provide a medium in which nearly all the flux through one coil passes through the other coil. Eddy current losses are reduced by using a laminated core. Iron is used as the core material because it is a soft ferromagnetic substance and hence reduces hysteresis losses. Transformation of energy to internal energy in the finite resistance of the coil wires is usually quite small. Typical transformers have power efficiencies from 90% to 99%. In the discussion that follows, we assume an *ideal transformer,* one in which the energy losses in the windings and core are zero.

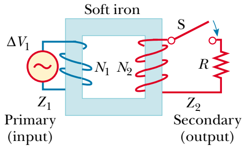


Figure 33.18 An ideal transformer consists of two coils wound on the same iron core. An alternating voltage ΔV_1 is applied to the primary coil, and the output voltage ΔV_2 is across the resistor of resistance R.

First, let us consider what happens in the primary circuit when the switch in the secondary circuit is open. If we assume that the resistance of the primary is negligible relative to its inductive reactance, then the primary circuit is equivalent to a simple circuit consisting of an inductor connected to an ac generator. Because the current is 90° out of phase with the voltage, the power factor cos ϕ is zero, and hence the average power delivered from the generator to the primary circuit is zero. Faraday's law states that the voltage ΔV_1 across the primary is

$$\Delta V_1 = -N_1 \frac{d\Phi_B}{dt} \qquad \textbf{(33.37)}$$

where Φ_B is the magnetic flux through each turn. If we assume that all magnetic field lines remain within the iron core, the flux through each turn of the primary equals the flux through each turn of the secondary. Hence, the voltage across the secondary is

$$\Delta V_2 = -N_2 \frac{d\Phi_B}{dt} \qquad \textbf{(33.38)}$$

Solving Equation 33.37 for $d\Phi_B/dt$ and substituting the result into Equation 33.38, we find that

$$\Delta V_2 = \frac{N_2}{N_1}\Delta V_1 \qquad \textbf{(33.39)}$$

When $N_2 > N_1$, the output voltage ΔV_2 exceeds the input voltage ΔV_1. This setup is referred to as a *step-up transformer.* When $N_2 < N_1$, the output voltage is less than the input voltage, and we have a *step-down transformer.*

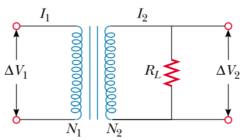


Figure 33.19 Circuit diagram for a transformer.

When the switch in the secondary circuit is thrown closed, a current I_2 is induced in the secondary. If the load in the secondary circuit is a pure resistance, the induced current is in phase with the induced voltage. The power supplied to the secondary circuit must be provided by the ac generator connected to the primary circuit, as shown in Figure 33.19. In an ideal transformer, where there are no losses, the power $I_1\,\Delta V_1$ supplied by the generator is equal to the power $I_2\,\Delta V_2$ in

This cylindrical step-down transformer drops the voltage from 4 000 V to 220 V for delivery to a group of residences. *(George Semple)*

the secondary circuit. That is,

$$I_1\,\Delta V_1 = I_2\,\Delta V_2 \qquad \textbf{(33.40)}$$

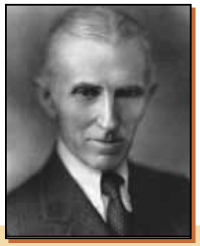
The value of the load resistance R_L determines the value of the secondary current because $I_2 = \Delta V_2/R_L$. Furthermore, the current in the primary is $I_1 = \Delta V_1/R_{\text{eq}}$, where

$$R_{\text{eq}} = \left(\frac{N_1}{N_2}\right)^2 R_L \qquad \textbf{(33.41)}$$

is the equivalent resistance of the load resistance when viewed from the primary side. From this analysis we see that a transformer may be used to match resistances between the primary circuit and the load. In this manner, maximum power transfer can be achieved between a given power source and the load resistance. For example, a transformer connected between the 1-kΩ output of an audio amplifier and an 8-Ω speaker ensures that as much of the audio signal as possible is transferred into the speaker. In stereo terminology, this is called *impedance matching.*

We can now also understand why transformers are useful for transmitting power over long distances. Because the generator voltage is stepped up, the current in the transmission line is reduced, and hence I^2R losses are reduced. In practice, the voltage is stepped up to around 230 000 V at the generating station, stepped down to around 20 000 V at a distributing station, then to 4 000 V for delivery to residential areas, and finally to 120–240 V at the customer's site. The power is supplied by a three-wire cable. In the United States, two of these wires are "hot," with voltages of 120 V with respect to a common ground wire. Home appliances operating on 120 V are connected in parallel between one of the hot wires and ground. Larger appliances, such as electric stoves and clothes dryers, require 240 V. This is obtained across the two hot wires, which are 180° out of phase so that the voltage difference between them is 240 V.

There is a practical upper limit to the voltages that can be used in transmission lines. Excessive voltages could ionize the air surrounding the transmission lines, which could result in a conducting path to ground or to other objects in the vicinity. This, of course, would present a serious hazard to any living creatures. For this reason, a long string of insulators is used to keep high-voltage wires away from their supporting metal towers. Other insulators are used to maintain separation between wires.



Nikola Tesla (1856–1943) Tesla was born in Croatia but spent most of his professional life as an inventor in the United States. He was a key figure in the development of alternating-current electricity, high-voltage transformers, and the transport of electric power via ac transmission lines. Tesla's viewpoint was at odds with the ideas of Thomas Edison, who committed himself to the use of direct current in power transmission. Tesla's ac approach won out. *(UPI/Bettmann)*

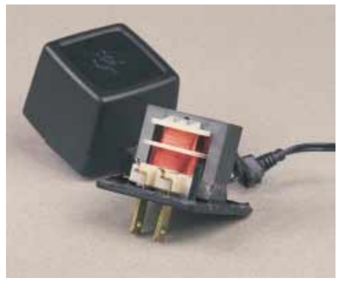


Figure 33.20 The primary winding in this transformer is directly attached to the prongs of the plug. The secondary winding is connected to the wire on the right, which runs to an electronic device. Many of these power-supply transformers also convert alternating current to direct current. *(George Semple)*

Many common household electronic devices require low voltages to operate properly. A small transformer that plugs directly into the wall, like the one illustrated in the photograph at the beginning of this chapter, can provide the proper voltage. Figure 33.20 shows the two windings wrapped around a common iron core that is found inside all these little "black boxes." This particular transformer converts the 120-V ac in the wall socket to 12.5-V ac. (Can you determine the ratio of the numbers of turns in the two coils?) Some black boxes also make use of diodes to convert the alternating current to direct current (see Section 33.9).

web
For information on how small transformers and hundreds of other everyday devices operate, visit
http://www.howstuffworks.com

EXAMPLE 33.9 The Economics of ac Power

An electricity-generating station needs to deliver 20 MW of power to a city 1.0 km away. (a) If the resistance of the wires is 2.0 Ω and the electricity costs about 10¢/kWh, estimate what it costs the utility company to send the power to the city for one day. A common voltage for commercial power generators is 22 kV, but a step-up transformer is used to boost the voltage to 230 kV before transmission.

Solution The power losses in the transmission line are the result of the resistance of the line. We can determine the loss from Equation 27.23, $\mathcal{P} = I^2R$. Because this is an estimate, we can use dc equations and calculate I from Equation 27.22:

$$I = \frac{\mathcal{P}}{\Delta V} = \frac{20 \times 10^6 \text{ W}}{230 \times 10^3 \text{ V}} = 87 \text{ A}$$

Therefore,

$$\mathcal{P} = I^2R = (87 \text{ A})^2(2.0\ \Omega) = 15 \text{ kW}$$

Over the course of a day, the energy loss due to the resistance of the wires is (15 kW)(24 h) = 360 kWh, at a cost of $36.

(b) Repeat the calculation for the situation in which the power plant delivers the electricity at its original voltage of 22 kV.

Solution

$$I = \frac{\mathcal{P}}{\Delta V} = \frac{20 \times 10^6 \text{ W}}{22 \times 10^3 \text{ V}} = 910 \text{ A}$$

$$\mathcal{P} = I^2R = (910 \text{ A})^2(2.0\ \Omega) = 1.7 \times 10^3 \text{ kW}$$

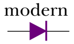
$$\text{Cost per day} = (1.7 \times 10^3 \text{ kW})(24 \text{ h})(\$0.10/\text{kWh}) = \$4\,100$$

The tremendous savings that are possible through the use of transformers and high-voltage transmission lines, along with the efficiency of using alternating current to operate motors, led to the universal adoption of alternating current instead of direct current for commercial power grids.

Optional Section

33.9 RECTIFIERS AND FILTERS

Portable electronic devices such as radios and compact disc (CD) players are often powered by direct current supplied by batteries. Many devices come with ac–dc converters that provide a readily available direct-current source if the batteries are low. Such a converter contains a transformer that steps the voltage down from 120 V to typically 9 V and a circuit that converts alternating current to direct current. The process of converting alternating current to direct current is called **rectification,** and the converting device is called a **rectifier.**

The most important element in a rectifier circuit is a **diode,** a circuit element that conducts current in one direction but not the other. Most diodes used in modern electronics are semiconductor devices. The circuit symbol for a diode is , where the arrow indicates the direction of the current through the diode. A diode has low resistance to current in one direction (the direction of the arrow) and high resistance to current in the opposite direction. We can understand how a diode rectifies a current by considering Figure 33.21a, which shows a

diode and a resistor connected to the secondary of a transformer. The transformer reduces the voltage from 120-V ac to the lower voltage that is needed for the device having a resistance R (the load resistance). Because current can pass through the diode in only one direction, the alternating current in the load resistor is reduced to the form shown by the solid curve in Figure 33.21b. The diode conducts current only when the side of the symbol containing the arrowhead has a positive potential relative to the other side. In this situation, the diode acts as a *half-wave rectifier* because current is present in the circuit during only half of each cycle.

When a capacitor is added to the circuit, as shown by the dashed lines and the capacitor symbol in Figure 33.21a, the circuit is a simple dc power supply. The time variation in the current in the load resistor (the dashed curve in Fig. 33.21b) is close to being zero, as determined by the RC time constant of the circuit.

The RC circuit in Figure 33.21a is one example of a **filter circuit,** which is used to smooth out or eliminate a time-varying signal. For example, radios are usually powered by a 60-Hz alternating voltage. After rectification, the voltage still contains a small ac component at 60 Hz (sometimes called *ripple*), which must be filtered. By "filtered," we mean that the 60-Hz ripple must be reduced to a value much less than that of the audio signal to be amplified, because without filtering, the resulting audio signal includes an annoying hum at 60 Hz.

To understand how a filter works, let us consider the simple series RC circuit shown in Figure 33.22a. The input voltage is across the two elements and is represented by $\Delta V_{max} \sin \omega t$. Because we are interested only in maximum values, we can use Equation 33.24, taking $X_L = 0$ and substituting $X_C = 1/\omega C$. This shows that the maximum input voltage is related to the maximum current by

$$\Delta V_{in} = I_{max} Z = I_{max} \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

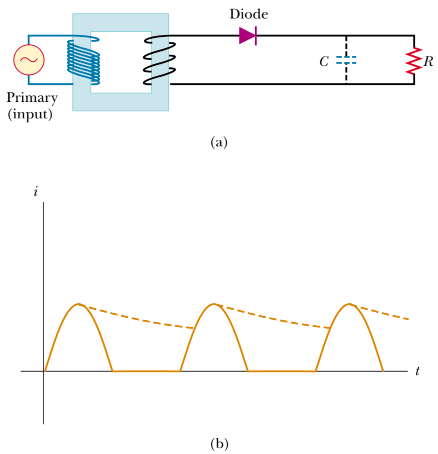


Figure 33.21 (a) A half-wave rectifier with an optional filter capacitor. (b) Current versus time in the resistor. The solid curve represents the current with no filter capacitor, and the dashed curve is the current when the circuit includes the capacitor.

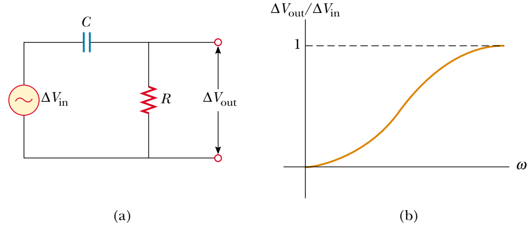


Figure 33.22 (a) A simple RC high-pass filter. (b) Ratio of output voltage to input voltage for an RC high-pass filter as a function of the angular frequency of the circuit.

If the voltage across the resistor is considered to be the output voltage, then the maximum output voltage is

$$\Delta V_{out} = I_{max}R$$

Therefore, the ratio of the output voltage to the input voltage is

$$\frac{\Delta V_{out}}{\Delta V_{in}} = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \quad \textbf{(33.42)}$$

High-pass filter

A plot of this ratio as a function of angular frequency (see Fig. 33.22b) shows that at low frequencies ΔV_{out} is much smaller than ΔV_{in}, whereas at high frequencies the two voltages are equal. Because the circuit preferentially passes signals of higher frequency while blocking low-frequency signals, the circuit is called an RC high-pass filter. Physically, a high-pass filter works because a capacitor "blocks out" direct current and ac current at low frequencies.

Now let us consider the circuit shown in Figure 33.23a, where the output voltage is taken across the capacitor. In this case, the maximum voltage equals the voltage across the capacitor. Because the impedance across the capacitor is $X_C = 1/\omega C$, we have

$$\Delta V_{out} = I_{max}X_C = \frac{I_{max}}{\omega C}$$

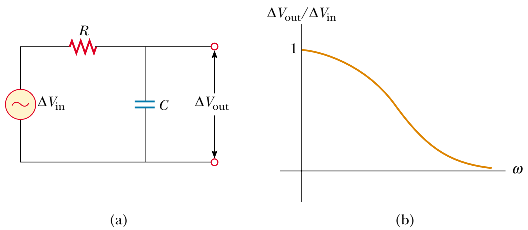


Figure 33.23 (a) A simple RC low-pass filter. (b) Ratio of output voltage to input voltage for an RC low-pass filter as a function of the angular frequency of the circuit.

Therefore, the ratio of the output voltage to the input voltage is

$$\frac{\Delta V_{out}}{\Delta V_{in}} = \frac{1/\omega C}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \quad \textbf{(33.43)}$$

Low-pass filter

This ratio, plotted as a function of ω in Figure 33.23b, shows that in this case the circuit preferentially passes signals of low frequency. Hence, the circuit is called an RC low-pass filter.

You may be familiar with crossover networks, which are an important part of the speaker systems for high-fidelity audio systems. These networks utilize low-pass filters to direct low frequencies to a special type of speaker, the "woofer," which is designed to reproduce the low notes accurately. The high frequencies are sent to the "tweeter" speaker.

Quick Quiz 33.6

Suppose you are designing a high-fidelity system containing both large loudspeakers (woofers) and small loudspeakers (tweeters). (a) What circuit element would you place in series with a woofer, which passes low-frequency signals? (b) What circuit element would you place in series with a tweeter, which passes high-frequency signals?

SUMMARY

If an ac circuit consists of a generator and a resistor, the current is in phase with the voltage. That is, the current and voltage reach their maximum values at the same time.

The **rms current** and **rms voltage** in an ac circuit in which the voltages and current vary sinusoidally are given by the expressions

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max} \quad \textbf{(33.4)}$$

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707 \Delta V_{max} \quad \textbf{(33.5)}$$

where I_{max} and ΔV_{max} are the maximum values.

If an ac circuit consists of a generator and an inductor, the current lags behind the voltage by 90°. That is, the voltage reaches its maximum value one quarter of a period before the current reaches its maximum value.

If an ac circuit consists of a generator and a capacitor, the current leads the voltage by 90°. That is, the current reaches its maximum value one quarter of a period before the voltage reaches its maximum value.

In ac circuits that contain inductors and capacitors, it is useful to define the **inductive reactance** X_L and the **capacitive reactance** X_C as

$$X_L = \omega L \quad \textbf{(33.10)}$$

$$X_C = \frac{1}{\omega C} \quad \textbf{(33.17)}$$

where ω is the angular frequency of the ac generator. The SI unit of reactance is the ohm.

The **impedance** Z of an RLC series ac circuit, which also has the ohm as its unit, is

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \tag{33.23}$$

This expression illustrates that we cannot simply add the resistance and reactances in a circuit. We must account for the fact that the applied voltage and current are out of phase, with the **phase angle** ϕ between the current and voltage being

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \tag{33.25}$$

The sign of ϕ can be positive or negative, depending on whether X_L is greater or less than X_C. The phase angle is zero when $X_L = X_C$.

The **average power** delivered by the generator in an RLC ac circuit is

$$\mathcal{P}_{\text{av}} = I_{\text{rms}}\,\Delta V_{\text{rms}} \cos\phi \tag{33.29}$$

An equivalent expression for the average power is

$$\mathcal{P}_{\text{av}} = I_{\text{rms}}^2 R \tag{33.30}$$

The average power delivered by the generator results in increasing internal energy in the resistor. No power loss occurs in an ideal inductor or capacitor.

The rms current in a series RLC circuit is

$$I_{\text{rms}} = \frac{\Delta V_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{33.32}$$

A series RLC circuit is in resonance when the inductive reactance equals the capacitive reactance. When this condition is met, the current given by Equation 33.32 reaches its maximum value. When $X_L = X_C$ in a circuit, the **resonance frequency** ω_0 of the circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{33.33}$$

The current in a series RLC circuit reaches its maximum value when the frequency of the generator equals ω_0—that is, when the "driving" frequency matches the resonance frequency.

Transformers allow for easy changes in alternating voltage. Because energy (and therefore power) are conserved, we can write

$$I_1\,\Delta V_1 = I_2\,\Delta V_2 \tag{33.40}$$

to relate the currents and voltages in the primary and secondary windings of a transformer.

QUESTIONS

1. Fluorescent lights flicker on and off 120 times every second. Explain what causes this. Why can't you see it happening?
2. Why does a capacitor act as a short circuit at high frequencies? Why does it act as an open circuit at low frequencies?
3. Explain how the acronyms in the mnemonic "ELI the ICE man" can be used to recall whether current leads voltage or voltage leads current in RLC circuits. (Note that "E" represents voltage.)
4. Why is the sum of the maximum voltages across the elements in a series RLC circuit usually greater than the maximum applied voltage? Doesn't this violate Kirchhoff's second rule?
5. Does the phase angle depend on frequency? What is the phase angle when the inductive reactance equals the capacitive reactance?
6. Energy is delivered to a series RLC circuit by a generator. This energy appears as internal energy in the resistor. What is the source of this energy?

7. Explain why the average power delivered to an RLC circuit by the generator depends on the phase between the current and the applied voltage.
8. A particular experiment requires a beam of light of very stable intensity. Why would an ac voltage be unsuitable for powering the light source?
9. Consider a series RLC circuit in which R is an incandescent lamp, C is some fixed capacitor, and L is a variable inductance. The source is 120-V ac. Explain why the lamp glows brightly for some values of L and does not glow at all for other values.
10. What determines the maximum voltage that can be used on a transmission line?
11. Will a transformer operate if a battery is used for the input voltage across the primary? Explain.
12. How can the average value of a current be zero and yet the square root of the average squared current not be zero?
13. What is the time average of the "square-wave" voltage shown in Figure Q33.13? What is its rms voltage?

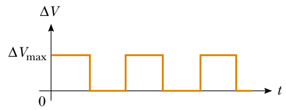


Figure Q33.13

14. Explain how the quality factor is related to the response characteristics of a radio receiver. Which variable most strongly determines the quality factor?
15. Why are the primary and secondary windings of a transformer wrapped on an iron core that passes through both coils?
16. With reference to Figure Q33.16, explain why the capacitor prevents a dc signal from passing between circuits A and B, yet allows an ac signal to pass from circuit A to circuit B. (The circuits are said to be capacitively coupled.)

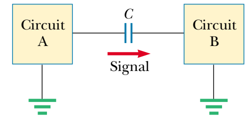


Figure Q33.16

17. With reference to Figure Q33.17, if C is made sufficiently large, an ac signal passes from circuit A to ground rather than from circuit A to circuit B. Hence, the capacitor acts as a filter. Explain.

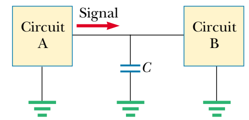


Figure Q33.17

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Note: Assume that all ac voltages and currents are sinusoidal unless stated otherwise.

Note that an ideal ammeter has zero resistance and that an ideal voltmeter has infinite resistance.

Section 33.1 ac Sources and Phasors

Section 33.2 Resistors in an ac Circuit

1. The rms output voltage of an ac generator is 200 V, and the operating frequency is 100 Hz. Write the equation giving the output voltage as a function of time.
2. (a) What is the resistance of a lightbulb that uses an average power of 75.0 W when connected to a 60.0-Hz power source having a maximum voltage of 170 V? (b) What is the resistance of a 100-W bulb?
3. An ac power supply produces a maximum voltage $\Delta V_{\text{max}} = 100$ V. This power supply is connected to a 24.0-Ω resistor, and the current and resistor voltage are measured with an ideal ac ammeter and voltmeter, as shown in Figure P33.3. What does each meter read?

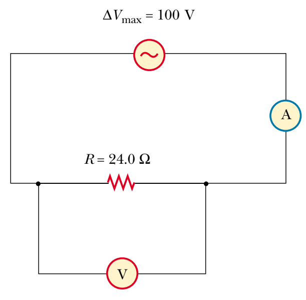


Figure P33.3

4. In the simple ac circuit shown in Figure 33.1, $R = 70.0\ \Omega$ and $\Delta v = \Delta V_{max} \sin \omega t$. (a) If $\Delta v_R = 0.250\ \Delta V_{max}$ for the first time at $t = 0.010\ 0$ s, what is the angular frequency of the generator? (b) What is the next value of t for which $\Delta v_R = 0.250\Delta V_{max}$?

5. The current in the circuit shown in Figure 33.1 equals 60.0% of the peak current at $t = 7.00$ ms. What is the smallest frequency of the generator that gives this current?

6. Figure P33.6 shows three lamps connected to a 120-V ac (rms) household supply voltage. Lamps 1 and 2 have 150-W bulbs; lamp 3 has a 100-W bulb. Find the rms current and the resistance of each bulb.

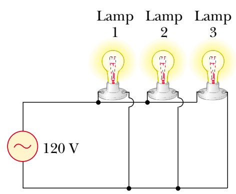


Figure P33.6

7. An audio amplifier, represented by the ac source and resistor in Figure P33.7, delivers to the speaker alternating voltage at audio frequencies. If the source voltage has an amplitude of 15.0 V, $R = 8.20\ \Omega$, and the speaker is equivalent to a resistance of 10.4 Ω, what time-averaged power is transferred to it?

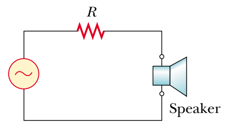


Figure P33.7

Section 33.3 Inductors in an ac Circuit

8. An inductor is connected to a 20.0-Hz power supply that produces a 50.0-V rms voltage. What inductance is needed to keep the instantaneous current in the circuit below 80.0 mA?

9. In a purely inductive ac circuit, such as that shown in Figure 33.4, $\Delta V_{max} = 100$ V. (a) If the maximum current is 7.50 A at 50.0 Hz, what is the inductance L? (b) At what angular frequency ω is the maximum current 2.50 A?

10. An inductor has a 54.0-Ω reactance at 60.0 Hz. What is the maximum current when this inductor is connected to a 50.0-Hz source that produces a 100-V rms voltage?

WEB 11. For the circuit shown in Figure 33.4, $\Delta V_{max} = 80.0$ V, $\omega = 65.0\pi$ rad/s, and $L = 70.0$ mH. Calculate the current in the inductor at $t = 15.5$ ms.

12. A 20.0-mH inductor is connected to a standard outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz). Determine the energy stored in the inductor at $t = (1/180)$ s, assuming that this energy is zero at $t = 0$.

13. **Review Problem.** Determine the maximum magnetic flux through an inductor connected to a standard outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz).

Section 33.4 Capacitors in an ac Circuit

14. (a) For what frequencies does a 22.0-μF capacitor have a reactance below 175 Ω? (b) Over this same frequency range, what is the reactance of a 44.0-μF capacitor?

15. What maximum current is delivered by a 2.20-μF capacitor when it is connected across (a) a North American outlet having $\Delta V_{rms} = 120$ V and $f = 60.0$ Hz? (b) a European outlet having $\Delta V_{rms} = 240$ V and $f = 50.0$ Hz?

16. A capacitor C is connected to a power supply that operates at a frequency f and produces an rms voltage ΔV. What is the maximum charge that appears on either of the capacitor plates?

17. What maximum current is delivered by an ac generator with $\Delta V_{max} = 48.0$ V and $f = 90.0$ Hz when it is connected across a 3.70-μF capacitor?

18. A 1.00-mF capacitor is connected to a standard outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz). Determine the current in the capacitor at $t = (1/180)$ s, assuming that at $t = 0$ the energy stored in the capacitor is zero.

Section 33.5 The *RLC* Series Circuit

19. An inductor ($L = 400$ mH), a capacitor ($C = 4.43\ \mu$F), and a resistor ($R = 500\ \Omega$) are connected in series. A 50.0-Hz ac generator produces a peak current of 250 mA in the circuit. (a) Calculate the required peak voltage ΔV_{max}. (b) Determine the phase angle by which the current leads or lags the applied voltage.

20. At what frequency does the inductive reactance of a 57.0-μH inductor equal the capacitive reactance of a 57.0-μF capacitor?

21. A series ac circuit contains the following components: $R = 150\ \Omega$, $L = 250$ mH, $C = 2.00\ \mu$F, and a generator with $\Delta V_{max} = 210$ V operating at 50.0 Hz. Calculate the (a) inductive reactance, (b) capacitive reactance, (c) impedance, (d) maximum current, and (e) phase angle between current and generator voltage.

22. A sinusoidal voltage $\Delta v(t) = (40.0\ \text{V}) \sin(100t)$ is applied to a series *RLC* circuit with $L = 160$ mH, $C = 99.0\ \mu$F, and $R = 68.0\ \Omega$. (a) What is the impedance of the circuit? (b) What is the maximum current? (c) Determine the numerical values for I_{max}, ω, and ϕ in the equation $i(t) = I_{max} \sin(\omega t - \phi)$.

WEB 23. An *RLC* circuit consists of a 150-Ω resistor, a 21.0-μF capacitor, and a 460-mH inductor, connected in series with a 120-V, 60.0-Hz power supply. (a) What is the phase angle between the current and the applied voltage? (b) Which reaches its maximum earlier, the current or the voltage?

24. A person is working near the secondary of a transformer, as shown in Figure P33.24. The primary voltage is 120 V at 60.0 Hz. The capacitance C_s, which is the stray capacitance between the person's hand and the secondary winding, is 20.0 pF. Assuming that the person has a body resistance to ground $R_b = 50.0$ kΩ, determine the rms voltage across the body. (*Hint:* Redraw the circuit with the secondary of the transformer as a simple ac source.)

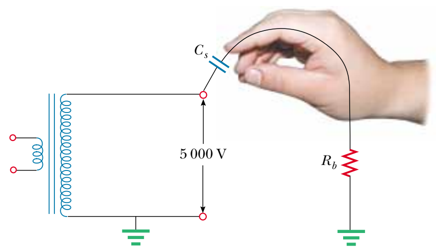


Figure P33.24

25. An ac source with $\Delta V_{max} = 150$ V and $f = 50.0$ Hz is connected between points a and d in Figure P33.25. Calculate the maximum voltages between points (a) a and b, (b) b and c, (c) c and d, and (d) b and d.

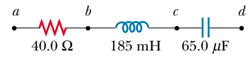


Figure P33.25 Problems 25 and 64.

26. Draw to scale a phasor diagram showing Z, X_L, X_C, and ϕ for an ac series circuit for which $R = 300\ \Omega$, $C = 11.0\ \mu$F, $L = 0.200$ H, and $f = (500/\pi)$ Hz.

27. A coil of resistance 35.0 Ω and inductance 20.5 H is in series with a capacitor and a 200-V (rms), 100-Hz source. The rms current in the circuit is 4.00 A. (a) Calculate the capacitance in the circuit. (b) What is ΔV_{rms} across the coil?

Section 33.6 Power in an ac Circuit

28. The voltage source in Figure P33.28 has an output $\Delta V_{rms} = 100$ V at $\omega = 1\ 000$ rad/s. Determine (a) the current in the circuit and (b) the power supplied by the source. (c) Show that the power delivered to the resistor is equal to the power supplied by the source.

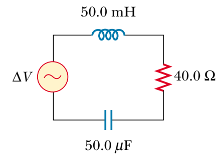


Figure P33.28

WEB 29. An ac voltage of the form $\Delta v = (100\ \text{V}) \sin(1\ 000t)$ is applied to a series *RLC* circuit. If $R = 400\ \Omega$, $C = 5.00\ \mu$F, and $L = 0.500$ H, what is the average power delivered to the circuit?

30. A series *RLC* circuit has a resistance of 45.0 Ω and an impedance of 75.0 Ω. What average power is delivered to this circuit when $\Delta V_{rms} = 210$ V?

31. In a certain series *RLC* circuit, $I_{rms} = 9.00$ A, $\Delta V_{rms} = 180$ V, and the current leads the voltage by 37.0°. (a) What is the total resistance of the circuit? (b) What is the reactance of the circuit $(X_L - X_C)$?

32. Suppose you manage a factory that uses many electric motors. The motors create a large inductive load to the electric power line, as well as a resistive load. The electric company builds an extra-heavy distribution line to supply you with a component of current that is 90° out of phase with the voltage, as well as with current in phase with the voltage. The electric company charges you an extra fee for "reactive volt-amps" in addition to the amount you pay for the energy you use. You can avoid the extra fee by installing a capacitor between the power line and your factory. The following problem models this solution.

In an *LR* circuit, a 120-V (rms), 60.0-Hz source is in series with a 25.0-mH inductor and a 20.0-Ω resistor. What are (a) the rms current and (b) the power factor? (c) What capacitor must be added in series to make the power factor 1? (d) To what value can the supply voltage be reduced if the power supplied is to be the same as that provided before installation of the capacitor?

33. **Review Problem.** Over a distance of 100 km, power of 100 MW is to be transmitted at 50.0 kV with only 1.00% loss. Copper wire of what diameter should be used for each of the two conductors of the transmission line? Assume that the current density in the conductors is uniform.

34. **Review Problem.** Suppose power $\mathcal{P}$ is to be transmitted over a distance d at a voltage ΔV, with only 1.00% loss. Copper wire of what diameter should be used for each of the two conductors of the transmission line? Assume that the current density in the conductors is uniform.

35. A diode is a device that allows current to pass in only one direction (the direction indicated by the arrowhead in its circuit-diagram symbol). Find, in terms of ΔV and

R, the average power delivered to the diode circuit shown in Figure P33.35.

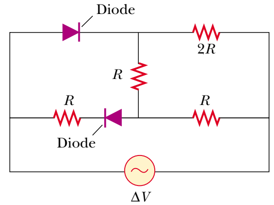


Figure P33.35

Section 33.7 Resonance in a Series *RLC* Circuit

36. The tuning circuit of an AM radio contains an LC combination. The inductance is 0.200 mH, and the capacitor is variable, so the circuit can resonate at any frequency between 550 kHz and 1 650 kHz. Find the range of values required for C.

37. An RLC circuit is used in a radio to tune in to an FM station broadcasting at 99.7 MHz. The resistance in the circuit is 12.0 Ω, and the inductance is 1.40 μH. What capacitance should be used?

38. A series RLC circuit has the following values: L = 20.0 mH, C = 100 nF, R = 20.0 Ω, and ΔV_{max} = 100 V, with $\Delta v = \Delta V_{max} \sin \omega t$. Find (a) the resonant frequency, (b) the amplitude of the current at the resonant frequency, (c) the Q of the circuit, and (d) the amplitude of the voltage across the inductor at resonance.

39. A 10.0-Ω resistor, a 10.0-mH inductor, and a 100-μF capacitor are connected in series to a 50.0-V (rms) source having variable frequency. What is the energy delivered to the circuit during one period if the operating frequency is twice the resonance frequency?

40. A resistor R, an inductor L, and a capacitor C are connected in series to an ac source of rms voltage ΔV and variable frequency. What is the energy delivered to the circuit during one period if the operating frequency is twice the resonance frequency?

41. Compute the quality factor for the circuits described in Problems 22 and 23. Which circuit has the sharper resonance?

Section 33.8 The Transformer and Power Transmission

42. A step-down transformer is used for recharging the batteries of portable devices such as tape players. The turns ratio inside the transformer is 13:1, and it is used with 120-V (rms) household service. If a particular ideal transformer draws 0.350 A from the house outlet, what (a) voltage and (b) current are supplied to a tape player from the transformer? (c) How much power is delivered?

43. A transformer has N_1 = 350 turns and N_2 = 2 000 turns. If the input voltage is $\Delta v(t)$ = (170 V) cos ωt, what rms voltage is developed across the secondary coil?

44. A step-up transformer is designed to have an output voltage of 2 200 V (rms) when the primary is connected across a 110-V (rms) source. (a) If there are 80 turns on the primary winding, how many turns are required on the secondary? (b) If a load resistor across the secondary draws a current of 1.50 A, what is the current in the primary under ideal conditions? (c) If the transformer actually has an efficiency of 95.0%, what is the current in the primary when the secondary current is 1.20 A?

45. In the transformer shown in Figure P33.45, the load resistor is 50.0 Ω. The turns ratio N_1:N_2 is 5:2, and the source voltage is 80.0 V (rms). If a voltmeter across the load measures 25.0 V (rms), what is the source resistance R_s?

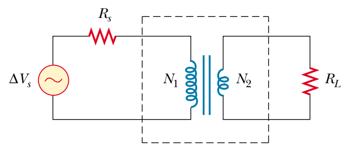


Figure P33.45

46. The secondary voltage of an ignition transformer in a furnace is 10.0 kV. When the primary operates at an rms voltage of 120 V, the primary impedance is 24.0 Ω and the transformer is 90.0% efficient. (a) What turns ratio is required? What are (b) the current in the secondary and (c) the impedance in the secondary?

47. A transmission line that has a resistance per unit length of 4.50×10^{-4} Ω/m is to be used to transmit 5.00 MW over 400 mi (6.44×10^5 m). The output voltage of the generator is 4.50 kV. (a) What is the power loss if a transformer is used to step up the voltage to 500 kV? (b) What fraction of the input power is lost to the line under these circumstances? (c) What difficulties would be encountered on attempting to transmit the 5.00 MW at the generator voltage of 4.50 kV?

(Optional)
Section 33.9 Rectifiers and Filters

48. The RC low-pass filter shown in Figure 33.23 has a resistance R = 90.0 Ω and a capacitance C = 8.00 nF. Calculate the gain ($\Delta V_{out}/\Delta V_{in}$) for input frequencies of (a) 600 Hz and (b) 600 kHz.

WEB **49.** The RC high-pass filter shown in Figure 33.22 has a resistance R = 0.500 Ω. (a) What capacitance gives an output signal that has one-half the amplitude of a 300-Hz input signal? (b) What is the gain ($\Delta V_{out}/\Delta V_{in}$) for a 600-Hz signal?

50. The circuit in Figure P33.50 represents a high-pass filter in which the inductor has internal resistance. What is the source frequency if the output voltage ΔV_2 is one-half the input voltage?

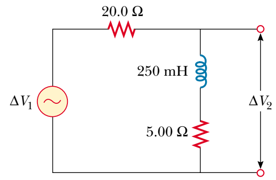


Figure P33.50

51. The resistor in Figure P33.51 represents the midrange speaker in a three-speaker system. Assume that its resistance is constant at 8.00 Ω. The source represents an audio amplifier producing signals of uniform amplitude ΔV_{in} = 10.0 V at all audio frequencies. The inductor and capacitor are to function as a bandpass filter with $\Delta V_{out}/\Delta V_{in} = \frac{1}{2}$ at 200 Hz and at 4 000 Hz. (a) Determine the required values of L and C. (b) Find the maximum value of the gain ratio $\Delta V_{out}/\Delta V_{in}$. (c) Find the frequency f_0 at which the gain ratio has its maximum value. (d) Find the phase shift between ΔV_{in} and ΔV_{out} at 200 Hz, at f_0, and at 4 000 Hz. (e) Find the average power transferred to the speaker at 200 Hz, at f_0, and at 4 000 Hz. (f) Treating the filter as a resonant circuit, find its quality factor.

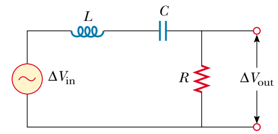


Figure P33.51

52. Show that two successive high-pass filters having the same values of R and C give a combined gain

$$\frac{\Delta V_{out}}{\Delta V_{in}} = \frac{1}{1 + (1/\omega RC)^2}$$

53. Consider a low-pass filter followed by a high-pass filter, as shown in Figure P33.53. If R = 1 000 Ω and C = 0.050 0 μF, determine $\Delta V_{out}/\Delta V_{in}$ for a 2.00-kHz input frequency.

ADDITIONAL PROBLEMS

54. Show that the rms value for the sawtooth voltage shown in Figure P33.54 is $\Delta V_{max}/\sqrt{3}$.

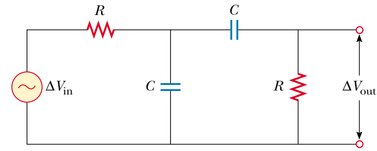


Figure P33.53

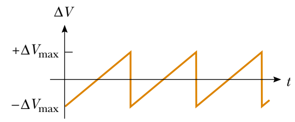


Figure P33.54

WEB **55.** A series RLC circuit consists of an 8.00-Ω resistor, a 5.00-μF capacitor, and a 50.0-mH inductor. A variable frequency source applies an emf of 400 V (rms) across the combination. Determine the power delivered to the circuit when the frequency is equal to one-half the resonance frequency.

56. To determine the inductance of a coil used in a research project, a student first connects the coil to a 12.0-V battery and measures a current of 0.630 A. The student then connects the coil to a 24.0-V (rms), 60.0-Hz generator and measures an rms current of 0.570 A. What is the inductance?

57. In Figure P33.57, find the current delivered by the 45.0-V (rms) power supply (a) when the frequency is very large and (b) when the frequency is very small.

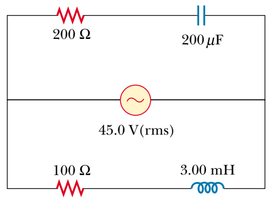


Figure P33.57

58. In the circuit shown in Figure P33.58, assume that all parameters except C are given. (a) Find the current as a function of time. (b) Find the power delivered to the circuit. (c) Find the current as a function of time after *only* switch 1 is opened. (d) After switch 2 is *also* opened, the current and voltage are in phase. Find the capacitance C. (e) Find the impedance of the circuit when both switches are open. (f) Find the maximum energy stored in the capacitor during oscillations. (g) Find the maximum energy stored in the inductor during oscillations. (h) Now the frequency of the voltage source is doubled. Find the phase difference between the current and the voltage. (i) Find the frequency that makes the inductive reactance one-half the capacitive reactance.

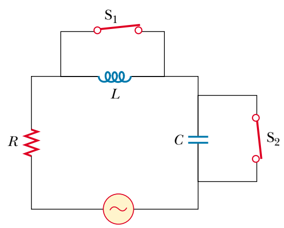


Figure P33.58

59. As an alternative to the RC filters described in Section 33.9, LC filters are used as both high- and low-pass filters. However, all real inductors have resistance, as indicated in Figure P33.59, and this resistance must be taken into account. (a) Determine which circuit in Figure P33.59 is the high-pass filter and which is the low-pass filter. (b) Derive the output/input ratio for each circuit, following the procedure used for the RC filters in Section 33.9.

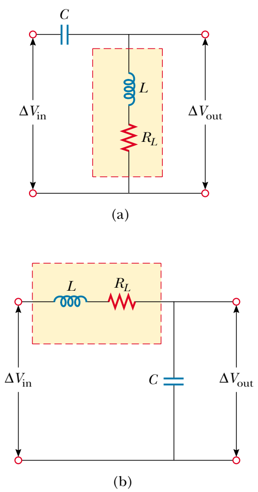


Figure P33.59

60. An 80.0-Ω resistor and a 200-mH inductor are connected in *parallel* across a 100-V (rms), 60.0-Hz source. (a) What is the rms current in the resistor? (b) By what angle does the total current lead or lag behind the voltage?

61. Make an order-of-magnitude estimate of the electric current that the electric company delivers to a town from a remote generating station. State the data that you measure or estimate. If you wish, you may consider a suburban bedroom community of 20 000 people.

62. A voltage $\Delta v = (100 \text{ V}) \sin \omega t$ (in SI units) is applied across a series combination of a 2.00-H inductor, a 10.0-μF capacitor, and a 10.0-Ω resistor. (a) Determine the angular frequency ω_0 at which the power delivered to the resistor is a maximum. (b) Calculate the power at that frequency. (c) Determine the two angular frequencies ω_1 and ω_2 at which the power delivered is one-half the maximum value. [The Q of the circuit is approximately $\omega_0/(\omega_2 - \omega_1)$.]

63. Consider a series RLC circuit having the following circuit parameters: $R = 200\ \Omega$, $L = 663$ mH, and $C = 26.5\ \mu$F. The applied voltage has an amplitude of 50.0 V and a frequency of 60.0 Hz. Find the following: (a) the current I_{max}, including its phase constant ϕ relative to the applied voltage Δv; (b) the voltage ΔV_R across the resistor and its phase relative to the current; (c) the voltage ΔV_C across the capacitor and its phase relative to the current; and (d) the voltage ΔV_L across the inductor and its phase relative to the current.

64. A power supply with $\Delta V_{rms} = 120$ V is connected between points a and d in Figure P33.25. At what frequency will it deliver a power of 250 W?

65. Example 28.2 showed that maximum power is transferred when the internal resistance of a dc source is equal to the resistance of the load. A transformer may be used to provide maximum power transfer between two ac circuits that have different impedances. (a) Show that the ratio of turns N_1/N_2 needed to meet this condition is

$$\frac{N_1}{N_2} = \sqrt{\frac{Z_1}{Z_2}}$$

(b) Suppose you want to use a transformer as an impedance-matching device between an audio amplifier that has an output impedance of 8.00 kΩ and a speaker that has an input impedance of 8.00 Ω. What should your N_1/N_2 ratio be?

66. Figure P33.66a shows a parallel RLC circuit, and the corresponding phasor diagram is provided in Figure P33.66b. The instantaneous voltages and rms voltages across the three circuit elements are the same, and each is in phase with the current through the resistor. The currents in C and L lead or lag behind the current in the resistor, as shown in Figure P33.66b. (a) Show that the rms current delivered by the source is

$$I_{rms} = \Delta V_{rms}\left[\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right]^{1/2}$$

(b) Show that the phase angle ϕ between ΔV_{rms} and I_{rms} is

$$\tan \phi = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)$$

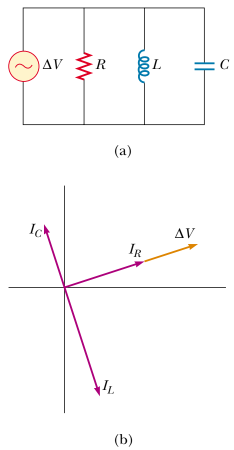


Figure P33.66

67. An 80.0-Ω resistor, a 200-mH inductor, and a 0.150-μF capacitor are connected *in parallel* across a 120-V (rms) source operating at 374 rad/s. (a) What is the resonant frequency of the circuit? (b) Calculate the rms current in the resistor, the inductor, and the capacitor. (c) What rms current is delivered by the source? (d) Is the current leading or lagging behind the voltage? By what angle?

68. Consider the phase-shifter circuit shown in Figure P33.68. The input voltage is described by the expression $\Delta v = (10.0 \text{ V}) \sin 200t$ (in SI units). Assuming that $L = 500$ mH, find (a) the value of R such that the output voltage lags behind the input voltage by 30.0° and (b) the amplitude of the output voltage.

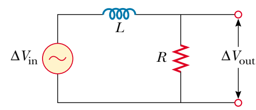


Figure P33.68

69. A series RLC circuit is operating at 2 000 Hz. At this frequency, $X_L = X_C = 1\,884\ \Omega$. The resistance of the circuit is 40.0 Ω. (a) Prepare a table showing the values of X_L, X_C, and Z for f = 300, 600, 800, 1 000, 1 500, 2 000, 3 000, 4 000, 6 000, and 10 000 Hz. (b) Plot on the same set of axes X_L, X_C, and Z as functions of $\ln f$.

70. A series RLC circuit in which $R = 1.00\ \Omega$, $L = 1.00$ mH, and $C = 1.00$ nF is connected to an ac generator delivering 1.00 V (rms). Make a precise graph of the power delivered to the circuit as a function of the frequency, and verify that the full width of the resonance peak at half-maximum is $R/2\pi L$.

71. Suppose the high-pass filter shown in Figure 33.22 has $R = 1\,000\ \Omega$ and $C = 0.050\,0\ \mu$F. (a) At what frequency does $\Delta V_{out}/\Delta V_{in} = \frac{1}{2}$? (b) Plot $\log_{10}(\Delta V_{out}/\Delta V_{in})$ versus $\log_{10}(f)$ over the frequency range from 1 Hz to 1 MHz. (This log–log plot of gain versus frequency is known as a *Bode plot*.)

ANSWERS TO QUICK QUIZZES

33.1 (c) $\mathcal{P}_{av} > 0$ and $i_{av} = 0$. The average power is proportional to the rms current—which, as Figure 33.3 shows, is nonzero even though the average current is zero. Condition (a) is valid only for an open circuit, and conditions (b) and (d) can never be true because $i_{av} = 0$ for ac circuits even though $i_{rms} > 0$.

33.2 (b) Sum of instantaneous voltages across elements. Choices (a) and (c) are incorrect because the unaligned sine curves in Figure 33.9b mean that the voltages are out of phase, so we cannot simply add the maximum (or rms) voltages across the elements. (In other words, $\Delta V \neq \Delta V_R + \Delta V_L + \Delta V_C$ even though it is true that $\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$.)

33.3 (a) $X_L < X_C$. (b) $X_L = X_C$. (c) $X_L > X_C$.

33.4 Equation 33.23 indicates that at resonance (when $X_L = X_C$) the impedance is due strictly to the resistor, $Z = R$. At resonance, the current is given by the expression $I_{rms} = \Delta V_{rms}/R$.

33.5 High. The higher the quality factor, the more sensitive the detector. As you can see from Figure 33.15a, when $Q = \omega_0/\Delta\omega$ is high, as it is in the $R = 3.5\ \Omega$ case, a slight change in the resonance frequency (as might happen when a small piece of metal passes through the portal) causes a large change in current that can be detected easily.

33.6 (a) An inductor. The current in an inductive circuit decreases with increasing frequency (see Eq. 33.9). Thus, an inductor connected in series with a woofer blocks high-frequency signals and passes low-frequency signals. (b) A capacitor. The current in a capacitive circuit decreases with decreasing frequency (see Eq. 33.16). When a capacitor is connected in series with a tweeter, the capacitor blocks low-frequency signals and passes high-frequency signals.

PUZZLER

This person is exposed to very bright sunlight at the beach. If he is wearing the wrong kind of sunglasses, he may be causing more permanent harm to his vision than he would be if he took the glasses off and squinted. What determines whether certain types of sunglasses are good for your eyes? *(Ron Chapple/FPG International)*

chapter

34

Electromagnetic Waves

Chapter Outline

The waves described in Chapters 16, 17, and 18 are mechanical waves. By definition, the propagation of mechanical disturbances—such as sound waves, water waves, and waves on a string—requires the presence of a medium. This chapter is concerned with the properties of electromagnetic waves, which (unlike mechanical waves) can propagate through empty space.

In Section 31.7 we gave a brief description of Maxwell's equations, which form the theoretical basis of all electromagnetic phenomena. The consequences of Maxwell's equations are far-reaching and dramatic. The Ampère–Maxwell law predicts that a time-varying electric field produces a magnetic field, just as Faraday's law tells us that a time-varying magnetic field produces an electric field. Maxwell's introduction of the concept of displacement current as a new source of a magnetic field provided the final important link between electric and magnetic fields in classical physics.

Astonishingly, Maxwell's equations also predict the existence of electromagnetic waves that propagate through space at the speed of light c. This chapter begins with a discussion of how Heinrich Hertz confirmed Maxwell's prediction when he generated and detected electromagnetic waves in 1887. That discovery has led to many practical communication systems, including radio, television, and radar. On a conceptual level, Maxwell unified the subjects of light and electromagnetism by developing the idea that light is a form of electromagnetic radiation.

Next, we learn how electromagnetic waves are generated by oscillating electric charges. The waves consist of oscillating electric and magnetic fields that are at right angles to each other and to the direction of wave propagation. Thus, electromagnetic waves are transverse waves. Maxwell's prediction of electromagnetic radiation shows that the amplitudes of the electric and magnetic fields in an electromagnetic wave are related by the expression $E = cB$. The waves radiated from the oscillating charges can be detected at great distances. Furthermore, electromagnetic waves carry energy and momentum and hence can exert pressure on a surface.

The chapter concludes with a look at the wide range of frequencies covered by electromagnetic waves. For example, radio waves (frequencies of about 10^7 Hz) are electromagnetic waves produced by oscillating currents in a radio tower's transmitting antenna. Light waves are a high-frequency form of electromagnetic radiation (about 10^{14} Hz) produced by oscillating electrons in atoms.

James Clerk Maxwell Scottish theoretical physicist (1831–1879) Maxwell developed the electromagnetic theory of light and the kinetic theory of gases, and he explained the nature of color vision and of Saturn's rings. His successful interpretation of the electromagnetic field produced the field equations that bear his name. Formidable mathematical ability combined with great insight enabled Maxwell to lead the way in the study of electromagnetism and kinetic theory. He died of cancer before he was 50. *(North Wind Picture Archives)*

34.1 MAXWELL'S EQUATIONS AND HERTZ'S DISCOVERIES

In his unified theory of electromagnetism, Maxwell showed that electromagnetic waves are a natural consequence of the fundamental laws expressed in the following four equations (see Section 31.7):

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \tag{34.1}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \tag{34.2}$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{34.3}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{34.4}$$

As we shall see in the next section, Equations 34.3 and 34.4 can be combined to obtain a wave equation for both the electric field and the magnetic field. In empty space ($Q = 0$, $I = 0$), the solution to these two equations shows that the speed at which electromagnetic waves travel equals the measured speed of light. This result led Maxwell to predict that light waves are a form of electromagnetic radiation.

The experimental apparatus that Hertz used to generate and detect electromagnetic waves is shown schematically in Figure 34.1. An induction coil is connected to a transmitter made up of two spherical electrodes separated by a narrow gap. The coil provides short voltage surges to the electrodes, making one positive and the other negative. A spark is generated between the spheres when the electric field near either electrode surpasses the dielectric strength for air (3×10^6 V/m; see Table 26.1). In a strong electric field, the acceleration of free electrons provides them with enough energy to ionize any molecules they strike. This ionization provides more electrons, which can accelerate and cause further ionizations. As the air in the gap is ionized, it becomes a much better conductor, and the discharge between the electrodes exhibits an oscillatory behavior at a very high frequency. From an electric-circuit viewpoint, this is equivalent to an LC circuit in which the inductance is that of the coil and the capacitance is due to the spherical electrodes.

Because L and C are quite small in Hertz's apparatus, the frequency of oscillation is very high, ≈ 100 MHz. (Recall from Eq. 32.22 that $\omega = 1/\sqrt{LC}$ for an LC circuit.) Electromagnetic waves are radiated at this frequency as a result of the oscillation (and hence acceleration) of free charges in the transmitter circuit. Hertz was able to detect these waves by using a single loop of wire with its own spark gap (the receiver). Such a receiver loop, placed several meters from the transmitter, has its own effective inductance, capacitance, and natural frequency of oscillation. In Hertz's experiment, sparks were induced across the gap of the receiving electrodes when the frequency of the receiver was adjusted to match that of the transmitter. Thus, Hertz demonstrated that the oscillating current induced in the receiver was produced by electromagnetic waves radiated by the transmitter. His experiment is analogous to the mechanical phenomenon in which a tuning fork responds to acoustic vibrations from an identical tuning fork that is oscillating.

Heinrich Rudolf Hertz German physicist (1857–1894) Hertz made his most important discovery—radio waves—in 1887. After finding that the speed of a radio wave was the same as that of light, he showed that radio waves, like light waves, could be reflected, refracted, and diffracted. Hertz died of blood poisoning at age 36. He made many contributions to science during his short life. The hertz, equal to one complete vibration or cycle per second, is named after him. *(The Bettmann Archive)*

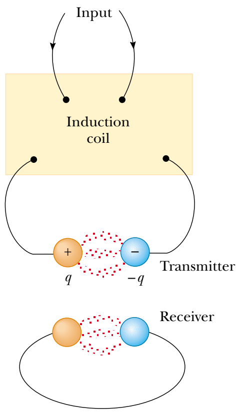
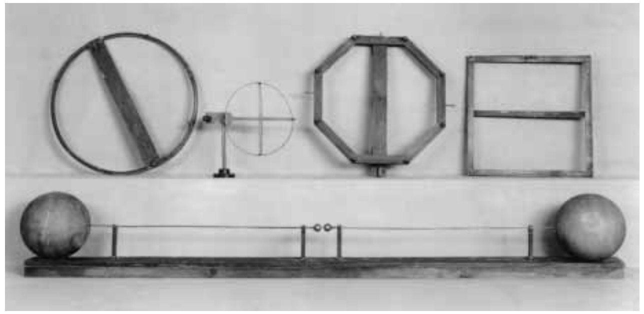


Figure 34.1 Schematic diagram of Hertz's apparatus for generating and detecting electromagnetic waves. The transmitter consists of two spherical electrodes connected to an induction coil, which provides short voltage surges to the spheres, setting up oscillations in the discharge between the electrodes (suggested by the red dots). The receiver is a nearby loop of wire containing a second spark gap.



A large oscillator (bottom) and circular, octagonal, and square receivers used by Heinrich Hertz.

QuickLab

Some electric motors use commutators that make and break electrical contact, creating sparks reminiscent of Hertz's method for generating electromagnetic waves. Try running an electric shaver or kitchen mixer near an AM radio. What happens to the reception?

Additionally, Hertz showed in a series of experiments that the radiation generated by his spark-gap device exhibited the wave properties of interference, diffraction, reflection, refraction, and polarization, all of which are properties exhibited by light. Thus, it became evident that the radio-frequency waves Hertz was generating had properties similar to those of light waves and differed only in frequency and wavelength. Perhaps his most convincing experiment was the measurement of the speed of this radiation. Radio-frequency waves of known frequency were reflected from a metal sheet and created a standing-wave interference pattern whose nodal points could be detected. The measured distance between the nodal points enabled determination of the wavelength λ. Using the relationship $v = \lambda f$ (Eq. 16.14), Hertz found that v was close to 3×10^8 m/s, the known speed c of visible light.

34.2 PLANE ELECTROMAGNETIC WAVES

The properties of electromagnetic waves can be deduced from Maxwell's equations. One approach to deriving these properties is to solve the second-order differential equation obtained from Maxwell's third and fourth equations. A rigorous mathematical treatment of that sort is beyond the scope of this text. To circumvent this problem, we assume that the vectors for the electric field and magnetic field in an electromagnetic wave have a specific space–time behavior that is simple but consistent with Maxwell's equations.

To understand the prediction of electromagnetic waves more fully, let us focus our attention on an electromagnetic wave that travels in the x direction (the *direction of propagation*). In this wave, the electric field $\mathbf{E}$ is in the y direction, and the magnetic field $\mathbf{B}$ is in the z direction, as shown in Figure 34.2. Waves such as this one, in which the electric and magnetic fields are restricted to being parallel to a pair of perpendicular axes, are said to be **linearly polarized waves.**[1] Furthermore, we assume that at any point P, the magnitudes E and B of the fields depend

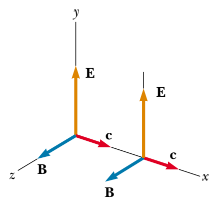


Figure 34.2 An electromagnetic wave traveling at velocity $\mathbf{c}$ in the positive x direction. The electric field is along the y direction, and the magnetic field is along the z direction. These fields depend only on x and t.

[1] Waves having other particular patterns of vibration of the electric and magnetic fields include circularly polarized waves. The most general polarization pattern is elliptical.

upon x and t only, and not upon the y or z coordinate. A collection of such waves from individual sources is called a **plane wave.** A surface connecting points of equal phase on all waves, which we call a **wave front,** would be a geometric plane. In comparison, a point source of radiation sends waves out in all directions. A surface connecting points of equal phase is a sphere for this situation, so we call this a **spherical wave.**

We can relate E and B to each other with Equations 34.3 and 34.4. In empty space, where $Q = 0$ and $I = 0$, Equation 34.3 remains unchanged and Equation 34.4 becomes

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad \textbf{(34.5)}$$

Using Equations 34.3 and 34.5 and the plane-wave assumption, we obtain the following differential equations relating E and B. (We shall derive these equations formally later in this section.) For simplicity, we drop the subscripts on the components E_y and B_z:

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \quad \textbf{(34.6)}$$

$$\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t} \quad \textbf{(34.7)}$$

Note that the derivatives here are partial derivatives. For example, when we evaluate $\partial E/\partial x$, we assume that t is constant. Likewise, when we evaluate $\partial B/\partial t$, x is held constant. Taking the derivative of Equation 34.6 with respect to x and combining the result with Equation 34.7, we obtain

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right) = -\frac{\partial}{\partial t}\left(-\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \textbf{(34.8)}$$

In the same manner, taking the derivative of Equation 34.7 with respect to x and combining it with Equation 34.6, we obtain

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} \quad \textbf{(34.9)}$$

Equations 34.8 and 34.9 both have the form of the general wave equation[2] with the wave speed v replaced by c, where

Speed of electromagnetic waves

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad \textbf{(34.10)}$$

Taking $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ and $\epsilon_0 = 8.854\,19 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$ in Equation 34.10, we find that $c = 2.997\,92 \times 10^8\ \mathrm{m/s}$. Because this speed is precisely the same as the speed of light in empty space, we are led to believe (correctly) that light is an electromagnetic wave.

[2] The general wave equation is of the form $(\partial^2 y/\partial x^2) = (1/v^2)(\partial^2 y/\partial t^2)$, where v is the speed of the wave and y is the wave function. The general wave equation was introduced as Equation 16.26, and it would be useful for you to review Section 16.9.

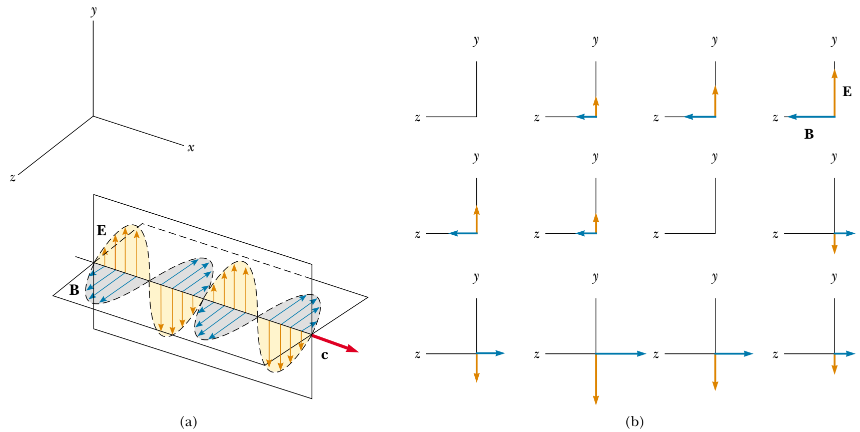


Figure 34.3 Representation of a sinusoidal, linearly polarized plane electromagnetic wave moving in the positive x direction with velocity **c**. (a) The wave at some instant. Note the sinusoidal variations of E and B with x. (b) A time sequence illustrating the electric and magnetic field vectors present in the yz plane, as seen by an observer looking in the negative x direction. Note the sinusoidal variations of E and B with t.

The simplest solution to Equations 34.8 and 34.9 is a sinusoidal wave, for which the field magnitudes E and B vary with x and t according to the expressions

Sinusoidal electric and magnetic fields

$$E = E_{\max} \cos(kx - \omega t) \quad \textbf{(34.11)}$$

$$B = B_{\max} \cos(kx - \omega t) \quad \textbf{(34.12)}$$

where $E_{\max}$ and $B_{\max}$ are the maximum values of the fields. The angular wave number is the constant $k = 2\pi/\lambda$, where λ is the wavelength. The angular frequency is $\omega = 2\pi f$, where f is the wave frequency. The ratio ω/k equals the speed c:

$$\frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = \lambda f = c$$

We have used Equation 16.14, $v = c = \lambda f$, which relates the speed, frequency, and wavelength of any continuous wave. Figure 34.3a is a pictorial representation, at one instant, of a sinusoidal, linearly polarized plane wave moving in the positive x direction. Figure 34.3b shows how the electric and magnetic field vectors at a fixed location vary with time.

Quick Quiz 34.1

What is the phase difference between B and E in Figure 34.3?

Taking partial derivatives of Equations 34.11 (with respect to x) and 34.12

(with respect to t), we find that

$$\frac{\partial E}{\partial x} = -kE_{\max}\sin(kx - \omega t)$$

$$\frac{\partial B}{\partial t} = \omega B_{\max}\sin(kx - \omega t)$$

Substituting these results into Equation 34.6, we find that at any instant

$$kE_{\max} = \omega B_{\max}$$

$$\frac{E_{\max}}{B_{\max}} = \frac{\omega}{k} = c$$

Using these results together with Equations 34.11 and 34.12, we see that

$$\frac{E_{\max}}{B_{\max}} = \frac{E}{B} = c \qquad \textbf{(34.13)}$$

That is, **at every instant the ratio of the magnitude of the electric field to the magnitude of the magnetic field in an electromagnetic wave equals the speed of light.**

Finally, note that electromagnetic waves obey the superposition principle (which we discussed in Section 16.4 with respect to mechanical waves) because the differential equations involving E and B are linear equations. For example, we can add two waves with the same frequency simply by adding the magnitudes of the two electric fields algebraically.

Properties of electromagnetic waves

- The solutions of Maxwell's third and fourth equations are wave-like, with both E and B satisfying a wave equation.
- Electromagnetic waves travel through empty space at the speed of light $c = 1/\sqrt{\mu_0\epsilon_0}$.
- The components of the electric and magnetic fields of plane electromagnetic waves are perpendicular to each other and perpendicular to the direction of wave propagation. We can summarize the latter property by saying that electromagnetic waves are transverse waves.
- The magnitudes of **E** and **B** in empty space are related by the expression $E/B = c$.
- Electromagnetic waves obey the principle of superposition.

EXAMPLE 34.1 An Electromagnetic Wave

A sinusoidal electromagnetic wave of frequency 40.0 MHz travels in free space in the x direction, as shown in Figure 34.4. (a) Determine the wavelength and period of the wave.

Solution Using Equation 16.14 for light waves, $c = \lambda f$, and given that $f = 40.0$ MHz $= 4.00 \times 10^7\ \text{s}^{-1}$, we have

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8\ \text{m/s}}{4.00 \times 10^7\ \text{s}^{-1}} = 7.50\ \text{m}$$

The period T of the wave is the inverse of the frequency:

$$T = \frac{1}{f} = \frac{1}{4.00 \times 10^7\ \text{s}^{-1}} = 2.50 \times 10^{-8}\ \text{s}$$

(b) At some point and at some instant, the electric field has its maximum value of 750 N/C and is along the y axis. Calculate the magnitude and direction of the magnetic field at this position and time.

Solution From Equation 34.13 we see that

$$B_{\max} = \frac{E_{\max}}{c} = \frac{750\ \text{N/C}}{3.00 \times 10^8\ \text{m/s}} = 2.50 \times 10^{-6}\ \text{T}$$

Because **E** and **B** must be perpendicular to each other and perpendicular to the direction of wave propagation (x in this case), we conclude that **B** is in the z direction.

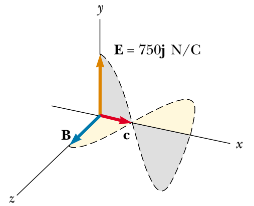


Figure 34.4 At some instant, a plane electromagnetic wave moving in the x direction has a maximum electric field of 750 N/C in the positive y direction. The corresponding magnetic field at that point has a magnitude E/c and is in the z direction.

(c) Write expressions for the space-time variation of the components of the electric and magnetic fields for this wave.

Solution We can apply Equations 34.11 and 34.12 directly:

$$E = E_{\max}\cos(kx - \omega t) = (750\ \text{N/C})\cos(kx - \omega t)$$

$$B = B_{\max}\cos(kx - \omega t) = (2.50 \times 10^{-6}\ \text{T})\cos(kx - \omega t)$$

where

$$\omega = 2\pi f = 2\pi(4.00 \times 10^7\ \text{s}^{-1}) = 2.51 \times 10^8\ \text{rad/s}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{7.50\ \text{m}} = 0.838\ \text{rad/m}$$

Let us summarize the properties of electromagnetic waves as we have described them:

Optional Section

Derivation of Equations 34.6 and 34.7

To derive Equation 34.6, we start with Faraday's law, Equation 34.3:

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

Let us again assume that the electromagnetic wave is traveling in the x direction, with the electric field **E** in the positive y direction and the magnetic field **B** in the positive z direction.

Consider a rectangle of width dx and height ℓ lying in the xy plane, as shown in Figure 34.5. To apply Equation 34.3, we must first evaluate the line integral of $\mathbf{E} \cdot d\mathbf{s}$ around this rectangle. The contributions from the top and bottom of the rectangle are zero because **E** is perpendicular to $d\mathbf{s}$ for these paths. We can express the electric field on the right side of the rectangle as

$$E(x + dx, t) \approx E(x, t) + \frac{dE}{dx}\bigg]_{t\ \text{constant}} dx = E(x, t) + \frac{\partial E}{\partial x}\,dx$$

while the field on the left side is simply $E(x, t)$.[3] Therefore, the line integral over this rectangle is approximately

$$\oint \mathbf{E} \cdot d\mathbf{s} = E(x + dx, t) \cdot \ell - E(x, t) \cdot \ell \approx (\partial E/\partial x)\,dx \cdot \ell \qquad \textbf{(34.14)}$$

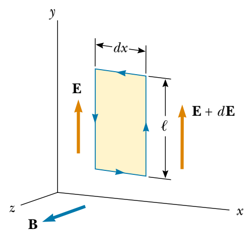


Figure 34.5 As a plane wave passes through a rectangular path of width dx lying in the xy plane, the electric field in the y direction varies from **E** to $\mathbf{E} + d\mathbf{E}$. This spatial variation in **E** gives rise to a time-varying magnetic field along the z direction, according to Equation 34.6.

Because the magnetic field is in the z direction, the magnetic flux through the rectangle of area $\ell\ dx$ is approximately $\Phi_B = B\ell\ dx$. (This assumes that dx is very small compared with the wavelength of the wave.) Taking the time derivative of

[3] Because dE/dx in this equation is expressed as the change in E with x at a given instant t, dE/dx is equivalent to the partial derivative $\partial E/\partial x$. Likewise, dB/dt means the change in B with time at a particular position x, so in Equation 34.15 we can replace dB/dt with $\partial B/\partial t$.

the magnetic flux gives

$$\frac{d\Phi_B}{dt} = \ell\, dx \frac{dB}{dt}\bigg]_{x\text{ constant}} = \ell\, dx \frac{\partial B}{\partial t} \tag{34.15}$$

Substituting Equations 34.14 and 34.15 into Equation 34.3, we obtain

$$\left(\frac{\partial E}{\partial x}\right) dx \cdot \ell = -\ell\, dx \frac{\partial B}{\partial t}$$

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

This expression is Equation 34.6.

In a similar manner, we can verify Equation 34.7 by starting with Maxwell's fourth equation in empty space (Eq. 34.5). In this case, we evaluate the line integral of $\mathbf{B} \cdot d\mathbf{s}$ around a rectangle lying in the xz plane and having width dx and length ℓ, as shown in Figure 34.6. Noting that the magnitude of the magnetic field changes from $B(x, t)$ to $B(x + dx, t)$ over the width dx, we find the line integral over this rectangle to be approximately

$$\oint \mathbf{B} \cdot d\mathbf{s} = B(x, t) \cdot \ell - B(x + dx, t) \cdot \ell \approx -(\partial B/\partial x)\, dx \cdot \ell \tag{34.16}$$

The electric flux through the rectangle is $\Phi_E = E\ell\, dx$, which, when differentiated with respect to time, gives

$$\frac{\partial \Phi_E}{\partial t} = \ell\, dx \frac{\partial E}{\partial t} \tag{34.17}$$

Substituting Equations 34.16 and 34.17 into Equation 34.5 gives

$$-(\partial B/\partial x)\, dx \cdot \ell = \mu_0\epsilon_0 \ell\, dx(\partial E/\partial t)$$

$$\frac{\partial B}{\partial x} = -\mu_0\epsilon_0 \frac{\partial E}{\partial t}$$

which is Equation 34.7.

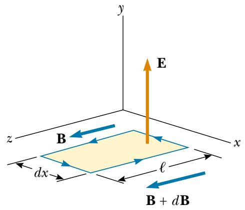


Figure 34.6 As a plane wave passes through a rectangular path of width dx lying in the xz plane, the magnetic field in the z direction varies from **B** to **B** + d**B**. This spatial variation in **B** gives rise to a time-varying electric field along the y direction, according to Equation 34.7.

34.3 ENERGY CARRIED BY ELECTROMAGNETIC WAVES

Electromagnetic waves carry energy, and as they propagate through space they can transfer energy to objects placed in their path. The rate of flow of energy in an electromagnetic wave is described by a vector **S**, called the **Poynting vector,** which is defined by the expression

Poynting vector

$$\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \tag{34.18}$$

The magnitude of the Poynting vector represents the rate at which energy flows through a unit surface area perpendicular to the direction of wave propagation. Thus, the magnitude of the Poynting vector represents *power per unit area*. The direction of the vector is along the direction of wave propagation (Fig. 34.7). The SI units of the Poynting vector are $\text{J/s}\cdot\text{m}^2 = \text{W/m}^2$.

As an example, let us evaluate the magnitude of **S** for a plane electromagnetic wave where $|\mathbf{E} \times \mathbf{B}| = EB$. In this case,

Magnitude of the Poynting vector for a plane wave

$$S = \frac{EB}{\mu_0} \tag{34.19}$$

Because $B = E/c$, we can also express this as

$$S = \frac{E^2}{\mu_0 c} = \frac{c}{\mu_0} B^2$$

These equations for S apply at any instant of time and represent the *instantaneous* rate at which energy is passing through a unit area.

What is of greater interest for a sinusoidal plane electromagnetic wave is the time average of S over one or more cycles, which is called the *wave intensity I.* (We discussed the intensity of sound waves in Chapter 17.) When this average is taken, we obtain an expression involving the time average of $\cos^2(kx - \omega t)$, which equals $\frac{1}{2}$. Hence, the average value of S (in other words, the intensity of the wave) is

Wave intensity

$$I = S_{\text{av}} = \frac{E_{\text{max}} B_{\text{max}}}{2\mu_0} = \frac{E_{\text{max}}^2}{2\mu_0 c} = \frac{c}{2\mu_0} B_{\text{max}}^2 \tag{34.20}$$

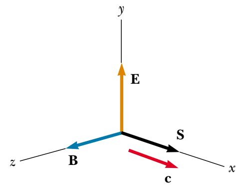


Figure 34.7 The Poynting vector **S** for a plane electromagnetic wave is along the direction of wave propagation.

Recall that the energy per unit volume, which is the instantaneous energy density u_E associated with an electric field, is given by Equation 26.13,

$$u_E = \tfrac{1}{2}\epsilon_0 E^2$$

and that the instantaneous energy density u_B associated with a magnetic field is given by Equation 32.14:

$$u_B = \frac{B^2}{2\mu_0}$$

Because E and B vary with time for an electromagnetic wave, the energy densities also vary with time. When we use the relationships $B = E/c$ and $c = 1/\sqrt{\mu_0\epsilon_0}$, Equation 32.14 becomes

$$u_B = \frac{(E/c)^2}{2\mu_0} = \frac{\mu_0\epsilon_0}{2\mu_0} E^2 = \tfrac{1}{2}\epsilon_0 E^2$$

Comparing this result with the expression for u_E, we see that

$$u_B = u_E = \tfrac{1}{2}\epsilon_0 E^2 = \frac{B^2}{2\mu_0}$$

That is, **for an electromagnetic wave, the instantaneous energy density associated with the magnetic field equals the instantaneous energy density associated with the electric field.** Hence, in a given volume the energy is equally shared by the two fields.

Total instantaneous energy density

The **total instantaneous energy density** u is equal to the sum of the energy densities associated with the electric and magnetic fields:

$$u = u_E + u_B = \epsilon_0 E^2 = \frac{B^2}{\mu_0}$$

Average energy density of an electromagnetic wave

When this total instantaneous energy density is averaged over one or more cycles of an electromagnetic wave, we again obtain a factor of $\frac{1}{2}$. Hence, for any electromagnetic wave, the total average energy per unit volume is

$$u_{av} = \epsilon_0 (E^2)_{av} = \tfrac{1}{2}\epsilon_0 E_{max}^2 = \frac{B_{max}^2}{2\mu_0} \qquad (34.21)$$

EXAMPLE 34.2 Fields on the Page

Estimate the maximum magnitudes of the electric and magnetic fields of the light that is incident on this page because of the visible light coming from your desk lamp. Treat the bulb as a point source of electromagnetic radiation that is about 5% efficient at converting electrical energy to visible light.

Solution Recall from Equation 17.8 that the wave intensity I a distance r from a point source is $I = \mathcal{P}_{av}/4\pi r^2$, where $\mathcal{P}_{av}$ is the average power output of the source and $4\pi r^2$ is the area of a sphere of radius r centered on the source. Because the intensity of an electromagnetic wave is also given by Equation 34.20, we have

$$I = \frac{\mathcal{P}_{av}}{4\pi r^2} = \frac{E_{max}^2}{2\mu_0 c}$$

We must now make some assumptions about numbers to enter in this equation. If we have a 60-W lightbulb, its output at 5% efficiency is approximately 3.0 W in the form of visible light. (The remaining energy transfers out of the bulb by conduction and invisible radiation.) A reasonable distance from the bulb to the page might be 0.30 m. Thus, we have

$$E_{max} = \sqrt{\frac{\mu_0 c \mathcal{P}_{av}}{2\pi r^2}}$$

$$= \sqrt{\frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(3.00 \times 10^8\ \text{m/s})(3.0\ \text{W})}{2\pi (0.30\ \text{m})^2}}$$

$$= 45\ \text{V/m}$$

From Equation 34.13,

$$B_{max} = \frac{E_{max}}{c} = \frac{45\ \text{V/m}}{3.00 \times 10^8\ \text{m/s}} = 1.5 \times 10^{-7}\ \text{T}$$

This value is two orders of magnitude smaller than the Earth's magnetic field, which, unlike the magnetic field in the light wave from your desk lamp, is not oscillating.

Exercise Estimate the energy density of the light wave just before it strikes this page.

Answer $9.0 \times 10^{-9}\ \text{J/m}^3$.

Comparing this result with Equation 34.20 for the average value of S, we see that

$$I = S_{av} = c u_{av} \qquad (34.22)$$

In other words, **the intensity of an electromagnetic wave equals the average energy density multiplied by the speed of light.**

34.4 MOMENTUM AND RADIATION PRESSURE

Electromagnetic waves transport linear momentum as well as energy. It follows that, as this momentum is absorbed by some surface, pressure is exerted on the surface. We shall assume in this discussion that the electromagnetic wave strikes the surface at normal incidence and transports a total energy U to the surface in a time t. Maxwell showed that, if the surface absorbs all the incident energy U in this time (as does a black body, introduced in Chapter 20), the total momentum $\mathbf{p}$ transported to the surface has a magnitude

Momentum transported to a perfectly absorbing surface

$$p = \frac{U}{c} \quad \text{(complete absorption)} \qquad (34.23)$$

The pressure exerted on the surface is defined as force per unit area F/A. Let us combine this with Newton's second law:

$$P = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt}$$

If we now replace p, the momentum transported to the surface by light, from Equation 34.23, we have

Radiation pressure exerted on a perfectly absorbing surface

$$P = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{d}{dt}\left(\frac{U}{c}\right) = \frac{1}{c}\frac{(dU/dt)}{A}$$

We recognize $(dU/dt)/A$ as the rate at which energy is arriving at the surface per unit area, which is the magnitude of the Poynting vector. Thus, the radiation pressure P exerted on the perfectly absorbing surface is

$$P = \frac{S}{c} \qquad (34.24)$$

QuickLab
Using Example 34.2 as a starting point, estimate the total force exerted on this page by the light from your desk lamp. Does it make a difference if the page contains large, dark photographs instead of mostly white space?

Note that Equation 34.24 is an expression for uppercase P, the pressure, while Equation 34.23 is an expression for lowercase p, linear momentum.

If the surface is a perfect reflector (such as a mirror) and incidence is normal, then the momentum transported to the surface in a time t is twice that given by Equation 34.23. That is, the momentum transferred to the surface by the incoming light is $p = U/c$, and that transferred by the reflected light also is $p = U/c$. Therefore,

Radiation pressure exerted on a perfectly reflecting surface

$$p = \frac{2U}{c} \quad \text{(complete reflection)} \qquad (34.25)$$

The momentum delivered to a surface having a reflectivity somewhere between these two extremes has a value between U/c and $2U/c$, depending on the properties of the surface. Finally, the radiation pressure exerted on a perfectly reflecting surface for normal incidence of the wave is[4]

$$P = \frac{2S}{c} \qquad (34.26)$$

web
Visit **http://pds.jpl.nasa.gov** for more information about missions to the planets. You may also want to read Arthur C. Clarke's 1963 science fiction story *The Wind from the Sun* about a solar yacht race.

Although radiation pressures are very small (about $5 \times 10^{-6}\ \text{N/m}^2$ for direct sunlight), they have been measured with torsion balances such as the one shown in Figure 34.8. A mirror (a perfect reflector) and a black disk (a perfect absorber) are connected by a horizontal rod suspended from a fine fiber. Normal-incidence light striking the black disk is completely absorbed, so all of the momentum of the

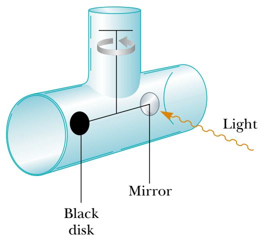


Figure 34.8 An apparatus for measuring the pressure exerted by light. In practice, the system is contained in a high vacuum.

Figure 34.9 Mariner 10 used its solar panels to "sail on sunlight."

[4] For oblique incidence on a perfectly reflecting surface, the momentum transferred is $(2U\cos\theta)/c$ and the pressure is $P = (2S\cos^2\theta)/c$, where θ is the angle between the normal to the surface and the direction of wave propagation.

light is transferred to the disk. Normal-incidence light striking the mirror is totally reflected, and hence the momentum transferred to the mirror is twice as great as that transferred to the disk. The radiation pressure is determined by measuring the angle through which the horizontal connecting rod rotates. The apparatus

CONCEPTUAL EXAMPLE 34.3 Sweeping the Solar System

A great amount of dust exists in interplanetary space. Although in theory these dust particles can vary in size from molecular size to much larger, very little of the dust in our solar system is smaller than about 0.2 μm. Why?

Solution The dust particles are subject to two significant forces—the gravitational force that draws them toward the Sun and the radiation-pressure force that pushes them away from the Sun. The gravitational force is proportional to the cube of the radius of a spherical dust particle because it is proportional to the mass and therefore to the volume $4\pi r^3/3$ of the particle. The radiation pressure is proportional to the square of the radius because it depends on the planar cross-section of the particle. For large particles, the gravitational force is greater than the force from radiation pressure. For particles having radii less than about 0.2 μm, the radiation-pressure force is greater than the gravitational force, and as a result these particles are swept out of the Solar System.

EXAMPLE 34.4 Pressure from a Laser Pointer

Many people giving presentations use a laser pointer to direct the attention of the audience. If a 3.0-mW pointer creates a spot that is 2.0 mm in diameter, determine the radiation pressure on a screen that reflects 70% of the light that strikes it. The power 3.0 mW is a time-averaged value.

Solution We certainly do not expect the pressure to be very large. Before we can calculate it, we must determine the Poynting vector of the beam by dividing the time-averaged power delivered via the electromagnetic wave by the cross-sectional area of the beam:

$$S = \frac{\mathcal{P}}{A} = \frac{\mathcal{P}}{\pi r^2} = \frac{3.0 \times 10^{-3}\ \text{W}}{\pi\left(\dfrac{2.0 \times 10^{-3}\ \text{m}}{2}\right)^2} = 955\ \text{W/m}^2$$

This is about the same as the intensity of sunlight at the Earth's surface. (Thus, it is not safe to shine the beam of a laser pointer into a person's eyes; that may be more dangerous than looking directly at the Sun.)

Now we can determine the radiation pressure from the laser beam. Equation 34.26 indicates that a completely reflected beam would apply a pressure of $P = 2S/c$. We can model the actual reflection as follows: Imagine that the surface absorbs the beam, resulting in pressure $P = S/c$. Then the surface emits the beam, resulting in additional pressure $P = S/c$. If the surface emits only a fraction f of the beam (so that f is the amount of the incident beam reflected), then the pressure due to the emitted beam is $P = fS/c$. Thus, the total pressure on the surface due to absorption and re-emission (reflection) is

$$P = \frac{S}{c} + f\frac{S}{c} = (1 + f)\frac{S}{c}$$

Notice that if $f = 1$, which represents complete reflection, this equation reduces to Equation 34.26. For a beam that is 70% reflected, the pressure is

$$P = (1 + 0.70)\frac{955\ \text{W/m}^2}{3.0 \times 10^8\ \text{m/s}} = \boxed{5.4 \times 10^{-6}\ \text{N/m}^2}$$

This is an extremely small value, as expected. (Recall from Section 15.2 that atmospheric pressure is approximately 10^5 N/m^2.)

EXAMPLE 34.5 Solar Energy

As noted in the preceding example, the Sun delivers about 1 000 W/m^2 of energy to the Earth's surface via electromagnetic radiation. (a) Calculate the total power that is incident on a roof of dimensions 8.00 m × 20.0 m.

Solution The magnitude of the Poynting vector for solar radiation at the surface of the Earth is $S = 1\,000$ W/m^2; this represents the power per unit area, or the light intensity. Assuming that the radiation is incident normal to the roof, we obtain

$$\mathcal{P} = SA = (1\,000\ \text{W/m}^2)(8.00 \times 20.0\ \text{m}^2) = \boxed{1.60 \times 10^5\ \text{W}}$$

If all of this power could be converted to electrical energy, it would provide more than enough power for the average home. However, solar energy is not easily harnessed, and the prospects for large-scale conversion are not as bright as may appear from this calculation. For example, the efficiency of conversion from solar to electrical energy is typically 10% for photovoltaic cells. Roof systems for converting solar energy to internal energy are approximately 50% efficient; however, solar energy is associated with other practical problems, such as overcast days, geographic location, and methods of energy storage.

(b) Determine the radiation pressure and the radiation force exerted on the roof, assuming that the roof covering is a perfect absorber.

Solution Using Equation 34.24 with $S = 1\,000$ W/m^2, we find that the radiation pressure is

$$P = \frac{S}{c} = \frac{1\,000\ \text{W/m}^2}{3.00 \times 10^8\ \text{m/s}} = \boxed{3.33 \times 10^{-6}\ \text{N/m}^2}$$

Because pressure equals force per unit area, this corresponds to a radiation force of

$$F = PA = (3.33 \times 10^{-6}\ \text{N/m}^2)(160\ \text{m}^2) = \boxed{5.33 \times 10^{-4}\ \text{N}}$$

Exercise How much solar energy is incident on the roof in 1 h?

Answer 5.76×10^8 J.

must be placed in a high vacuum to eliminate the effects of air currents.

NASA is exploring the possibility of *solar sailing* as a low-cost means of sending spacecraft to the planets. Large reflective sheets would be used in much the way canvas sheets are used on earthbound sailboats. In 1973 NASA engineers took advantage of the momentum of the sunlight striking the solar panels of Mariner 10 (Fig. 34.9) to make small course corrections when the spacecraft's fuel supply was running low. (This procedure was carried out when the spacecraft was in the vicinity of the planet Mercury. Would it have worked as well near Pluto?)

Optional Section

34.5 RADIATION FROM AN INFINITE CURRENT SHEET

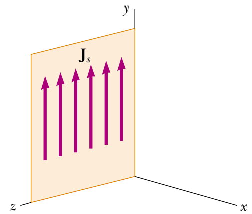


Figure 34.10 A portion of an infinite current sheet lying in the yz plane. The current density is sinusoidal and is given by the expression $J_s = J_{\max} \cos \omega t$. The magnetic field is everywhere parallel to the sheet and lies along z.

In this section, we describe the electric and magnetic fields radiated by a flat conductor carrying a time-varying current. In the symmetric plane geometry employed here, the mathematics is less complex than that required in lower-symmetry situations.

Consider an infinite conducting sheet lying in the yz plane and carrying a surface current in the y direction, as shown in Figure 34.10. The current is distributed across the z direction such that the current per unit length is $\mathbf{J}_s$. Let us assume that J_s varies sinusoidally with time as

$$J_s = J_{\max} \cos \omega t$$

where $J_{\max}$ is the amplitude of the current variation and ω is the angular frequency of the variation. A similar problem concerning the case of a steady current was treated in Example 30.6, in which we found that the magnetic field outside the sheet is everywhere parallel to the sheet and lies along the z axis. The magnetic field was found to have a magnitude

Radiated magnetic field

$$B_z = \mu_0 \frac{J_s}{2}$$

[5] Note that the solution could also be written in the form $\cos(\omega t - kx)$, which is equivalent to $\cos(kx - \omega t)$. That is, $\cos\theta$ is an even function, which means that $\cos(-\theta) = \cos\theta$.

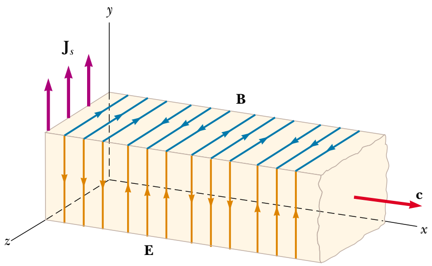


Figure 34.11 Representation of the plane electromagnetic wave radiated by an infinite current sheet lying in the yz plane. The vector $\mathbf{B}$ is in the z direction, the vector $\mathbf{E}$ is in the y direction, and the direction of wave motion is along x. Both vector $\mathbf{B}$ and vector $\mathbf{E}$ behave according to the expression $\cos(kx - \omega t)$. Compare this drawing with Figure 34.3a.

In the present situation, where J_s varies with time, this equation for B_z is valid only for distances close to the sheet. Substituting the expression for J_s, we have

$$B_z = \frac{\mu_0}{2} J_{\max} \cos \omega t \qquad \text{(for small values of } x\text{)}$$

To obtain the expression valid for B_z for arbitrary values of x, we can investigate the solution:[5]

$$B_z = \frac{\mu_0 J_{\max}}{2} \cos(kx - \omega t) \tag{34.27}$$

Radiated electric field

You should note two things about this solution, which is unique to the geometry under consideration. First, when x is very small, it agrees with our original solution. Second, it satisfies the wave equation as expressed in Equation 34.9. We conclude that the magnetic field lies along the z axis, varies with time, and is characterized by a transverse traveling wave having an angular frequency ω and an angular wave number $k = 2\pi/\lambda$.

We can obtain the electric field radiating from our infinite current sheet by using Equation 34.13:

$$E_y = cB_z = \frac{\mu_0 J_{\max} c}{2} \cos(kx - \omega t) \tag{34.28}$$

That is, the electric field is in the y direction, perpendicular to $\mathbf{B}$, and has the same space and time dependencies. These expressions for B_z and E_y show that the radiation field of an infinite current sheet carrying a sinusoidal current is a plane electromagnetic wave propagating with a speed c along the x axis, as shown in Figure 34.11.

We can calculate the Poynting vector for this wave from Equations 34.19,

EXAMPLE 34.6 An Infinite Sheet Carrying a Sinusoidal Current

An infinite current sheet lying in the yz plane carries a sinusoidal current that has a maximum density of 5.00 A/m. (a) Find the maximum values of the radiated magnetic and electric fields.

Solution From Equations 34.27 and 34.28, we see that the maximum values of B_z and E_y are

$$B_{\max} = \frac{\mu_0 J_{\max}}{2} \qquad \text{and} \qquad E_{\max} = \frac{\mu_0 J_{\max} c}{2}$$

Using the values $\mu_0 = 4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A}$, $J_{\max} = 5.00$ A/m, and $c = 3.00 \times 10^8$ m/s, we get

$$B_{\max} = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(5.00\ \text{A/m})}{2} = 3.14 \times 10^{-6}\ \text{T}$$

$$E_{\max} = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(5.00\ \text{A/m})(3.00 \times 10^8\ \text{m/s})}{2} = 942\ \text{V/m}$$

(b) What is the average power incident on a flat surface that is parallel to the sheet and has an area of 3.00 m^2? (The length and width of this surface are both much greater than the wavelength of the radiation.)

Solution The intensity, or power per unit area, radiated in each direction by the current sheet is given by Equation 34.30:

$$I = \frac{\mu_0 J_{\max}^2 c}{8} = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(5.00\ \text{A/m})^2(3.00 \times 10^8\ \text{m/s})}{8} = 1.18 \times 10^3\ \text{W/m}^2$$

Multiplying this by the area of the surface, we obtain the incident power:

$$\mathcal{P} = IA = (1.18 \times 10^3\ \text{W/m}^2)(3.00\ \text{m}^2) = 3.54 \times 10^3\ \text{W}$$

The result is independent of the distance from the current sheet because we are dealing with a plane wave.

34.27, and 34.28:

$$S = \frac{EB}{\mu_0} = \frac{\mu_0 J_{\max}^2 c}{4} \cos^2(kx - \omega t) \tag{34.29}$$

The intensity of the wave, which equals the average value of S, is

$$I = S_{\text{av}} = \frac{\mu_0 J_{\max}^2 c}{8} \tag{34.30}$$

Accelerating charges produce electromagnetic radiation

This intensity represents the power per unit area of the outgoing wave on each side of the sheet. The total rate of energy emitted per unit area of the conductor is $2S_{\text{av}} = \mu_0 J_{\max}^2 c/4$.

Optional Section

34.6 PRODUCTION OF ELECTROMAGNETIC WAVES BY AN ANTENNA

Neither stationary charges nor steady currents can produce electromagnetic waves. Whenever the current through a wire changes with time, however, the wire emits

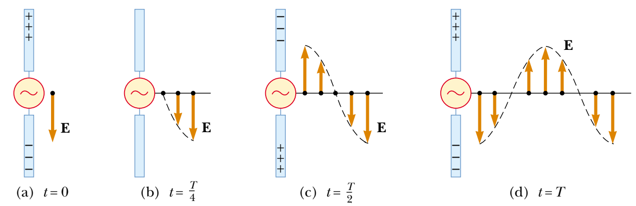


Figure 34.12 The electric field set up by charges oscillating in an antenna. The field moves away from the antenna with the speed of light.

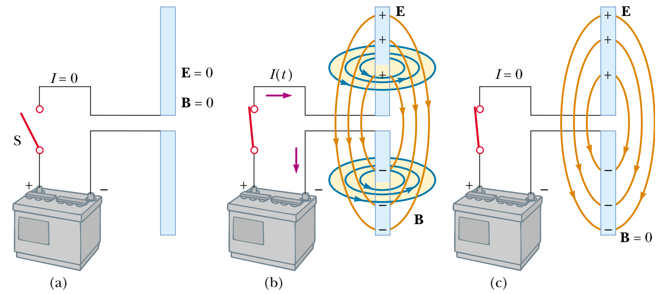


Figure 34.13 A pair of metal rods connected to a battery. (a) When the switch is open and no current exists, the electric and magnetic fields are both zero. (b) Immediately after the switch is closed, the rods are being charged (so a current exists). Because the current is changing, the rods generate changing electric and magnetic fields. (c) When the rods are fully charged, the current is zero, the electric field is a maximum, and the magnetic field is zero.

electromagnetic radiation. **The fundamental mechanism responsible for this radiation is the acceleration of a charged particle. Whenever a charged particle accelerates, it must radiate energy.**

An alternating voltage applied to the wires of an antenna forces an electric charge in the antenna to oscillate. This is a common technique for accelerating charges and is the source of the radio waves emitted by the transmitting antenna of a radio station. Figure 34.12 shows how this is done. Two metal rods are connected to a generator that provides a sinusoidally oscillating voltage. This causes charges to oscillate in the two rods. At $t = 0$, the upper rod is given a maximum positive charge and the bottom rod an equal negative charge, as shown in Figure 34.12a. The electric field near the antenna at this instant is also shown in Figure 34.12a. As the positive and negative charges decrease from their maximum values, the rods become less charged, the field near the rods decreases in strength, and the downward-directed maximum electric field produced at $t = 0$ moves away from the rod. (A magnetic field oscillating in a direction perpendicular to the plane of the diagram in Fig. 34.12 accompanies the oscillating electric field, but it is not shown for the sake of clarity.) When the charges on the rods are momentarily zero (Fig. 34.12b), the electric field at the rod has dropped to zero. This occurs at a time equal to one quarter of the period of oscillation.

As the generator charges the rods in the opposite sense from that at the beginning, the upper rod soon obtains a maximum negative charge and the lower rod a maximum positive charge (Fig. 34.12c); this results in an electric field near the rod that is directed upward after a time equal to one-half the period of oscillation. The oscillations continue as indicated in Figure 34.12d. The electric field near the antenna oscillates in phase with the charge distribution. That is, the field points down when the upper rod is positive and up when the upper rod is negative. Furthermore, the magnitude of the field at any instant depends on the amount of charge on the rods at that instant.

As the charges continue to oscillate (and accelerate) between the rods, the

[6] We have neglected the fields caused by the wires leading to the rods. This is a good approximation if the circuit dimensions are much less than the length of the rods.

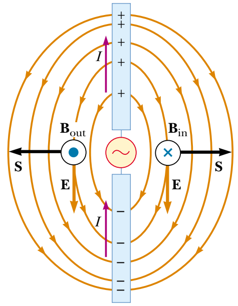


Figure 34.14 A half-wave antenna consists of two metal rods connected to an alternating voltage source. This diagram shows **E** and **B** at an instant when the current is upward. Note that the electric field lines resemble those of a dipole (shown in Fig. 23.21).

electric field they set up moves away from the antenna at the speed of light. As you can see from Figure 34.12, one cycle of charge oscillation produces one wavelength in the electric-field pattern.

Next, consider what happens when two conducting rods are connected to the terminals of a battery (Fig. 34.13). Before the switch is closed, the current is zero, so no fields are present (Fig. 34.13a). Just after the switch is closed, positive charge begins to build up on one rod and negative charge on the other (Fig. 34.13b), a situation that corresponds to a time-varying current. The changing charge distribution causes the electric field to change; this in turn produces a magnetic field around the rods.[6] Finally, when the rods are fully charged, the current is zero; hence, no magnetic field exists at that instant (Fig. 34.13c).

Now let us consider the production of electromagnetic waves by a *half-wave antenna.* In this arrangement, two conducting rods are connected to a source of alternating voltage (such as an *LC* oscillator), as shown in Figure 34.14. The length of each rod is equal to one quarter of the wavelength of the radiation that will be emitted when the oscillator operates at frequency *f.* The oscillator forces charges to accelerate back and forth between the two rods. Figure 34.14 shows the configuration of the electric and magnetic fields at some instant when the current is upward. The electric field lines resemble those of an electric dipole. (As a result, this type of antenna is sometimes called a *dipole antenna.*) Because these charges are continuously oscillating between the two rods, the antenna can be approximated by an oscillating electric dipole. The magnetic field lines form concentric circles around the antenna and are perpendicular to the electric field lines at all points. The magnetic field is zero at all points along the axis of the antenna. Furthermore, **E** and **B** are 90° out of phase in time because the current is zero when the charges at the outer ends of the rods are at a maximum.

At the two points where the magnetic field is shown in Figure 34.14, the Poynting vector **S** is directed radially outward. This indicates that energy is flowing away from the antenna at this instant. At later times, the fields and the Poynting vector change direction as the current alternates. Because **E** and **B** are 90° out of phase at points near the dipole, the net energy flow is zero. From this, we might conclude (incorrectly) that no energy is radiated by the dipole.

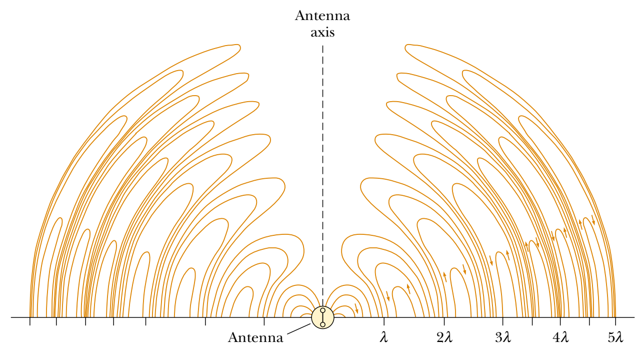


Figure 34.15 Electric field lines surrounding a dipole antenna at a given instant. The radiation fields propagate outward from the antenna with a speed *c*.

However, we find that energy is indeed radiated. Because the dipole fields fall off as $1/r^3$ (as shown in Example 23.6 for the electric field of a static dipole), they are not important at great distances from the antenna. However, at these great distances, something else causes a type of radiation different from that close to the antenna. The source of this radiation is the continuous induction of an electric field by the time-varying magnetic field and the induction of a magnetic field by the time-varying electric field, predicted by Equations 34.3 and 34.4. The electric and magnetic fields produced in this manner are in phase with each other and vary as $1/r$. The result is an outward flow of energy at all times.

The electric field lines produced by a dipole antenna at some instant are shown in Figure 34.15 as they propagate away from the antenna. Note that the intensity and the power radiated are a maximum in a plane that is perpendicular to the antenna and passing through its midpoint. Furthermore, the power radiated is zero along the antenna's axis. A mathematical solution to Maxwell's equations for the dipole antenna shows that the intensity of the radiation varies as $(\sin^2\theta)/r^2$, where θ is measured from the axis of the antenna. The angular dependence of the radiation intensity is sketched in Figure 34.16.

Electromagnetic waves can also induce currents in a receiving antenna. The response of a dipole receiving antenna at a given position is a maximum when the antenna axis is parallel to the electric field at that point and zero when the axis is perpendicular to the electric field.

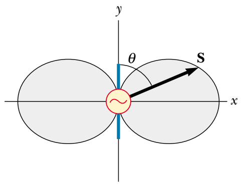


Figure 34.16 Angular dependence of the intensity of radiation produced by an oscillating electric dipole.

QuickLab

Rotate a portable radio (with a telescoping antenna) about a horizontal axis while it is tuned to a weak station. Can you use what you learn from this movement to verify the answer to Quick Quiz 34.2?

Quick Quiz 34.2

If the plane electromagnetic wave in Figure 34.11 represents the signal from a distant radio station, what would be the best orientation for your portable radio antenna—(a) along the *x* axis, (b) along the *y* axis, or (c) along the *z* axis?

34.7 THE SPECTRUM OF ELECTROMAGNETIC WAVES

The various types of electromagnetic waves are listed in Figure 34.17, which shows the **electromagnetic spectrum.** Note the wide ranges of frequencies and wavelengths. No sharp dividing point exists between one type of wave and the next. Remember that **all forms of the various types of radiation are produced by the same phenomenon—accelerating charges.** The names given to the types of waves are simply for convenience in describing the region of the spectrum in which they lie.

Radio waves

Radio waves are the result of charges accelerating through conducting wires. Ranging from more than 10^4 m to about 0.1 m in wavelength, they are generated by such electronic devices as *LC* oscillators and are used in radio and television communication systems.

Microwaves

Microwaves have wavelengths ranging from approximately 0.3 m to 10^{-4} m and are also generated by electronic devices. Because of their short wavelengths, they are well suited for radar systems and for studying the atomic and molecular properties of matter. Microwave ovens (in which the wavelength of the radiation is $\lambda = 0.122$ m) are an interesting domestic application of these waves. It has been suggested that solar energy could be harnessed by beaming microwaves to the Earth from a solar collector in space.[7]

Infrared waves

Visible light waves

[7] P. Glaser, "Solar Power from Satellites," *Phys. Today,* February 1977, p. 30.

Satellite-dish television antennas receive television-station signals from satellites in orbit around the Earth.

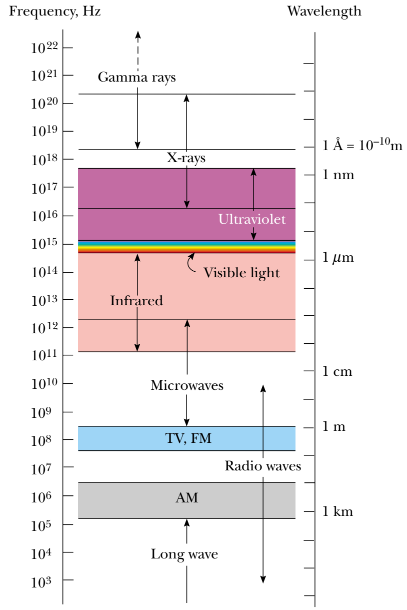


Figure 34.17 The electromagnetic spectrum. Note the overlap between adjacent wave types.

Ultraviolet waves

Infrared waves have wavelengths ranging from 10^{-3} m to the longest wavelength of visible light, 7×10^{-7} m. These waves, produced by molecules and room-temperature objects, are readily absorbed by most materials. The infrared (IR) energy absorbed by a substance appears as internal energy because the energy agitates the atoms of the object, increasing their vibrational or translational motion, which results in a temperature increase. Infrared radiation has practical and scientific applications in many areas, including physical therapy, IR photography, and vibrational spectroscopy.

Visible light, the most familiar form of electromagnetic waves, is the part of the electromagnetic spectrum that the human eye can detect. Light is produced by the rearrangement of electrons in atoms and molecules. The various wavelengths of visible light, which correspond to different colors, range from red ($\lambda \approx 7 \times 10^{-7}$ m) to violet ($\lambda \approx 4 \times 10^{-7}$ m). The sensitivity of the human eye is a function of wavelength, being a maximum at a wavelength of about 5.5×10^{-7} m. With this in mind, why do you suppose tennis balls often have a yellow-green color?

Ultraviolet waves cover wavelengths ranging from approximately 4×10^{-7} m to 6×10^{-10} m. The Sun is an important source of ultraviolet (UV) light, which is the main cause of sunburn. Sunscreen lotions are transparent to visible light but absorb most UV light. The higher a sunscreen's solar protection factor (SPF), the greater the percentage of UV light absorbed. Ultraviolet rays have also been implicated in the formation of cataracts, a clouding of the lens inside the eye. Wearing sunglasses that do not block UV light is worse for your eyes than wearing no sunglasses. The lenses of any sunglasses absorb some visible light, thus causing the wearer's pupils to dilate. If the glasses do not also block UV light, then more damage may be done to the lens of the eye because of the dilated pupils. If you wear no sunglasses at all, your pupils are contracted, you squint, and a lot less UV light enters your eyes. High-quality sunglasses block nearly all the eye-damaging UV light.

X-rays

Most of the UV light from the Sun is absorbed by ozone (O_3) molecules in the Earth's upper atmosphere, in a layer called the stratosphere. This ozone shield converts lethal high-energy UV radiation to infrared radiation, which in turn warms the stratosphere. Recently, a great deal of controversy has arisen concerning the possible depletion of the protective ozone layer as a result of the chemicals emitted from aerosol spray cans and used as refrigerants.

Gamma rays

X-rays have wavelengths in the range from approximately 10^{-8} m to 10^{-12} m. The most common source of x-rays is the deceleration of high-energy electrons bombarding a metal target. X-rays are used as a diagnostic tool in medicine and as a treatment for certain forms of cancer. Because x-rays damage or destroy living tissues and organisms, care must be taken to avoid unnecessary exposure or overexposure. X-rays are also used in the study of crystal structure because x-ray wavelengths are comparable to the atomic separation distances in solids (about 0.1 nm).

EXAMPLE 34.7 A Half-Wave Antenna

A half-wave antenna works on the principle that the optimum length of the antenna is one-half the wavelength of the radiation being received. What is the optimum length of a car antenna when it receives a signal of frequency 94.0 MHz?

Solution Equation 16.14 tells us that the wavelength of the signal is

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{9.40 \times 10^7 \text{ Hz}} = 3.19 \text{ m}$$

Thus, to operate most efficiently, the antenna should have a length of (3.19 m)/2 = 1.60 m. For practical reasons, car antennas are usually one-quarter wavelength in size.

Gamma rays are electromagnetic waves emitted by radioactive nuclei (such as ^{60}Co and ^{137}Cs) and during certain nuclear reactions. High-energy gamma rays are a component of cosmic rays that enter the Earth's atmosphere from space. They have wavelengths ranging from approximately 10^{-10} m to less than 10^{-14} m. They are highly penetrating and produce serious damage when absorbed by living tissues. Consequently, those working near such dangerous radiation must be protected with heavily absorbing materials, such as thick layers of lead.

Quick Quiz 34.3

The *AM* in *AM radio* stands for *amplitude modulation,* and *FM* stands for *frequency modulation.* (The word *modulate* means "to change.") If our eyes could see the electromagnetic waves from a radio antenna, how could you tell an AM wave from an FM wave?

SUMMARY

Electromagnetic waves, which are predicted by Maxwell's equations, have the

following properties:

- The electric field and the magnetic field each satisfy a wave equation. These two wave equations, which can be obtained from Maxwell's third and fourth equations, are

$$\frac{\partial^2 E}{\partial x^2} = \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad \textbf{(34.8)}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0\epsilon_0 \frac{\partial^2 B}{\partial t^2} \qquad \textbf{(34.9)}$$

- The waves travel through a vacuum with the speed of light c, where

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3.00 \times 10^8 \text{ m/s} \qquad \textbf{(34.10)}$$

- The electric and magnetic fields are perpendicular to each other and perpendicular to the direction of wave propagation. (Hence, electromagnetic waves are transverse waves.)
- The instantaneous magnitudes of **E** and **B** in an electromagnetic wave are related by the expression

$$\frac{E}{B} = c \qquad \textbf{(34.13)}$$

- The waves carry energy. The rate of flow of energy crossing a unit area is described by the Poynting vector **S**, where

$$\mathbf{S} \equiv \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \qquad \textbf{(34.18)}$$

- They carry momentum and hence exert pressure on surfaces. If an electromagnetic wave whose Poynting vector is **S** is completely absorbed by a surface upon which it is normally incident, the radiation pressure on that surface is

$$P = \frac{S}{c} \quad \text{(complete absorption)} \qquad \textbf{(34.24)}$$

If the surface totally reflects a normally incident wave, the pressure is doubled.

The electric and magnetic fields of a sinusoidal plane electromagnetic wave propagating in the positive x direction can be written

$$E = E_{\text{max}} \cos(kx - \omega t) \qquad \textbf{(34.11)}$$

$$B = B_{\text{max}} \cos(kx - \omega t) \qquad \textbf{(34.12)}$$

where ω is the angular frequency of the wave and k is the angular wave number. These equations represent special solutions to the wave equations for E and B. Be-

QUESTIONS

1. For a given incident energy of an electromagnetic wave, why is the radiation pressure on a perfectly reflecting surface twice as great as that on a perfectly absorbing surface?
2. Describe the physical significance of the Poynting vector.
3. Do all current-carrying conductors emit electromagnetic waves? Explain.
4. What is the fundamental cause of electromagnetic radiation?
5. Electrical engineers often speak of the radiation resistance of an antenna. What do you suppose they mean by this phrase?
6. If a high-frequency current is passed through a solenoid containing a metallic core, the core warms up by induc-

tion. This process also cooks foods in microwave ovens. Explain why the materials warm up in these situations.

7. Before the advent of cable television and satellite dishes, homeowners either mounted a television antenna on the roof or used "rabbit ears" atop their sets (Fig. Q34.7). Certain orientations of the receiving antenna on a television set gave better reception than others. Furthermore, the best orientation varied from station to station. Explain.

Figure Q34.7 Questions 7, 12, 13, and 14. The V-shaped antenna is the VHF antenna. *(George Semple)*

8. Does a wire connected to the terminals of a battery emit an electromagnetic wave? Explain.

9. If you charge a comb by running it through your hair and then hold the comb next to a bar magnet, do the electric and magnetic fields that are produced constitute an electromagnetic wave?

10. An empty plastic or glass dish is cool to the touch right after it is removed from a microwave oven. How can this be possible? (Assume that your electric bill has been paid.)

11. Often when you touch the indoor antenna on a radio or television receiver, the reception instantly improves. Why?

12. Explain how the (dipole) VHF antenna of a television set works. (See Fig. Q34.7.)

13. Explain how the UHF (loop) antenna of a television set works. (See Fig. Q34.7.)

14. Explain why the voltage induced in a UHF (loop) antenna depends on the frequency of the signal, whereas the voltage in a VHF (dipole) antenna does not. (See Fig. Q34.7.)

15. List as many similarities and differences between sound waves and light waves as you can.

16. What does a radio wave do to the charges in the receiving antenna to provide a signal for your car radio?

17. What determines the height of an AM radio station's broadcast antenna?

18. Some radio transmitters use a "phased array" of antennas. What is their purpose?

19. What happens to the radio reception in an airplane as it flies over the (vertical) dipole antenna of the control tower?

20. When light (or other electromagnetic radiation) travels across a given region, what oscillates?

21. Why should an infrared photograph of a person look different from a photograph of that person taken with visible light?

22. Suppose a creature from another planet had eyes that were sensitive to infrared radiation. Describe what the creature would see if it looked around the room you are now in. That is, what would be bright and what would be dim?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*

WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics

☐ = paired numerical/symbolic problems

Section 34.1 Maxwell's Equations and Hertz's Discoveries

Section 34.2 Plane Electromagnetic Waves

Note: Assume that the medium is vacuum unless specified otherwise.

1. If the North Star, Polaris, were to burn out today, in what year would it disappear from our vision? Take the distance from the Earth to Polaris as 6.44×10^{18} m.

2. The speed of an electromagnetic wave traveling in a transparent nonmagnetic substance is $v = 1/\sqrt{\kappa\mu_0\epsilon_0}$, where κ is the dielectric constant of the substance. Determine the speed of light in water, which has a dielectric constant at optical frequencies of 1.78.

3. An electromagnetic wave in vacuum has an electric field amplitude of 220 V/m. Calculate the amplitude of the corresponding magnetic field.

4. Calculate the maximum value of the magnetic field of an electromagnetic wave in a medium where the speed of light is two thirds of the speed of light in vacuum and where the electric field amplitude is 7.60 mV/m.

WEB 5. Figure 34.3a shows a plane electromagnetic sinusoidal wave propagating in what we choose as the x direction. Suppose that the wavelength is 50.0 m, and the electric field vibrates in the xy plane with an amplitude of 22.0 V/m. Calculate (a) the frequency of the wave and (b) the magnitude and direction of $\mathbf{B}$ when the electric field has its maximum value in the negative y direction. (c) Write an expression for B in the form

$$B = B_{\max} \cos(kx - \omega t)$$

with numerical values for $B_{\max}$, k, and ω.

6. Write down expressions for the electric and magnetic fields of a sinusoidal plane electromagnetic wave having a frequency of 3.00 GHz and traveling in the positive x direction. The amplitude of the electric field is 300 V/m.

7. In SI units, the electric field in an electromagnetic wave is described by

$$E_y = 100 \sin(1.00 \times 10^7 x - \omega t)$$

Find (a) the amplitude of the corresponding magnetic field, (b) the wavelength λ, and (c) the frequency f.

8. Verify by substitution that the following equations are solutions to Equations 34.8 and 34.9, respectively:

$$E = E_{\max} \cos(kx - \omega t)$$

$$B = B_{\max} \cos(kx - \omega t)$$

9. **Review Problem.** A standing-wave interference pattern is set up by radio waves between two metal sheets 2.00 m apart. This is the shortest distance between the plates that will produce a standing-wave pattern. What is the fundamental frequency?

10. A microwave oven is powered by an electron tube called a magnetron, which generates electromagnetic waves of frequency 2.45 GHz. The microwaves enter the oven and are reflected by the walls. The standing-wave pattern produced in the oven can cook food unevenly, with hot spots in the food at antinodes and cool spots at nodes, so a turntable is often used to rotate the food and distribute the energy. If a microwave oven intended for use with a turntable is instead used with a cooking dish in a fixed position, the antinodes can appear as burn marks on foods such as carrot strips or cheese. The separation distance between the burns is measured to be 6 cm ± 5%. From these data, calculate the speed of the microwaves.

Section 34.3 Energy Carried by Electromagnetic Waves

11. How much electromagnetic energy per cubic meter is contained in sunlight, if the intensity of sunlight at the Earth's surface under a fairly clear sky is 1 000 W/m^2?

12. An AM radio station broadcasts isotropically (equally in all directions) with an average power of 4.00 kW. A dipole receiving antenna 65.0 cm long is at a location 4.00 miles from the transmitter. Compute the emf that is induced by this signal between the ends of the receiving antenna.

13. What is the average magnitude of the Poynting vector 5.00 miles from a radio transmitter broadcasting isotropically with an average power of 250 kW?

14. A monochromatic light source emits 100 W of electromagnetic power uniformly in all directions. (a) Calculate the average electric-field energy density 1.00 m from the source. (b) Calculate the average magnetic-field energy density at the same distance from the source. (c) Find the wave intensity at this location.

WEB 15. A community plans to build a facility to convert solar radiation to electric power. They require 1.00 MW of power, and the system to be installed has an efficiency of 30.0% (that is, 30.0% of the solar energy incident on the surface is converted to electrical energy). What must be the effective area of a perfectly absorbing surface used in such an installation, assuming a constant intensity of 1 000 W/m^2?

16. Assuming that the antenna of a 10.0-kW radio station radiates spherical electromagnetic waves, compute the maximum value of the magnetic field 5.00 km from the antenna, and compare this value with the surface magnetic field of the Earth.

WEB 17. The filament of an incandescent lamp has a 150-Ω resistance and carries a direct current of 1.00 A. The filament is 8.00 cm long and 0.900 mm in radius. (a) Calculate the Poynting vector at the surface of the filament. (b) Find the magnitude of the electric and magnetic fields at the surface of the filament.

18. In a region of free space the electric field at an instant of time is $\mathbf{E} = (80.0\mathbf{i} + 32.0\mathbf{j} - 64.0\mathbf{k})$ N/C and the magnetic field is $\mathbf{B} = (0.200\mathbf{i} + 0.080\,0\mathbf{j} + 0.290\mathbf{k})$ μT. (a) Show that the two fields are perpendicular to each other. (b) Determine the Poynting vector for these fields.

19. A lightbulb filament has a resistance of 110 Ω. The bulb is plugged into a standard 120-V (rms) outlet and emits 1.00% of the electric power delivered to it as electromagnetic radiation of frequency f. Assuming that the bulb is covered with a filter that absorbs all other frequencies, find the amplitude of the magnetic field 1.00 m from the bulb.

20. A certain microwave oven contains a magnetron that has an output of 700 W of microwave power for an electrical input power of 1.40 kW. The microwaves are entirely transferred from the magnetron into the oven chamber through a waveguide, which is a metal tube of rectangular cross-section with a width of 6.83 cm and a height of 3.81 cm. (a) What is the efficiency of the magnetron? (b) Assuming that the food is absorbing all the microwaves produced by the magnetron and that no energy is reflected back into the waveguide, find the direction and magnitude of the Poynting vector, averaged over time, in the waveguide near the entrance to the oven chamber. (c) What is the maximum electric field magnitude at this point?

21. High-power lasers in factories are used to cut through cloth and metal (Fig. P34.21). One such laser has a beam diameter of 1.00 mm and generates an electric field with an amplitude of 0.700 MV/m at the target. Find (a) the amplitude of the magnetic field produced, (b) the intensity of the laser, and (c) the power delivered by the laser.

Figure P34.21 A laser cutting device mounted on a robot arm is being used to cut through a metallic plate. *(Philippe Plailly/SPL/Photo Researchers)*

22. At what distance from a 100-W electromagnetic-wave point source does $E_{max} = 15.0$ V/m?

23. A 10.0-mW laser has a beam diameter of 1.60 mm. (a) What is the intensity of the light, assuming it is uniform across the circular beam? (b) What is the average energy density of the beam?

24. At one location on the Earth, the rms value of the magnetic field caused by solar radiation is 1.80 μT. From this value, calculate (a) the average electric field due to solar radiation, (b) the average energy density of the solar component of electromagnetic radiation at this location, and (c) the magnitude of the Poynting vector for the Sun's radiation. (d) Compare the value found in part (c) with the value of the solar intensity given in Example 34.5.

Section 34.4 Momentum and Radiation Pressure

25. A radio wave transmits 25.0 W/m^2 of power per unit area. A flat surface of area A is perpendicular to the direction of propagation of the wave. Calculate the radiation pressure on it if the surface is a perfect absorber.

26. A plane electromagnetic wave of intensity 6.00 W/m^2 strikes a small pocket mirror, of area 40.0 cm^2, held perpendicular to the approaching wave. (a) What momentum does the wave transfer to the mirror each second? (b) Find the force that the wave exerts on the mirror.

27. A possible means of space flight is to place a perfectly reflecting aluminized sheet into orbit around the Earth and then use the light from the Sun to push this "solar sail." Suppose a sail of area 6.00×10^5 m^2 and mass 6 000 kg is placed in orbit facing the Sun. (a) What force is exerted on the sail? (b) What is the sail's acceleration? (c) How long does it take the sail to reach the Moon, 3.84×10^8 m away? Ignore all gravitational effects, assume that the acceleration calculated in part (b) remains constant, and assume a solar intensity of 1 340 W/m^2.

28. A 100-mW laser beam is reflected back upon itself by a mirror. Calculate the force on the mirror.

WEB **29.** A 15.0-mW helium–neon laser ($\lambda = 632.8$ nm) emits a beam of circular cross-section with a diameter of 2.00 mm. (a) Find the maximum electric field in the beam. (b) What total energy is contained in a 1.00-m length of the beam? (c) Find the momentum carried by a 1.00-m length of the beam.

30. Given that the intensity of solar radiation incident on the upper atmosphere of the Earth is 1 340 W/m^2, determine (a) the solar radiation incident on Mars, (b) the total power incident on Mars, and (c) the total force acting on the planet. (d) Compare this force to the gravitational attraction between Mars and the Sun (see Table 14.2).

31. A plane electromagnetic wave has an intensity of 750 W/m^2. A flat rectangular surface of dimensions 50.0 cm × 100 cm is placed perpendicular to the direction of the wave. If the surface absorbs half of the energy and reflects half, calculate (a) the total energy absorbed by the surface in 1.00 min and (b) the momentum absorbed in this time.

(Optional)
Section 34.5 Radiation from an Infinite Current Sheet

32. A large current-carrying sheet emits radiation in each direction (normal to the plane of the sheet) with an intensity of 570 W/m^2. What maximum value of sinusoidal current density is required?

33. A rectangular surface of dimensions 120 cm × 40.0 cm is parallel to and 4.40 m away from a much larger conducting sheet in which a sinusoidally varying surface current exists that has a maximum value of 10.0 A/m. (a) Calculate the average power that is incident on the smaller sheet. (b) What power per unit area is radiated by the larger sheet?

(Optional)
Section 34.6 Production of Electromagnetic Waves by an Antenna

34. Two hand-held radio transceivers with dipole antennas are separated by a great fixed distance. Assuming that the transmitting antenna is vertical, what fraction of the maximum received power will occur in the receiving antenna when it is inclined from the vertical by (a) 15.0°? (b) 45.0°? (c) 90.0°?

35. Two radio-transmitting antennas are separated by half the broadcast wavelength and are driven in phase with each other. In which directions are (a) the strongest and (b) the weakest signals radiated?

36. Figure 34.14 shows a Hertz antenna (also known as a half-wave antenna, since its length is $\lambda/2$). The antenna is far enough from the ground that reflections do not significantly affect its radiation pattern. Most AM radio stations, however, use a Marconi antenna, which consists of the top half of a Hertz antenna. The lower end of this (quarter-wave) antenna is connected to earth ground, and the ground itself serves as the missing lower half. What are the heights of the Marconi antennas for radio stations broadcasting at (a) 560 kHz and (b) 1 600 kHz?

37. Review Problem. Accelerating charges radiate electromagnetic waves. Calculate the wavelength of radiation produced by a proton in a cyclotron with a radius of 0.500 m and a magnetic field with a magnitude of 0.350 T.

38. Review Problem. Accelerating charges radiate electromagnetic waves. Calculate the wavelength of radiation produced by a proton in a cyclotron of radius R and magnetic field B.

Section 34.7 The Spectrum of Electromagnetic Waves

39. (a) Classify waves with frequencies of 2 Hz, 2 kHz, 2 MHz, 2 GHz, 2 THz, 2 PHz, 2 EHz, 2 ZHz, and 2 YHz on the electromagnetic spectrum. (b) Classify waves with wavelengths of 2 km, 2 m, 2 mm, 2 μm, 2 nm, 2 pm, 2 fm, and 2 am.

40. Compute an order-of-magnitude estimate for the frequency of an electromagnetic wave with a wavelength equal to (a) your height; (b) the thickness of this sheet of paper. How is each wave classified on the electromagnetic spectrum?

41. The human eye is most sensitive to light having a wavelength of 5.50×10^{-7} m, which is in the green–yellow region of the visible electromagnetic spectrum. What is the frequency of this light?

42. Suppose you are located 180 m from a radio transmitter. (a) How many wavelengths are you from the transmitter if the station calls itself 1150 AM? (The AM band frequencies are in kilohertz.) (b) What if this station were 98.1 FM? (The FM band frequencies are in megahertz.)

43. What are the wavelengths of electromagnetic waves in free space that have frequencies of (a) 5.00×10^{19} Hz and (b) 4.00×10^{9} Hz?

44. A radar pulse returns to the receiver after a total travel time of 4.00×10^{-4} s. How far away is the object that reflected the wave?

45. *This just in!* An important news announcement is transmitted by radio waves to people sitting next to their radios, 100 km from the station, and by sound waves to people sitting across the newsroom, 3.00 m from the newscaster. Who receives the news first? Explain. Take the speed of sound in air to be 343 m/s.

46. The U.S. Navy has long proposed the construction of extremely low-frequency (ELF) communication systems. Such waves could penetrate the oceans to reach distant submarines. Calculate the length of a quarter-wavelength antenna for a transmitter generating ELF waves with a frequency of 75.0 Hz. How practical is this?

47. What are the wavelength ranges in (a) the AM radio band (540–1 600 kHz), and (b) the FM radio band (88.0–108 MHz)?

48. There are 12 VHF television channels (Channels 2–13) that lie in the range of frequencies between 54.0 MHz and 216 MHz. Each channel is assigned a width of 6.0 MHz, with the two ranges 72.0–76.0 MHz and 88.0–174 MHz reserved for non-TV purposes. (Channel 2, for example, lies between 54.0 and 60.0 MHz.) Calculate the wavelength ranges for (a) Channel 4, (b) Channel 6, and (c) Channel 8.

ADDITIONAL PROBLEMS

49. Assume that the intensity of solar radiation incident on the cloud tops of Earth is 1 340 W/m^2. (a) Calculate the total power radiated by the Sun, taking the average Earth–Sun separation to be 1.496×10^{11} m. (b) Determine the maximum values of the electric and magnetic fields at the Earth's location due to solar radiation.

50. The intensity of solar radiation at the top of the Earth's atmosphere is 1 340 W/m^2. Assuming that 60% of the incoming solar energy reaches the Earth's surface and assuming that you absorb 50% of the incident energy, make an order-of-magnitude estimate of the amount of solar energy you absorb in a 60-min sunbath.

WEB **51. Review Problem.** In the absence of cable input or a satellite dish, a television set can use a dipole-receiving antenna for VHF channels and a loop antenna for UHF channels (see Fig. Q34.7). The UHF antenna produces an emf from the changing magnetic flux through the loop. The TV station broadcasts a signal with a frequency f, and the signal has an electric-field amplitude E_{max} and a magnetic-field amplitude B_{max} at the location of the receiving antenna. (a) Using Faraday's law, derive an expression for the amplitude of the emf that appears in a single-turn circular loop antenna with a radius r, which is small compared to the wavelength of the wave. (b) If the electric field in the signal points vertically, what should be the orientation of the loop for best reception?

52. Consider a small, spherical particle of radius r located in space a distance R from the Sun. (a) Show that the ratio F_{rad}/F_{grav} is proportional to $1/r$, where F_{rad} is the force exerted by solar radiation and F_{grav} is the force of gravitational attraction. (b) The result of part (a) means that, for a sufficiently small value of r, the force exerted on the particle by solar radiation exceeds the force of gravitational attraction. Calculate the value of r for which the particle is in equilibrium under the two forces. (Assume that the particle has a perfectly absorbing surface and a mass density of 1.50 g/cm^3. Let the particle be located 3.75×10^{11} m from the Sun, and use 214 W/m^2 as the value of the solar intensity at that point.)

53. A dish antenna with a diameter of 20.0 m receives (at normal incidence) a radio signal from a distant source, as shown in Figure P34.53. The radio signal is a continuous sinusoidal wave with amplitude E_{max} = 0.200 μV/m. Assume that the antenna absorbs all the radiation that falls on the dish. (a) What is the amplitude of the magnetic field in this wave? (b) What is the intensity of the radiation received by this antenna? (c) What power is received by the antenna? (d) What force is exerted on the antenna by the radio waves?

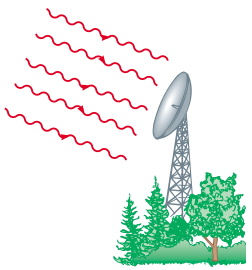


Figure P34.53

54. A parallel-plate capacitor has circular plates of radius r separated by distance ℓ. It has been charged to voltage ΔV and is being discharged as current i is drawn from it. Assume that the plate separation ℓ is very small compared to r, so the electric field is essentially constant in the volume between the plates and is zero outside this volume. Note that the displacement current between the capacitor plates creates a magnetic field. (a) Determine the magnitude and direction of the Poynting vector at the cylindrical surface surrounding the electric field volume. (b) Use the value of the Poynting vector and the lateral surface area of the cylinder to find the total power transfer for the capacitor. (c) What are the changes to these results if the direction of the current is reversed, so the capacitor is charging?

55. A section of a very long air-core solenoid, far from either end, forms an inductor with radius r, length ℓ, and n turns of wire per unit length. At a particular instant, the solenoid current is i and is increasing at the rate di/dt. Ignore the resistance of the wire. (a) Find the magnitude and direction of the Poynting vector over the interior surface of this section of solenoid. (b) Find the rate at which the energy stored in the magnetic field of the inductor is increasing. (c) Express the power in terms of the voltage ΔV across the inductor.

56. A goal of the Russian space program is to illuminate dark northern cities with sunlight reflected to Earth from a 200-m-diameter mirrored surface in orbit. Several smaller prototypes have already been constructed and put into orbit. (a) Assume that sunlight with an intensity of 1 340 W/m^2 falls on the mirror nearly perpendicularly, and that the atmosphere of the Earth allows 74.6% of the energy of sunlight to pass through it in clear weather. What power is received by a city when the space mirror is reflecting light to it? (b) The plan is for the reflected sunlight to cover a circle with a diameter of 8.00 km. What is the intensity of the light (the average magnitude of the Poynting vector) received by the city? (c) This intensity is what percentage of the vertical component of sunlight at Saint Petersburg in January, when the sun reaches an angle of 7.00° above the horizon at noon?

57. In 1965 Arno Penzias and Robert Wilson discovered the cosmic microwave radiation that was left over from the Big Bang expansion of the Universe. Suppose the energy density of this background radiation is equal to 4.00×10^{-14} J/m^3. Determine the corresponding electric-field amplitude.

58. A hand-held cellular telephone operates in the 860- to 900-MHz band and has a power output of 0.600 W from an antenna 10.0 cm long (Fig. P34.58). (a) Find the average magnitude of the Poynting vector 4.00 cm from the antenna, at the location of a typical person's head. Assume that the antenna emits energy with cylindrical wave fronts. (The actual radiation from antennas follows a more complicated pattern, as suggested by Fig. 34.15.) (b) The ANSI/IEEE C95.1-1991 maximum exposure standard is 0.57 mW/cm^2 for persons living near

Figure P34.58. *(©1998 Adam Smith/FPG International)*

cellular telephone base stations, who would be continuously exposed to the radiation. Compare the answer to part (a) with this standard.

59. A linearly polarized microwave with a wavelength of 1.50 cm is directed along the positive x axis. The electric field vector has a maximum value of 175 V/m and vibrates in the xy plane. (a) Assume that the magnetic-field component of the wave can be written in the form $B = B_{max} \sin(kx - \omega t)$, and give values for B_{max}, k, and ω. Also, determine in which plane the magnetic-field vector vibrates. (b) Calculate the magnitude of the Poynting vector for this wave. (c) What maximum radiation pressure would this wave exert if it were directed at normal incidence onto a perfectly reflecting sheet? (d) What maximum acceleration would be imparted to a 500-g sheet (perfectly reflecting and at normal incidence) with dimensions of 1.00 m $\times$ 0.750 m?

60. *Review Section 20.7 on thermal radiation.* (a) An elderly couple have installed a solar water heater on the roof of their house (Fig. P34.60). The solar-energy collector consists of a flat closed box with extraordinarily good thermal insulation. Its interior is painted black, and its front face is made of insulating glass. Assume that its emissivity for visible light is 0.900 and its emissivity for infrared light is 0.700. Assume that the noon Sun shines in perpendicular to the glass, with intensity 1 000 W/m^2, and that no water is then entering or leaving the box. Find the steady-state temperature of the interior of the box. (b) The couple have built an identical box with no water tubes. It lies flat on the ground in front of the house. They use it as a cold frame, where they plant seeds in early spring. If the same noon Sun is at an elevation angle of 50.0°, find the steady-state temperature of the interior of this box, assuming that the ventilation slots are tightly closed.

Figure P34.60 *(©Bill Banaszewski/Visuals Unlimited)*

61. An astronaut, stranded in space 10.0 m from his spacecraft and at rest relative to it, has a mass (including equipment) of 110 kg. Since he has a 100-W light source that forms a directed beam, he decides to use the beam as a photon rocket to propel himself continuously toward the spacecraft. (a) Calculate how long it takes him to reach the spacecraft by this method. (b) Suppose, instead, that he decides to throw the light source away in a direction opposite the spacecraft. If the light source has a mass of 3.00 kg and, after being thrown, moves at 12.0 m/s *relative to the recoiling astronaut*, how long does it take for the astronaut to reach the spacecraft?

62. The Earth reflects approximately 38.0% of the incident sunlight from its clouds and surface. (a) Given that the intensity of solar radiation is 1 340 W/m^2, what is the radiation pressure on the Earth, in pascals, when the Sun is straight overhead? (b) Compare this to normal atmospheric pressure at the Earth's surface, which is 101 kPa.

63. Lasers have been used to suspend spherical glass beads in the Earth's gravitational field. (a) If a bead has a mass of 1.00 μg and a density of 0.200 g/cm^3, determine the radiation intensity needed to support the bead. (b) If the beam has a radius of 0.200 cm, what power is required for this laser?

64. Lasers have been used to suspend spherical glass beads in the Earth's gravitational field. (a) If a bead has a mass m and a density ρ, determine the radiation intensity needed to support the bead. (b) If the beam has a radius r, what power is required for this laser?

65. **Review Problem.** A 1.00-m-diameter mirror focuses the Sun's rays onto an absorbing plate 2.00 cm in radius, which holds a can containing 1.00 L of water at 20.0°C. (a) If the solar intensity is 1.00 kW/m^2, what is the intensity on the absorbing plate? (b) What are the maximum magnitudes of the fields **E** and **B**? (c) If 40.0% of the energy is absorbed, how long would it take to bring the water to its boiling point?

66. A microwave source produces pulses of 20.0-GHz radiation, with each pulse lasting 1.00 ns. A parabolic reflector (R = 6.00 cm) is used to focus these pulses into a parallel beam of radiation, as shown in Figure P34.66. The average power during each pulse is 25.0 kW. (a) What is the wavelength of these microwaves? (b) What is the total energy contained in each pulse? (c) Compute the average energy density inside each pulse. (d) Determine the amplitude of the electric and magnetic fields in these microwaves. (e) Compute the force exerted on the surface during the 1.00-ns duration of each pulse if the pulsed beam strikes an absorbing surface.

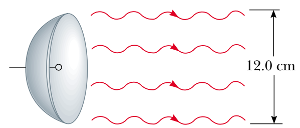


Figure P34.66

67. The electromagnetic power radiated by a nonrelativistic moving point charge q having an acceleration a is

$$\mathcal{P} = \frac{q^2a^2}{6\pi\epsilon_0 c^3}$$

where ϵ_0 is the permittivity of vacuum (free space) and c is the speed of light in vacuum. (a) Show that the right side of this equation is in watts. (b) If an electron is placed in a constant electric field of 100 N/C, determine the acceleration of the electron and the electromagnetic power radiated by this electron. (c) If a proton is placed in a cyclotron with a radius of 0.500 m and a magnetic field of magnitude 0.350 T, what electromagnetic power is radiated by this proton?

68. A thin tungsten filament with a length of 1.00 m radiates 60.0 W of power in the form of electromagnetic waves. A perfectly absorbing surface, in the form of a hollow cylinder with a radius of 5.00 cm and a length of 1.00 m, is placed concentrically with the filament. Calculate the radiation pressure acting on the cylinder. (Assume that the radiation is emitted in the radial direction, and neglect end effects.)

69. The torsion balance shown in Figure 34.8 is used in an experiment to measure radiation pressure. The suspension fiber exerts an elastic restoring torque. Its torque constant is 1.00×10^{-11} N · m/degree, and the length of the horizontal rod is 6.00 cm. The beam from a 3.00-mW helium–neon laser is incident on the black disk, and the mirror disk is completely shielded. Calculate the angle between the equilibrium positions of the horizontal bar when the beam is switched from "off" to "on."

70. **Review Problem.** The study of Creation suggests a Creator with a remarkable liking for beetles and for small red stars. A red star, typical of the most common kind, radiates electromagnetic waves with a power of 6.00×10^{23} W, which is only 0.159% of the luminosity of the Sun. Consider a spherical planet in a circular orbit around this star. Assume that the emissivity of the planet, as defined in Section 20.7, is equal for infrared and visible light. Assume that the planet has a uniform surface temperature. Identify the projected area over which the planet absorbs starlight, and the radiating area of the planet. If beetles thrive at a temperature of 310 K, what should the radius of the planet's orbit be?

71. A "laser cannon" of a spacecraft has a beam of cross-sectional area A. The maximum electric field in the beam is E. At what rate a will an asteroid accelerate away from the spacecraft if the laser beam strikes the asteroid perpendicularly to its surface, and the surface is nonreflecting? The mass of the asteroid is m. Neglect the acceleration of the spacecraft.

72. A plane electromagnetic wave varies sinusoidally at 90.0 MHz as it travels along the $+x$ direction. The peak value of the electric field is 2.00 mV/m, and it is directed along the $\pm y$ direction. (a) Find the wavelength, the period, and the maximum value of the magnetic field. (b) Write expressions in SI units for the space and time variations of the electric field and of the magnetic field. Include numerical values, and include subscripts to indicate coordinate directions. (c) Find the average power per unit area that this wave propagates through space. (d) Find the average energy density in the radiation (in joules per cubic meter). (e) What radiation pressure would this wave exert upon a perfectly reflecting surface at normal incidence?

Answers to Quick Quizzes

34.1 Zero. Figure 34.3b shows that the **B** and **E** vectors reach their maximum and minimum values at the same time.

34.2 (b) Along the y axis because that is the orientation of the electric field. The electric field moves electrons in the antenna, thus inducing a current that is detected and amplified.

34.3 The AM wave, because its amplitude is changing, would appear to vary in brightness. The FM wave would have changing colors because the color we perceive is related to the frequency of the light.

Figure Q35.10

areas of the Earth see a total eclipse, other areas see a partial eclipse, and most areas see no eclipse.

17. The display windows of some department stores are slanted slightly inward at the bottom. This is to decrease the glare from streetlights or the Sun, which would make it difficult for shoppers to see the display inside. Sketch a light ray reflecting off such a window to show how this technique works.

18. When two colors of light X and Y are sent through a glass prism, X is bent more than Y. Which color travels more slowly in the prism?

19. Why does the arc of a rainbow appear with red on top and violet on the bottom?

20. Under what conditions is a mirage formed? On a hot day, what are we seeing when we observe "water on the road"?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 35.1 The Nature of Light

Section 35.2 Measurements of the Speed of Light

1. The Apollo 11 astronauts set up a highly reflecting panel on the Moon's surface. The speed of light can be found by measuring the time it takes a laser beam to travel from Earth, reflect from the retroreflector, and return to Earth. If this interval is measured to be 2.51 s, what is the measured speed of light? Take the center-to-center distance from the Earth to the Moon to be 3.84×10^8 m, and do not neglect the sizes of the Earth and the Moon.

2. As a result of his observations, Roemer concluded that eclipses of Io by Jupiter were delayed by 22 min during a six-month period as the Earth moved from the point in its orbit where it is closest to Jupiter to the diametrically opposite point where it is farthest from Jupiter. Using 1.50×10^8 km as the average radius of the Earth's orbit around the Sun, calculate the speed of light from these data.

3. In an experiment to measure the speed of light using the apparatus of Fizeau (see Fig. 35.2), the distance between light source and mirror was 11.45 km and the wheel had 720 notches. The experimentally determined value of c was 2.998×10^8 m/s. Calculate the minimum angular speed of the wheel for this experiment.

4. Figure P35.4 shows an apparatus used to measure the speed distribution of gas molecules. It consists of two slotted rotating disks separated by a distance d, with the slots displaced by the angle θ. Suppose that the speed of light is measured by sending a light beam from the left through this apparatus. (a) Show that a light beam will be seen in the detector (that is, will make it through both slots) only if its speed is given by $c = \omega d/\theta$, where ω is the angular speed of the disks and θ is measured in radians. (b) What is the measured speed of light if the distance between the two slotted rotating disks is 2.50 m, the slot in the second disk is displaced 1/60 of 1° from the slot in the first disk, and the disks are rotating at 5 555 rev/s?

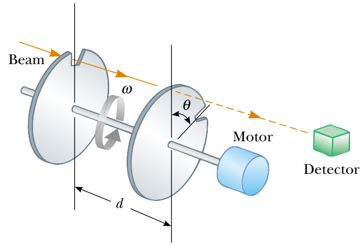


Figure P35.4

Section 35.3 The Ray Approximation in Geometric Optics

Section 35.4 Reflection

Section 35.5 Refraction

Section 35.6 Huygens's Principle

Note: In this section, if an index of refraction value is not given, refer to Table 35.1.

5. A narrow beam of sodium yellow light, with wavelength 589 nm in vacuum, is incident from air onto a smooth water surface at an angle $\theta_1 = 35.0°$. Determine the angle of refraction θ_2 and the wavelength of the light in water.

6. The wavelength of red helium–neon laser light in air is 632.8 nm. (a) What is its frequency? (b) What is its wavelength in glass that has an index of refraction of 1.50? (c) What is its speed in the glass?

7. An underwater scuba diver sees the Sun at an apparent angle of 45.0° from the vertical. What is the actual direction of the Sun?

8. A laser beam is incident at an angle of 30.0° from the vertical onto a solution of corn syrup in water. If the beam is refracted to 19.24° from the vertical, (a) what is the index of refraction of the syrup solution? Suppose that the light is red, with a vacuum wavelength of 632.8 nm. Find its (b) wavelength, (c) frequency, and (d) speed in the solution.

9. Find the speed of light in (a) flint glass, (b) water, and (c) cubic zirconia.

10. A light ray initially in water enters a transparent substance at an angle of incidence of 37.0°, and the transmitted ray is refracted at an angle of 25.0°. Calculate the speed of light in the transparent substance.

WEB **11.** A ray of light strikes a flat block of glass ($n = 1.50$) of thickness 2.00 cm at an angle of 30.0° with the normal. Trace the light beam through the glass, and find the angles of incidence and refraction at each surface.

12. Light of wavelength 436 nm in air enters a fishbowl filled with water and then exits through the crown glass wall of the container. What is the wavelength of the light (a) in the water and (b) in the glass?

13. An opaque cylindrical tank with an open top has a diameter of 3.00 m and is completely filled with water. When the setting Sun reaches an angle of 28.0° above the horizon, sunlight ceases to illuminate any part of the bottom of the tank. How deep is the tank?

14. The angle between the two mirrors illustrated in Figure P35.14 is a right angle. The beam of light in the vertical plane P strikes mirror 1 as shown. (a) Determine the distance that the reflected light beam travels before striking mirror 2. (b) In what direction does the light beam travel after being reflected from mirror 2?

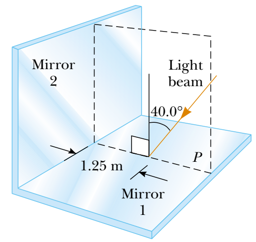


Figure P35.14

15. How many times will the incident beam shown in Figure P35.15 be reflected by each of the parallel mirrors?

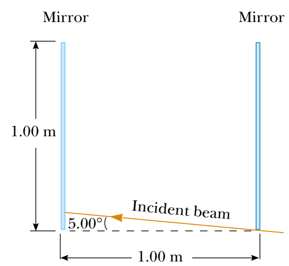


Figure P35.15

16. When the light illustrated in Figure P35.16 passes through the glass block, it is shifted laterally by the distance d. If $n = 1.50$, what is the value of d?

17. Find the time required for the light to pass through the glass block described in Problem 16.

18. The light beam shown in Figure P35.18 makes an angle of 20.0° with the normal line NN' in the linseed oil. Determine the angles θ and θ'. (The index of refraction for linseed oil is 1.48.)

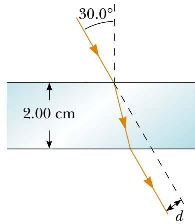


Figure P35.16 Problems 16 and 17.

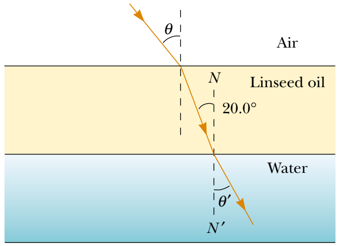


Figure P35.18

19. Two light pulses are emitted simultaneously from a source. Both pulses travel to a detector, but one first passes through 6.20 m of ice. Determine the difference in the pulses' times of arrival at the detector.

20. When you look through a window, by how much time is the light you see delayed by having to go through glass instead of air? Make an order-of-magnitude estimate on the basis of data you specify. By how many wavelengths is it delayed?

21. Light passes from air into flint glass. (a) What angle of incidence must the light have if the component of its velocity perpendicular to the interface is to remain constant? (b) Can the component of velocity parallel to the interface remain constant during refraction?

22. The reflecting surfaces of two intersecting flat mirrors are at an angle of θ ($0° < \theta < 90°$), as shown in Figure

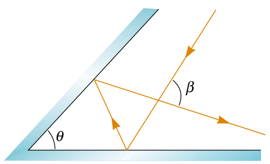


Figure P35.22

P35.22. If a light ray strikes the horizontal mirror, show that the emerging ray will intersect the incident ray at an angle of $\beta = 180° - 2\theta$.

23. A light ray enters the atmosphere of a planet and descends vertically 20.0 km to the surface. The index of refraction where the light enters the atmosphere is 1.000, and it increases linearly to the surface where it has a value of 1.005. (a) How long does it take the ray to traverse this path? (b) Compare this to the time it takes in the absence of an atmosphere.

24. A light ray enters the atmosphere of a planet and descends vertically to the surface a distance h. The index of refraction where the light enters the atmosphere is 1.000, and it increases linearly to the surface where it has a value of n. (a) How long does it take the ray to traverse this path? (b) Compare this to the time it takes in the absence of an atmosphere.

Section 35.7 Dispersion and Prisms

25. A narrow white light beam is incident on a block of fused quartz at an angle of 30.0°. Find the angular width of the light beam inside the quartz.

26. A ray of light strikes the midpoint of one face of an equiangular glass prism ($n = 1.50$) at an angle of incidence of 30.0°. Trace the path of the light ray through the glass, and find the angles of incidence and refraction at each surface.

27. A prism that has an apex angle of 50.0° is made of cubic zirconia, with $n = 2.20$. What is its angle of minimum deviation?

28. Light with a wavelength of 700 nm is incident on the face of a fused quartz prism at an angle of 75.0° (with respect to the normal to the surface). The apex angle of the prism is 60.0°. Using the value of n from Figure 35.20, calculate the angle (a) of refraction at this first surface, (b) of incidence at the second surface, (c) of refraction at the second surface, and (d) between the incident and emerging rays.

WEB **29.** The index of refraction for violet light in silica flint glass is 1.66, and that for red light is 1.62. What is the angular dispersion of visible light passing through a prism of apex angle 60.0° if the angle of incidence is 50.0°? (See Fig. P35.29.)

30. Show that if the apex angle Φ of a prism is small, an approximate value for the angle of minimum deviation is $\delta_{\min} = (n - 1)\Phi$.

31. A triangular glass prism with an apex angle of $\Phi = 60.0°$ has an index of refraction $n = 1.50$ (Fig. P35.31). What is the smallest angle of incidence θ_1 for which a light ray can emerge from the other side?

32. A triangular glass prism with an apex angle of Φ has an index of refraction n (Fig. P35.31). What is the smallest angle of incidence θ_1 for which a light ray can emerge from the other side?

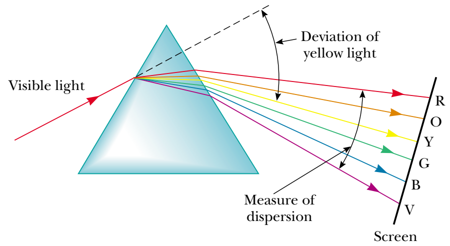


Figure P35.29

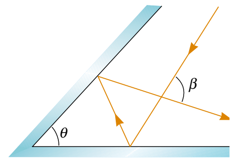


Figure P35.31

33. An experimental apparatus includes a prism made of sodium chloride. The angle of minimum deviation for light of wavelength 589 nm is to be 10.0°. What is the required apex angle of the prism?

34. A triangular glass prism with an apex angle of 60.0° has an index of refraction of 1.50. (a) Show that if its angle of incidence on the first surface is $\theta_1 = 48.6°$, light will pass symmetrically through the prism, as shown in Figure 35.26. (b) Find the angle of deviation $\delta_{\min}$ for $\theta_1 = 48.6°$. (c) Find the angle of deviation if the angle of incidence on the first surface is 45.6°. (d) Find the angle of deviation if $\theta_1 = 51.6°$.

Section 35.8 Total Internal Reflection

35. For 589-nm light, calculate the critical angle for the following materials surrounded by air: (a) diamond, (b) flint glass, and (c) ice.

36. Repeat Problem 35 for the situation in which the materials are surrounded by water.

37. Consider a common mirage formed by super-heated air just above a roadway. A truck driver whose eyes are 2.00 m above the road, where $n = 1.000\,3$, looks forward. She perceives the illusion of a patch of water ahead on the road, where her line of sight makes an angle of 1.20° below the horizontal. Find the index of refraction of the air just above the road surface. (*Hint:* Treat this as a problem in total internal reflection.)

38. Determine the maximum angle θ for which the light

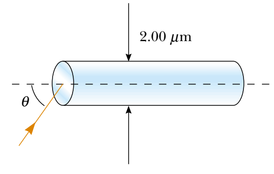


Figure P35.38

rays incident on the end of the pipe shown in Figure P35.38 are subject to total internal reflection along the walls of the pipe. Assume that the pipe has an index of refraction of 1.36 and the outside medium is air.

39. A glass fiber ($n = 1.50$) is submerged in water ($n = 1.33$). What is the critical angle for light to stay inside the optical fiber?

40. A glass cube is placed on a newspaper, which rests on a table. A person reads all of the words the cube covers, through all of one vertical side. Determine the maximum possible index of refraction of the glass.

41. A large Lucite cube ($n = 1.59$) has a small air bubble (a defect in the casting process) below one surface. When a penny (diameter, 1.90 cm) is placed directly over the bubble on the outside of the cube, one cannot see the bubble by looking down into the cube at any angle. However, when a dime (diameter, 1.75 cm) is placed directly over it, one can see the bubble by looking down into the cube. What is the range of the possible depths of the air bubble beneath the surface?

42. A room contains air in which the speed of sound is 343 m/s. The walls of the room are made of concrete, in which the speed of sound is 1 850 m/s. (a) Find the critical angle for total internal reflection of sound at the concrete–air boundary. (b) In which medium must the sound be traveling to undergo total internal reflection? (c) "A bare concrete wall is a highly efficient mirror for sound." Give evidence for or against this statement.

43. In about 1965, engineers at the Toro Company invented a gasoline gauge for small engines, diagrammed in Figure P35.43. The gauge has no moving parts. It consists of a flat slab of transparent plastic fitting vertically into a slot in the cap on the gas tank. None of the plastic has a reflective coating. The plastic projects from the horizontal top down nearly to the bottom of the opaque tank. Its lower edge is cut with facets making angles of 45° with the horizontal. A lawnmower operator looks down from above and sees a boundary between bright and dark on the gauge. The location of the boundary, across the width of the plastic, indicates the quantity of gasoline in the tank. Explain how the gauge works. Explain the design requirements, if any, for the index of refraction of the plastic.

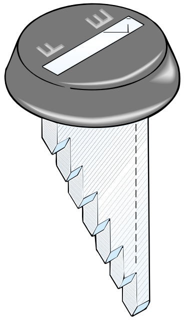


Figure P35.43

(Optional)

Section 35.9 Fermat's Principle

44. The shoreline of a lake runs from east to west. A swimmer gets into trouble 20.0 m out from shore and 26.0 m to the east of a lifeguard, whose station is 16.0 m in from the shoreline. The lifeguard takes a negligible amount of time to accelerate. He can run at 7.00 m/s and swim at 1.40 m/s. To reach the swimmer as quickly as possible, in what direction should the lifeguard start running? You will need to solve a transcendental equation numerically.

ADDITIONAL PROBLEMS

45. A narrow beam of light is incident from air onto a glass surface with an index of refraction of 1.56. Find the angle of incidence for which the corresponding angle of refraction is one half the angle of incidence. (*Hint:* You might want to use the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$.)

46. (a) Consider a horizontal interface between air above and glass with an index of 1.55 below. Draw a light ray incident from the air at an angle of incidence of 30.0°. Determine the angles of the reflected and refracted rays and show them on the diagram. (b) Suppose instead that the light ray is incident from the glass at an angle of incidence of 30.0°. Determine the angles of the reflected and refracted rays and show all three rays on a new diagram. (c) For rays incident from the air onto the air–glass surface, determine and tabulate the angles of reflection and refraction for all the angles of incidence at 10.0° intervals from 0 to 90.0°. (d) Do the same for light rays traveling up to the interface through the glass.

WEB **47.** A small underwater pool light is 1.00 m below the surface. The light emerging from the water forms a circle on the water's surface. What is the diameter of this circle?

48. One technique for measuring the angle of a prism is shown in Figure P35.48. A parallel beam of light is directed on the angle so that the beam reflects from opposite sides. Show that the angular separation of the two beams is given by $B = 2A$.

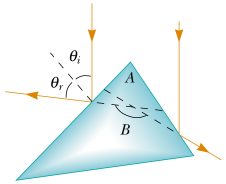


Figure P35.48

49. The walls of a prison cell are perpendicular to the four cardinal compass directions. On the first day of spring, light from the rising Sun enters a rectangular window in the eastern wall. The light traverses 2.37 m horizontally to shine perpendicularly on the wall opposite the window. A young prisoner observes the patch of light moving across this western wall and for the first time forms his own understanding of the rotation of the Earth. (a) With what speed does the illuminated rectangle move? (b) The prisoner holds a small square mirror flat against the wall at one corner of the rectangle of light. The mirror reflects light back to a spot on the eastern wall close beside the window. How fast does the smaller square of light move across that wall? (c) Seen from a latitude of 40.0° north, the rising Sun moves through the sky along a line making a 50.0° angle with the southeastern horizon. In what direction does the rectangular patch of light on the western wall of the prisoner's cell move? (d) In what direction does the smaller square of light on the eastern wall move?

50. The laws of refraction and reflection are the same for sound as for light. The speed of sound in air is 340 m/s, and that of sound in water is 1 510 m/s. If a sound wave approaches a plane water surface at an angle of incidence of 12.0°, what is the angle of refraction?

51. Cold sodium atoms (near absolute zero) in a state called a *Bose–Einstein condensate* can slow the speed of light from its normally high value to a speed approaching that of an automobile in a city. The speed of light in one such medium was recorded as 61.15 km/h. (a) Find the index of refraction of this medium. (b) What is the critical angle for total internal reflection if the condensate is surrounded by vacuum?

52. A narrow beam of white light is incident at 25.0° onto a slab of heavy flint glass 5.00 cm thick. The indices of refraction of the glass at wavelengths of 400 nm and 700 nm are 1.689 and 1.642, respectively. Find the width of the visible beam as it emerges from the slab.

53. A hiker stands on a mountain peak near sunset and observes a rainbow caused by water droplets in the air 8.00 km away. The valley is 2.00 km below the mountain peak and entirely flat. What fraction of the complete circular arc of the rainbow is visible to the hiker? (See Fig. 35.25.)

54. A fish is at a depth d under water. Take the index of refraction of water as 4/3. Show that when the fish is viewed at an angle of refraction θ_1, the apparent depth z of the fish is

$$z = \frac{3d\cos\theta_1}{\sqrt{7 + 9\cos^2\theta_1}}$$

WEB **55.** A laser beam strikes one end of a slab of material, as shown in Figure P35.55. The index of refraction of the slab is 1.48. Determine the number of internal reflections of the beam before it emerges from the opposite end of the slab.

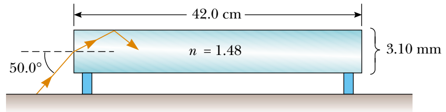


Figure P35.55

56. When light is normally incident on the interface between two transparent optical media, the intensity of the reflected light is given by the expression

$$S_1' = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 S_1$$

In this equation, S_1 represents the average magnitude of the Poynting vector in the incident light (the incident intensity), S_1' is the reflected intensity, and n_1 and n_2 are the refractive indices of the two media. (a) What fraction of the incident intensity is reflected for 589-nm light normally incident on an interface between air and crown glass? (b) In part (a), does it matter whether the light is in the air or in the glass as it strikes the interface? (c) A Bose–Einstein condensate (see Problem 51) has an index of refraction of 1.76×10^7. Find the percent reflection for light falling perpendicularly on its surface. What would the condensate look like?

57. Refer to Problem 56 for a description of the reflected intensity of light normally incident on an interface between two transparent media. (a) When light is normally incident on an interface between vacuum and a transparent medium of index n, show that the intensity S_2 of the transmitted light is given by the expression $S_2/S_1 = 4n/(n+1)^2$. (b) Light travels perpendicularly through a diamond slab, surrounded by air, with parallel surfaces of entry and exit. Apply the transmission fraction in part (a) to find the approximate overall transmission through the slab of diamond as a percentage. Ignore light reflected back and forth within the slab.

58. This problem builds upon the results of Problems 56 and 57. Light travels perpendicularly through a diamond slab, surrounded by air, with parallel surfaces of entry and exit. What fraction of the incident intensity is the intensity of the transmitted light? Include the effects of light reflected back and forth inside the slab.

59. The light beam shown in Figure P35.59 strikes surface 2 at the critical angle. Determine the angle of incidence, θ_1.

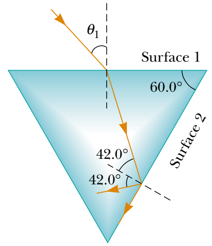


Figure P35.59

60. A 4.00-m-long pole stands vertically in a lake having a depth of 2.00 m. When the Sun is 40.0° above the horizontal, determine the length of the pole's shadow on the bottom of the lake. Take the index of refraction for water to be 1.33.

WEB **61.** A light ray of wavelength 589 nm is incident at an angle θ on the top surface of a block of polystyrene, as shown in Figure P35.61. (a) Find the maximum value of θ for which the refracted ray undergoes total internal reflec-

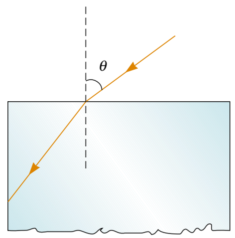


Figure P35.61

tion at the left vertical face of the block. Repeat the calculation for the case in which the polystyrene block is immersed in (b) water and (c) carbon disulfide.

62. A ray of light passes from air into water. For its deviation angle $\delta = |\theta_1 - \theta_2|$ to be 10.0°, what must be its angle of incidence?

63. A shallow glass dish is 4.00 cm wide at the bottom, as shown in Figure P35.63. When an observer's eye is positioned as shown, the observer sees the edge of the bottom of the empty dish. When this dish is filled with water, the observer sees the center of the bottom of the dish. Find the height of the dish.

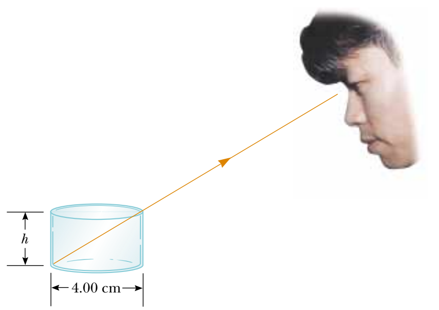


Figure P35.63

64. A material having an index of refraction n is surrounded by a vacuum and is in the shape of a quarter circle of radius R (Fig. P35.64). A light ray parallel to the base of the material is incident from the left at a distance of L above the base and emerges out of the material at the angle θ. Determine an expression for θ.

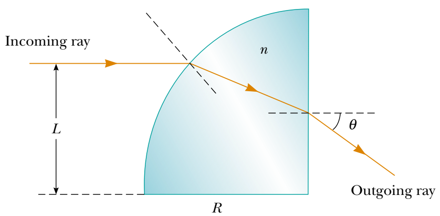


Figure P35.64

65. Derive the law of reflection (Eq. 35.2) from Fermat's principle of least time. (See the procedure outlined in Section 35.9 for the derivation of the law of refraction from Fermat's principle.)

66. A transparent cylinder of radius $R = 2.00$ m has a mirrored surface on its right half, as shown in Figure P35.66. A light ray traveling in air is incident on the left side of the cylinder. The incident light ray and exiting light ray are parallel and $d = 2.00$ m. Determine the index of refraction of the material.

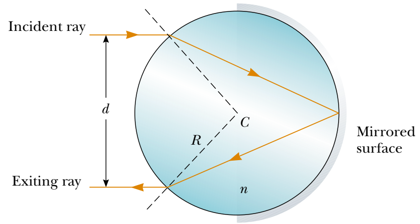


Figure P35.66

67. A. H. Pfund's method for measuring the index of refraction of glass is illustrated in Figure P35.67. One face of a slab of thickness t is painted white, and a small hole scraped clear at point P serves as a source of diverging rays when the slab is illuminated from below. Ray PBB' strikes the clear surface at the critical angle and is totally reflected, as are rays such as PCC'. Rays such as PAA' emerge from the clear surface. On the painted surface there appears a dark circle of diameter d, surrounded by an illuminated region, or halo. (a) Derive a formula for n in terms of the measured quantities d and t. (b) What is the diameter of the dark circle if $n = 1.52$ for a slab 0.600 cm thick? (c) If white light is used, the critical angle depends on color caused by dispersion. Is the inner edge of the white halo tinged with red light or violet light? Explain.

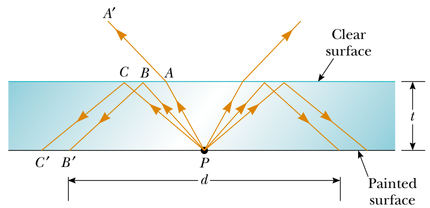


Figure P35.67

68. A light ray traveling in air is incident on one face of a right-angle prism with an index of refraction of $n = 1.50$, as shown in Figure P35.68, and the ray follows the path shown in the figure. If $\theta = 60.0°$ and the base of the prism is mirrored, what is the angle ϕ made by the outgoing ray with the normal to the right face of the prism?

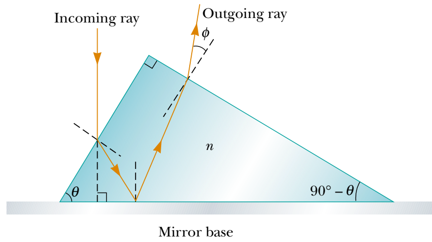


Figure P35.68

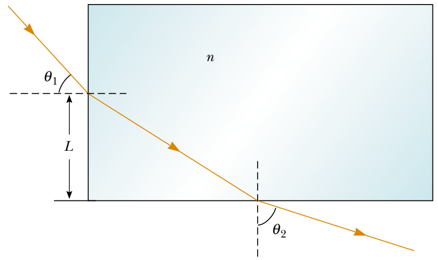


Figure P35.69

69. A light ray enters a rectangular block of plastic at an angle of $\theta_1 = 45.0°$ and emerges at an angle of $\theta_2 = 76.0°$, as shown in Figure P35.69. (a) Determine the index of refraction for the plastic. (b) If the light ray enters the plastic at a point $L = 50.0$ cm from the bottom edge, how long does it take the light ray to travel through the plastic?

70. Students allow a narrow beam of laser light to strike a water surface. They arrange to measure the angle of refraction for selected angles of incidence and record the data shown in the accompanying table. Use the data to verify Snell's law of refraction by plotting the sine of the angle of incidence versus the sine of the angle of refraction. Use the resulting plot to deduce the index of refraction of water.

Angle of Incidence (degrees)	Angle of Refraction (degrees)
10.0	7.5
20.0	15.1
30.0	22.3
40.0	28.7
50.0	35.2
60.0	40.3
70.0	45.3
80.0	47.7

Answers to Quick Quizzes

35.1 Beams ② and ④ are reflected; beams ③ and ⑤ are refracted.

35.2 Fused quartz. An ideal lens would have an index of refraction that does not vary with wavelength so that all colors would be bent through the same angle by the lens. Of the three choices, fused quartz has the least variation in n across the visible spectrum. Thus, it is the best choice for a single-element lens.

35.3 The two rays on the right result from total internal reflection at the right face of the prism. Because all of the light in these rays is reflected (rather than partly refracted), these two rays are brightest. The light from the other three rays is divided into reflected and refracted parts.

PUZZLER

Most car headlights have lines across their faces, like those shown here. Without these lines, the headlights either would not function properly or would be much more likely to break from the jarring of the car on a bumpy road. What is the purpose of the lines? *(George Semple)*

chapter

36

Geometric Optics

Chapter Outline

This chapter is concerned with the images that result when spherical waves fall on flat and spherical surfaces. We find that images can be formed either by reflection or by refraction and that mirrors and lenses work because of reflection and refraction. We continue to use the ray approximation and to assume that light travels in straight lines. Both of these steps lead to valid predictions in the field called *geometric optics.* In subsequent chapters, we shall concern ourselves with interference and diffraction effects—the objects of study in the field of *wave optics.*

36.1 IMAGES FORMED BY FLAT MIRRORS

14.6

We begin by considering the simplest possible mirror, the flat mirror. Consider a point source of light placed at O in Figure 36.1, a distance p in front of a flat mirror. The distance p is called the **object distance.** Light rays leave the source and are reflected from the mirror. Upon reflection, the rays continue to diverge (spread apart), but they appear to the viewer to come from a point I behind the mirror. Point I is called the **image** of the object at O. Regardless of the system under study, we always locate images by extending diverging rays back to a point from which they appear to diverge. **Images are located either at the point from which rays of light *actually* diverge or at the point from which they *appear* to diverge.** Because the rays in Figure 36.1 appear to originate at I, which is a distance q behind the mirror, this is the location of the image. The distance q is called the **image distance.**

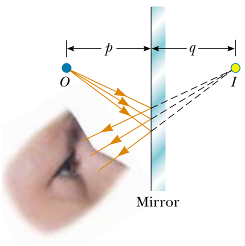


Figure 36.1 An image formed by reflection from a flat mirror. The image point I is located behind the mirror a perpendicular distance q from the mirror (the image distance). Study of Figure 36.2 shows that this image distance has the same magnitude as the object distance p.

Images are classified as real or virtual. **A real image is formed when light rays pass through and diverge from the image point; a virtual image is formed when the light rays do not pass through the image point but appear to diverge from that point.** The image formed by the mirror in Figure 36.1 is virtual. The image of an object seen in a flat mirror is always virtual. Real images can be displayed on a screen (as at a movie), but virtual images cannot be displayed on a screen.

We can use the simple geometric techniques shown in Figure 36.2 to examine the properties of the images formed by flat mirrors. Even though an infinite number of light rays leave each point on the object, we need to follow only two of them to determine where an image is formed. One of those rays starts at P, follows a horizontal path to the mirror, and reflects back on itself. The second ray follows the oblique path PR and reflects as shown, according to the law of reflection. An observer in front of the mirror would trace the two reflected rays back to the point at which they appear to have originated, which is point P' behind the mirror. A continuation of this process for points other than P on the object would result in a virtual image (represented by a yellow arrow) behind the mirror. Because triangles PQR and $P'QR$ are congruent, $PQ = P'Q$. We conclude that **the image formed by an object placed in front of a flat mirror is as far behind the mirror as the object is in front of the mirror.**

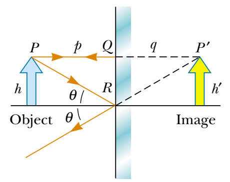


Figure 36.2 A geometric construction that is used to locate the image of an object placed in front of a flat mirror. Because the triangles PQR and $P'QR$ are congruent, $|p| = |q|$ and $h = h'$.

Geometry also reveals that the object height h equals the image height h'. Let us define **lateral magnification** M as follows:

Lateral magnification

$$M \equiv \frac{\text{Image height}}{\text{Object height}} = \frac{h'}{h} \qquad \textbf{(36.1)}$$

Mt. Hood reflected in Trillium Lake. Why is the image inverted and the same size as the mountain?

QuickLab

View yourself in a full-length mirror. Standing close to the mirror, place one piece of tape at the top of the image of your head and another piece at the very bottom of the image of your feet. Now step back a few meters and observe your image. How big is it relative to its original size? How does the distance between the pieces of tape compare with your actual height? You may want to refer to Problem 3.

This is a general definition of the lateral magnification for any type of mirror. For a flat mirror, $M = 1$ because $h' = h$.

Finally, note that a flat mirror produces an image that has an *apparent* left–right reversal. You can see this reversal by standing in front of a mirror and raising your right hand, as shown in Figure 36.3. The image you see raises its left hand. Likewise, your hair appears to be parted on the side opposite your real part, and a mole on your right cheek appears to be on your left cheek.

This reversal is not *actually* a left–right reversal. Imagine, for example, lying on your left side on the floor, with your body parallel to the mirror surface. Now your head is on the left and your feet are on the right. If you shake your feet, the image does not shake its head! If you raise your right hand, however, the image again raises its left hand. Thus, the mirror again appears to produce a left–right reversal but in the up–down direction!

Figure 36.3 The image in the mirror of a person's right hand is reversed front to back. This makes the right hand appear to be a left hand. Notice that the thumb is on the left side of both real hands and on the left side of the image. That the thumb is not on the right side of the image indicates that there is no left-to-right reversal.

The reversal is actually a *front–back reversal,* caused by the light rays going forward toward the mirror and then reflecting back from it. An interesting exercise is to stand in front of a mirror while holding an overhead transparency in front of you so that you can read the writing on the transparency. You will be able to read the writing on the image of the transparency, also. You may have had a similar experience if you have attached a transparent decal with words on it to the rear window of your car. If the decal can be read from outside the car, you can also read it when looking into your rearview mirror from inside the car.

We conclude that the image that is formed by a flat mirror has the following properties.

- The image is as far behind the mirror as the object is in front of the mirror.
- The image is unmagnified, virtual, and upright. (By *upright* we mean that, if the object arrow points upward as in Figure 36.2, so does the image arrow.)
- The image has front–back reversal.

Quick Quiz 36.1

In the overhead view of Figure 36.4, the image of the stone seen by observer 1 is at *C*. Where does observer 2 see the image—at *A*, at *B*, at *C*, at *D*, at *E*, or not at all?

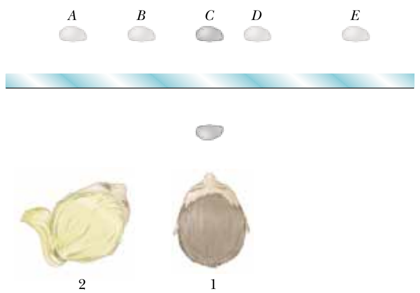


Figure 36.4

Conceptual Example 36.1 Multiple Images Formed by Two Mirrors

Two flat mirrors are at right angles to each other, as illustrated in Figure 36.5, and an object is placed at point *O*. In this situation, multiple images are formed. Locate the positions of these images.

Solution The image of the object is at I_1 in mirror 1 and at I_2 in mirror 2. In addition, a third image is formed at I_3. This third image is the image of I_1 in mirror 2 or, equivalently, the image of I_2 in mirror 1. That is, the image at I_1 (or I_2) serves as the object for I_3. Note that to form this image at I_3, the rays reflect twice after leaving the object at *O*.

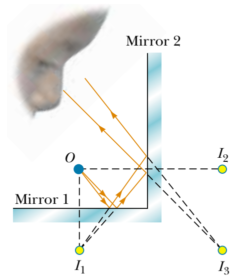


Figure 36.5 When an object is placed in front of two mutually perpendicular mirrors as shown, three images are formed.

Conceptual Example 36.2 The Levitated Professor

The professor in the box shown in Figure 36.6 appears to be balancing himself on a few fingers, with his feet off the floor. He can maintain this position for a long time, and he appears to defy gravity. How was this illusion created?

Solution This is one of many magicians' optical illusions that make use of a mirror. The box in which the professor stands is a cubical frame that contains a flat vertical mirror positioned in a diagonal plane of the frame. The professor straddles the mirror so that one foot, which you see, is in front of the mirror, and one foot, which you cannot see, is behind the mirror. When he raises the foot in front of the mirror, the reflection of that foot also rises, so he appears to float in air.

Figure 36.6 An optical illusion.

Conceptual Example 36.3 The Tilting Rearview Mirror

Most rearview mirrors in cars have a day setting and a night setting. The night setting greatly diminishes the intensity of the image in order that lights from trailing vehicles do not blind the driver. How does such a mirror work?

Solution Figure 36.7 shows a cross-sectional view of a rearview mirror for each setting. The unit consists of a reflective coating on the back of a wedge of glass. In the day setting (Fig. 36.7a), the light from an object behind the car strikes the glass wedge at point 1. Most of the light enters the wedge, refracting as it crosses the front surface, and reflects from the back surface to return to the front surface, where it is refracted again as it re-enters the air as ray *B* (for *bright*). In addition, a small portion of the light is reflected at the front surface of the glass, as indicated by ray *D* (for *dim*).

This dim reflected light is responsible for the image that is observed when the mirror is in the night setting (Fig. 36.7b). In this case, the wedge is rotated so that the path followed by the bright light (ray *B*) does not lead to the eye. Instead, the dim light reflected from the front surface of the wedge travels to the eye, and the brightness of trailing headlights does not become a hazard.

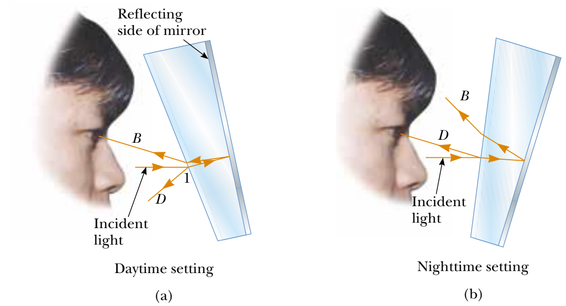


Figure 36.7 Cross-sectional views of a rearview mirror. (a) With the day setting, the silvered back surface of the mirror reflects a bright ray *B* into the driver's eyes. (b) With the night setting, the glass of the unsilvered front surface of the mirror reflects a dim ray *D* into the driver's eyes.

36.2 Images Formed by Spherical Mirrors

Concave Mirrors

14.7 A **spherical mirror,** as its name implies, has the shape of a section of a sphere. This type of mirror focuses incoming parallel rays to a point, as demonstrated by the colored light rays in Figure 36.8. Figure 36.9a shows a cross-section of a spherical mirror, with its surface represented by the solid, curved black line. (The blue band represents the structural support for the mirrored surface, such as a curved piece of glass on which the silvered surface is deposited.) Such a mirror, in which light is reflected from the inner, concave surface, is called a **concave mirror.** The mirror has a radius of curvature *R*, and its center of curvature is point *C*. Point *V* is the center of the spherical section, and a line through *C* and *V* is called the **principal axis** of the mirror.

Now consider a point source of light placed at point *O* in Figure 36.9b, where *O* is any point on the principal axis to the left of *C*. Two diverging rays that originate at *O* are shown. After reflecting from the mirror, these rays converge (come together) at the image point *I*. They then continue to diverge from *I* as if an object were there. As a result, we have at point *I* a real image of the light source at *O*.

We shall consider in this section only rays that diverge from the object and make a small angle with the principal axis. Such rays are called **paraxial rays.** All

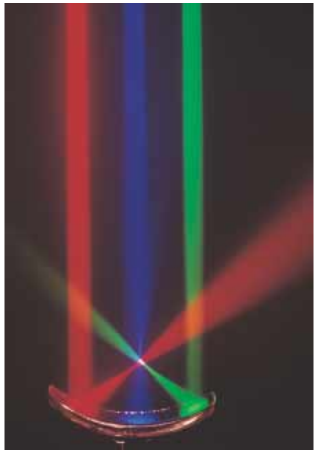


Figure 36.8 Red, blue, and green light rays are reflected by a curved mirror. Note that the point where the three colors meet is white.

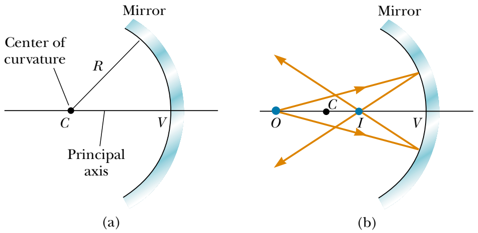


Figure 36.9 (a) A concave mirror of radius R. The center of curvature C is located on the principal axis. (b) A point object placed at O in front of a concave spherical mirror of radius R, where O is any point on the principal axis farther than R from the mirror surface, forms a real image at I. If the rays diverge from O at small angles, they all reflect through the same image point.

such rays reflect through the image point, as shown in Figure 36.9b. Rays that are far from the principal axis, such as those shown in Figure 36.10, converge to other points on the principal axis, producing a blurred image. This effect, which is called **spherical aberration,** is present to some extent for any spherical mirror and is discussed in Section 36.5.

We can use Figure 36.11 to calculate the image distance q from a knowledge of the object distance p and radius of curvature R. By convention, these distances are measured from point V. Figure 36.11 shows two rays leaving the tip of the object. One of these rays passes through the center of curvature C of the mirror, hitting the mirror perpendicular to the mirror surface and reflecting back on itself. The second ray strikes the mirror at its center (point V) and reflects as shown, obeying the law of reflection. The image of the tip of the arrow is located at the point where these two rays intersect. From the gold right triangle in Figure 36.11, we see that $\tan\theta = h/p$, and from the blue right triangle we see that $\tan\theta = -h'/q$. The negative sign is introduced because the image is inverted, so h' is taken to be negative. Thus, from Equation 36.1 and these results, we find that the magnification of the mirror is

$$M = \frac{h'}{h} = -\frac{q}{p} \tag{36.2}$$

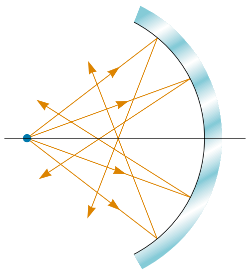


Figure 36.10 Rays diverging from the object at large angles from the principal axis reflect from a spherical concave mirror to intersect the principal axis at different points, resulting in a blurred image. This condition is called *spherical aberration.*

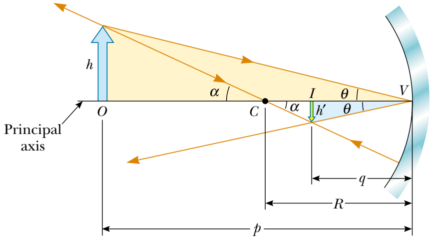


Figure 36.11 The image formed by a spherical concave mirror when the object O lies outside the center of curvature C.

We also note from the two triangles in Figure 36.11 that have α as one angle that

$$\tan\alpha = \frac{h}{p-R} \qquad \text{and} \qquad \tan\alpha = -\frac{h'}{R-q}$$

from which we find that

$$\frac{h'}{h} = -\frac{R-q}{p-R} \tag{36.3}$$

If we compare Equations 36.2 and 36.3, we see that

$$\frac{R-q}{p-R} = \frac{q}{p}$$

Simple algebra reduces this to

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R} \tag{36.4}$$

Mirror equation in terms of R

This expression is called the **mirror equation.** It is applicable only to paraxial rays.

If the object is very far from the mirror—that is, if p is so much greater than R that p can be said to approach infinity—then $1/p \approx 0$, and we see from Equation 36.4 that $q \approx R/2$. That is, when the object is very far from the mirror, the image point is halfway between the center of curvature and the center point on the mirror, as shown in Figure 36.12a. The incoming rays from the object are essentially parallel in this figure because the source is assumed to be very far from the mirror. We call the image point in this special case the **focal point** F and the image distance the **focal length** f, where

$$f = \frac{R}{2} \tag{36.5}$$

Focal length

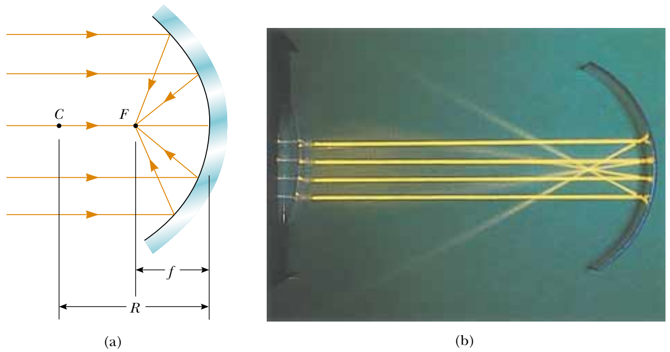


Figure 36.12 (a) Light rays from a distant object ($p \approx \infty$) reflect from a concave mirror through the focal point F. In this case, the image distance $q \approx R/2 = f$, where f is the focal length of the mirror. (b) Reflection of parallel rays from a concave mirror.

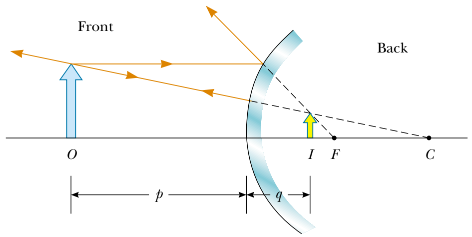


Figure 36.13 Formation of an image by a spherical convex mirror. The image formed by the real object is virtual and upright.

Focal length is a parameter particular to a given mirror and therefore can be used to compare one mirror with another. The mirror equation can be expressed in terms of the focal length:

Mirror equation in terms of f

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \qquad \textbf{(36.6)}$$

Notice that the focal length of a mirror depends only on the curvature of the mirror and not on the material from which the mirror is made. This is because the formation of the image results from rays reflected from the surface of the material. We shall find in Section 36.4 that the situation is different for lenses; in that case the light actually passes through the material.

Convex Mirrors

Figure 36.13 shows the formation of an image by a **convex mirror**—that is, one silvered so that light is reflected from the outer, convex surface. This is sometimes called a **diverging mirror** because the rays from any point on an object diverge after reflection as though they were coming from some point behind the mirror. The image in Figure 36.13 is virtual because the reflected rays only appear to originate at the image point, as indicated by the dashed lines. Furthermore, the image is always upright and smaller than the object. This type of mirror is often used in stores to foil shoplifters. A single mirror can be used to survey a large field of view because it forms a smaller image of the interior of the store.

We do not derive any equations for convex spherical mirrors because we can use Equations 36.2, 36.4, and 36.6 for either concave or convex mirrors if we adhere to the following procedure. Let us refer to the region in which light rays move toward the mirror as the *front side* of the mirror, and the other side as the *back side*. For example, in Figures 36.10 and 36.12, the side to the left of the mirrors is the front side, and the side to the right of the mirrors is the back side. Figure 36.14 states the sign conventions for object and image distances, and Table 36.1 summarizes the sign conventions for all quantities.

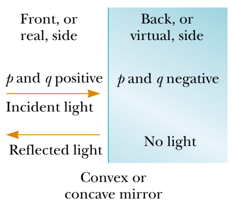


Figure 36.14 Signs of p and q for convex and concave mirrors.

TABLE 36.1 Sign Conventions for Mirrors

p is **positive** if object is in **front** of mirror (real object). p is **negative** if object is in **back** of mirror (virtual object).
q is **positive** if image is in **front** of mirror (real image). q is **negative** if image is in **back** of mirror (virtual image).
Both f and R are **positive** if center of curvature is in **front** of mirror (concave mirror). Both f and R are **negative** if center of curvature is in **back** of mirror (convex mirror).
If M is **positive,** image is **upright.** If M is **negative,** image is **inverted.**

Reflection of parallel lines from a convex cylindrical mirror. The image is virtual, upright, and reduced in size.

Ray Diagrams for Mirrors

The positions and sizes of images formed by mirrors can be conveniently determined with *ray diagrams*. These graphical constructions reveal the nature of the image and can be used to check results calculated from the mirror and magnification equations. To draw a ray diagram, we need to know the position of the object and the locations of the mirror's focal point and center of curvature. We then draw three rays to locate the image, as shown by the examples in Figure 36.15. These rays all start from the same object point and are drawn as follows. We may choose any point on the object; here, we choose the top of the object for simplicity:

- Ray 1 is drawn from the top of the object parallel to the principal axis and is reflected through the focal point F.
- Ray 2 is drawn from the top of the object through the focal point and is reflected parallel to the principal axis.
- Ray 3 is drawn from the top of the object through the center of curvature C and is reflected back on itself.

The intersection of any two of these rays locates the image. The third ray serves as a check of the construction. The image point obtained in this fashion must always agree with the value of q calculated from the mirror equation.

With concave mirrors, note what happens as the object is moved closer to the mirror. The real, inverted image in Figure 36.15a moves to the left as the object approaches the focal point. When the object is at the focal point, the image is infinitely far to the left. However, when the object lies between the focal point and the mirror surface, as shown in Figure 36.15b, the image is virtual, upright, and enlarged. This latter situation applies in the use of a shaving mirror or a makeup mirror. Your face is closer to the mirror than the focal point, and you see an upright, enlarged image of your face.

In a convex mirror (see Fig. 36.15c), the image of an object is always virtual, upright, and reduced in size. In this case, as the object distance increases, the virtual image decreases in size and approaches the focal point as p approaches infinity. You should construct other diagrams to verify how image position varies with object position.

QuickLab

Compare the images formed of your face when you look first at the front side and then at the back side of a shiny soup spoon. Why do the two images look so different from each other?

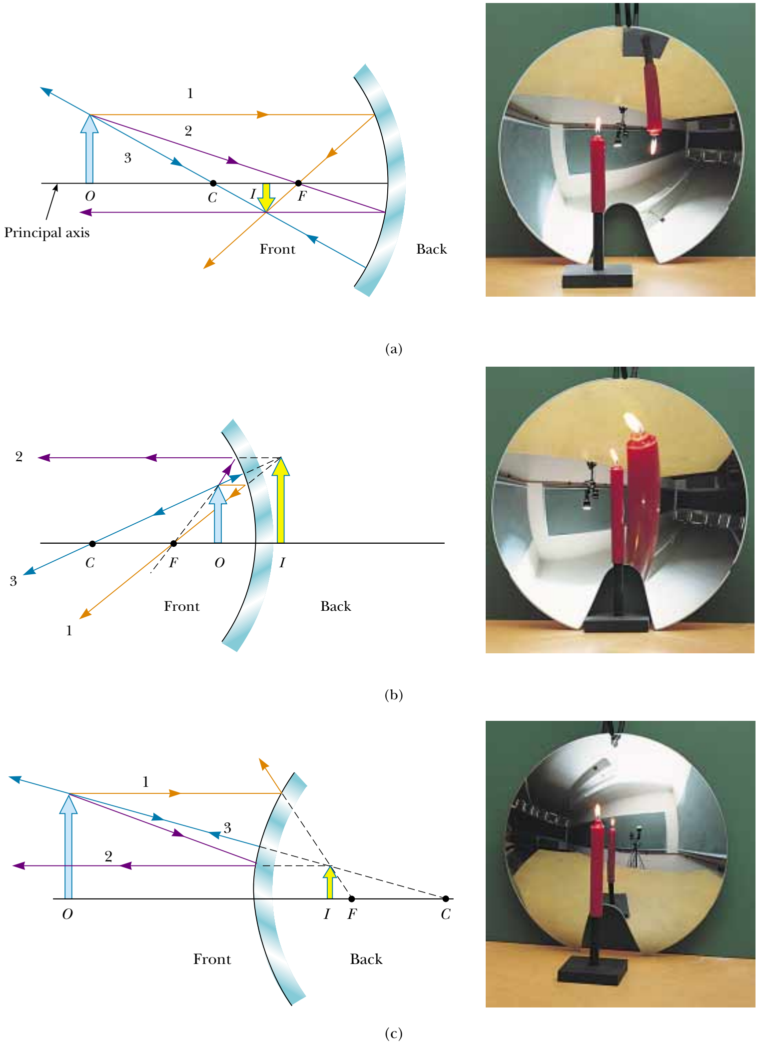


Figure 36.15 Ray diagrams for spherical mirrors, along with corresponding photographs of the images of candles. (a) When the object is located so that the center of curvature lies between the object and a concave mirror surface, the image is real, inverted, and reduced in size. (b) When the object is located between the focal point and a concave mirror surface, the image is virtual, upright, and enlarged. (c) When the object is in front of a convex mirror, the image is virtual, upright, and reduced in size.

EXAMPLE 36.4 The Image from a Mirror

Assume that a certain spherical mirror has a focal length of +10.0 cm. Locate and describe the image for object distances of (a) 25.0 cm, (b) 10.0 cm, and (c) 5.00 cm.

Solution Because the focal length is positive, we know that this is a concave mirror (see Table 36.1). (a) This situation is analogous to that in Figure 36.15a; hence, we expect the image to be real and closer to the mirror than the object. According to the figure, it should also be inverted and reduced in size. We find the image distance by using the Equation 36.6 form of the mirror equation:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\frac{1}{25.0 \text{ cm}} + \frac{1}{q} = \frac{1}{10.0 \text{ cm}}$$

$$q = \boxed{16.7 \text{ cm}}$$

The magnification is given by Equation 36.2:

$$M = -\frac{q}{p} = -\frac{16.7 \text{ cm}}{25.0 \text{ cm}} = -0.668$$

The fact that the absolute value of M is less than unity tells us that the image is smaller than the object, and the negative sign for M tells us that the image is inverted. Because q is positive, the image is located on the front side of the mirror and is real. Thus, we see that our predictions were correct.

(b) When the object distance is 10.0 cm, the object is located at the focal point. Now we find that

$$\frac{1}{10.0 \text{ cm}} + \frac{1}{q} = \frac{1}{10.0 \text{ cm}}$$

$$q = \boxed{\infty}$$

which means that rays originating from an object positioned at the focal point of a mirror are reflected so that the image is formed at an infinite distance from the mirror; that is, the rays travel parallel to one another after reflection. This is the situation in a flashlight, where the bulb filament is placed at the focal point of a reflector, producing a parallel beam of light.

(c) When the object is at $p = 5.00$ cm, it lies between the focal point and the mirror surface, as shown in Figure 36.15b. Thus, we expect a magnified, virtual, upright image. In this case, the mirror equation gives

$$\frac{1}{5.00 \text{ cm}} + \frac{1}{q} = \frac{1}{10.0 \text{ cm}}$$

$$q = \boxed{-10.0 \text{ cm}}$$

The image is virtual because it is located behind the mirror, as expected. The magnification is

$$M = -\frac{q}{p} = -\left(\frac{-10.0 \text{ cm}}{5.00 \text{ cm}}\right) = 2.00$$

The image is twice as large as the object, and the positive sign for M indicates that the image is upright (see Fig. 36.15b).

Exercise At what object distance is the magnification −1.00?

Answer 20.0 cm.

EXAMPLE 36.5 The Image from a Convex Mirror

A woman who is 1.5 m tall is located 3.0 m from an antishoplifting mirror, as shown in Figure 36.16. The focal length of the mirror is −0.25 m. Find (a) the position of her image and (b) the magnification.

Solution (a) This situation is depicted in Figure 36.15c. We should expect to find an upright, reduced, virtual image. To find the image position, we use Equation 36.6:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} = \frac{1}{-0.25 \text{ m}}$$

$$\frac{1}{q} = \frac{1}{-0.25 \text{ m}} - \frac{1}{3.0 \text{ m}}$$

$$q = \boxed{-0.23 \text{ m}}$$

Figure 36.16 Convex mirrors, often used for security in department stores, provide wide-angle viewing.

The negative value of q indicates that her image is virtual, or behind the mirror, as shown in Figure 36.15c.

(b) The magnification is

$$M = -\frac{q}{p} = -\left(\frac{-0.23 \text{ m}}{3.0 \text{ m}}\right) = 0.077$$

The image is much smaller than the woman, and it is upright because M is positive.

Exercise Find the height of the image.

Answer 0.12 m.

36.3 IMAGES FORMED BY REFRACTION

In this section we describe how images are formed when light rays are refracted at the boundary between two transparent materials. Consider two transparent media having indices of refraction n_1 and n_2, where the boundary between the two media is a spherical surface of radius R (Fig. 36.17). We assume that the object at O is in the medium for which the index of refraction is n_1, where $n_1 < n_2$. Let us consider the paraxial rays leaving O. As we shall see, all such rays are refracted at the spherical surface and focus at a single point I, the image point.

Figure 36.18 shows a single ray leaving point O and focusing at point I. Snell's law of refraction applied to this refracted ray gives

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Because θ_1 and θ_2 are assumed to be small, we can use the small-angle approximation $\sin \theta \approx \theta$ (angles in radians) and say that

$$n_1\theta_1 = n_2\theta_2$$

Now we use the fact that an exterior angle of any triangle equals the sum of the two opposite interior angles. Applying this rule to triangles OPC and PIC in Figure 36.18 gives

$$\theta_1 = \alpha + \beta$$

$$\beta = \theta_2 + \gamma$$

If we combine all three expressions and eliminate θ_1 and θ_2, we find that

$$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta \qquad \textbf{(36.7)}$$

Looking at Figure 36.18, we see three right triangles that have a common vertical leg of length d. For paraxial rays (unlike the relatively large-angle ray shown in Fig. 36.18), the horizontal legs of these triangles are approximately p for the triangle containing angle α, R for the triangle containing angle β, and q for the triangle containing angle γ. In the small-angle approximation, $\tan \theta \approx \theta$, so we can write the approximate relationships from these triangles as follows:

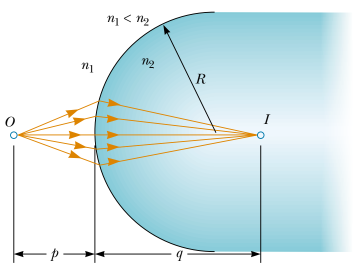


Figure 36.17 An image formed by refraction at a spherical surface. Rays making small angles with the principal axis diverge from a point object at O and are refracted through the image point I.

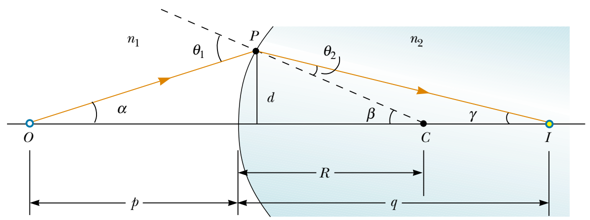


Figure 36.18 Geometry used to derive Equation 36.8.

$$\tan \alpha \approx \alpha \approx \frac{d}{p} \qquad \tan \beta \approx \beta \approx \frac{d}{R} \qquad \tan \gamma \approx \gamma \approx \frac{d}{q}$$

We substitute these expressions into Equation 36.7 and divide through by d to get

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \qquad \textbf{(36.8)}$$

For a fixed object distance p, the image distance q is independent of the angle that the ray makes with the axis. This result tells us that all paraxial rays focus at the same point I.

As with mirrors, we must use a sign convention if we are to apply this equation to a variety of cases. We define the side of the surface in which light rays originate as the front side. The other side is called the back side. Real images are formed by refraction in back of the surface, in contrast with mirrors, where real images are formed in front of the reflecting surface. Because of the difference in location of real images, the refraction sign conventions for q and R are opposite the reflection sign conventions. For example, q and R are both positive in Figure 36.18. The sign conventions for spherical refracting surfaces are summarized in Table 36.2.

We derived Equation 36.8 from an assumption that $n_1 < n_2$. This assumption is not necessary, however. Equation 36.8 is valid regardless of which index of refraction is greater.

TABLE 36.2 Sign Conventions for Refracting Surfaces

p is **positive** if object is in **front** of surface (real object). p is **negative** if object is in **back** of surface (virtual object).
q is **positive** if image is in **back** of surface (real image). q is **negative** if image is in **front** of surface (virtual image).
R is **positive** if center of curvature is in **back** of convex surface. R is **negative** if center of curvature is in **front** of concave surface.

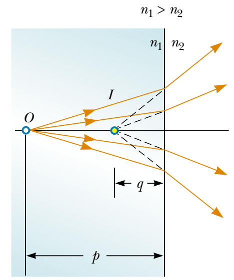


Figure 36.19 The image formed by a flat refracting surface is virtual and on the same side of the surface as the object. All rays are assumed to be paraxial.

Flat Refracting Surfaces

If a refracting surface is flat, then R is infinite and Equation 36.8 reduces to

$$\frac{n_1}{p} = -\frac{n_2}{q}$$

$$q = -\frac{n_2}{n_1}p \qquad \textbf{(36.9)}$$

From this expression we see that the sign of q is opposite that of p. Thus, according to Table 36.2, **the image formed by a flat refracting surface is on the same side of the surface as the object.** This is illustrated in Figure 36.19 for the situation in which the object is in the medium of index n_1 and n_1 is greater than n_2. In this case, a virtual image is formed between the object and the surface. If n_1 is less than n_2, the rays in the back side diverge from each other at lesser angles than those in Figure 36.19. As a result, the virtual image is formed to the left of the object.

CONCEPTUAL EXAMPLE 36.6 Let's Go Scuba Diving!

It is well known that objects viewed under water with the naked eye appear blurred and out of focus. However, a scuba diver using a mask has a clear view of underwater objects. (a) Explain how this works, using the facts that the indices of refraction of the cornea, water, and air are 1.376, 1.333, and 1.000 29, respectively.

Solution Because the cornea and water have almost identical indices of refraction, very little refraction occurs when a person under water views objects with the naked eye. In this case, light rays from an object focus behind the retina, resulting in a blurred image. When a mask is used, the air space between the eye and the mask surface provides the normal amount of refraction at the eye–air interface, and the light from the object is focused on the retina.

(b) If a lens prescription is ground into the glass of a mask, should the curved surface be on the inside of the mask, the outside, or both?

Solution If a lens prescription is ground into the glass of the mask so that the wearer can see without eyeglasses, only the inside surface is curved. In this way the prescription is accurate whether the mask is used under water or in air. If the curvature were on the outer surface, the refraction at the outer surface of the glass would change depending on whether air or water were present on the outside of the mask.

EXAMPLE 36.7 Gaze into the Crystal Ball

A dandelion seed ball 4.0 cm in diameter is embedded in the center of a spherical plastic paperweight having a diameter of 6.0 cm (Fig. 36.20a). The index of refraction of the plastic is $n_1 = 1.50$. Find the position of the image of the near edge of the seed ball.

Solution Because $n_1 > n_2$, where $n_2 = 1.00$ is the index of refraction for air, the rays originating from the seed ball are refracted away from the normal at the surface and diverge outward, as shown in Figure 36.20b. Hence, the image is formed inside the paperweight and is virtual. From the given dimensions, we know that the near edge of the seed ball is 1.0 cm beneath the surface of the paperweight. Applying Equation 36.8 and noting from Table 36.2 that R is negative, we obtain

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}$$

$$\frac{1.50}{1.0\text{ cm}} + \frac{1}{q} = \frac{1.00 - 1.50}{-3.0\text{ cm}}$$

$$q = \boxed{-0.75\text{ cm}}$$

The negative sign for q indicates that the image is in front of the surface—in other words, in the same medium as the object, as shown in Figure 36.20b. Being in the same medium as the object, the image must be virtual (see Table 36.2). The surface of the seed ball appears to be closer to the paperweight surface than it actually is.

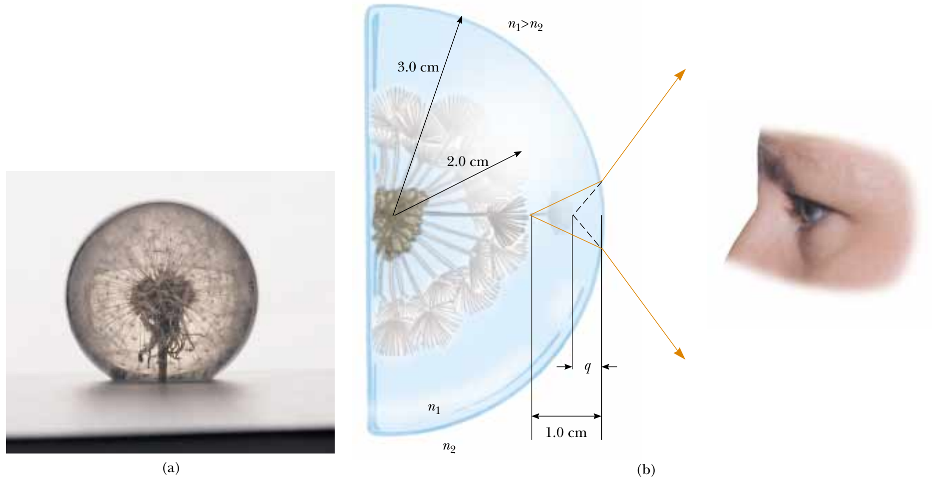


Figure 36.20 (a) An object embedded in a plastic sphere forms a virtual image between the surface of the object and the sphere surface. All rays are assumed paraxial. Because the object is inside the sphere, the front of the refracting surface is the *interior* of the sphere. (b) Rays from the surface of the object form an image that is still inside the plastic sphere but closer to the plastic surface.

EXAMPLE 36.8 The One That Got Away

A small fish is swimming at a depth d below the surface of a pond (Fig. 36.21). What is the apparent depth of the fish, as viewed from directly overhead?

Solution Because the refracting surface is flat, R is infinite. Hence, we can use Equation 36.9 to determine the location of the image with $p = d$. Using the indices of refraction given in Figure 36.21, we obtain

$$q = -\frac{n_2}{n_1}p = -\frac{1.00}{1.33}d = \boxed{-0.752d}$$

Because q is negative, the image is virtual, as indicated by the dashed lines in Figure 36.21. The apparent depth is three-fourths the actual depth.

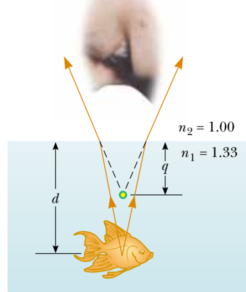


Figure 36.21 The apparent depth q of the fish is less than the true depth d. All rays are assumed to be paraxial.

36.4 THIN LENSES

14.8 Lenses are commonly used to form images by refraction in optical instruments, such as cameras, telescopes, and microscopes. We can use what we just learned about images formed by refracting surfaces to help us locate the image formed by a lens. We recognize that light passing through a lens experiences refraction at two surfaces. The development we shall follow is based on the notion that **the image formed by one refracting surface serves as the object for the second surface.** We shall analyze a thick lens first and then let the thickness of the lens be approximately zero.

Consider a lens having an index of refraction n and two spherical surfaces with radii of curvature R_1 and R_2, as in Figure 36.22. (Note that R_1 is the radius of curvature of the lens surface that the light leaving the object reaches first and that R_2 is the radius of curvature of the other surface of the lens.) An object is placed at point O at a distance p_1 in front of surface 1. If the object were far from surface 1, the light rays from the object that struck the surface would be almost parallel to each other. The refraction at the surface would focus these rays, forming a real image to the right of surface 1 in Figure 36.22 (as in Fig. 36.17). If the object is placed close to surface 1, as shown in Figure 36.22, the rays diverging from the object and striking the surface cover a wide range of angles and are not parallel to each other. In this case, the refraction at the surface is not sufficient to cause the rays to converge on the right side of the surface. They still diverge, although they are closer to parallel than they were before they struck the surface. This results in a virtual image of the object at I_1 to the left of the surface, as shown in Figure 36.22. This image is then used as the object for surface 2, which results in a real image I_2 to the right of the lens.

Let us begin with the virtual image formed by surface 1. Using Equation 36.8 and assuming that $n_1 = 1$ because the lens is surrounded by air, we find that the image I_1 formed by surface 1 satisfies the equation

$$(1) \qquad \frac{1}{p_1} + \frac{n}{q_1} = \frac{n-1}{R_1}$$

where q_1 is a negative number because it represents a virtual image formed on the front side of surface 1.

Now we apply Equation 36.8 to surface 2, taking $n_1 = n$ and $n_2 = 1$. (We make this switch in index because the light rays from I_1 approaching surface 2 are *in the material of the lens,* and this material has index n. We could also imagine removing the object at O, filling all of the space to the left of surface 1 with the material of the lens, and placing the object at I_1; the light rays approaching surface 2 would be the same as in the actual situation in Fig. 36.22.) Taking p_2 as the object distance for surface 2 and q_2 as the image distance gives

$$(2) \qquad \frac{n}{p_2} + \frac{1}{q_2} = \frac{1-n}{R_2}$$

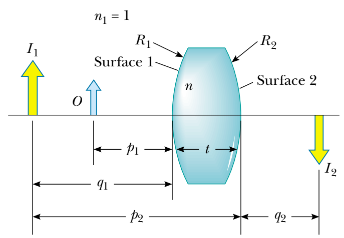


Figure 36.22 To locate the image formed by a lens, we use the virtual image at I_1 formed by surface 1 as the object for the image formed by surface 2. The final image is real and is located at I_2.

We now introduce mathematically the fact that the image formed by the first surface acts as the object for the second surface. We do this by noting from Figure 36.22 that p_2 is the sum of q_1 and t and by setting $p_2 = -q_1 + t$, where t is the thickness of the lens. (Remember that q_1 is a negative number and that p_2 must be positive by our sign convention—thus, we must introduce a negative sign for q_1.) For a *thin* lens (for which the thickness is small compared to the radii of curvature), we can neglect t. In this approximation, we see that $p_2 = -q_1$. Hence, Equation (2) becomes

$$(3) \qquad -\frac{n}{q_1} + \frac{1}{q_2} = \frac{1-n}{R_2}$$

Adding Equations (1) and (3), we find that

$$(4) \qquad \frac{1}{p_1} + \frac{1}{q_2} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

For a thin lens, we can omit the subscripts on p_1 and q_2 in Equation (4) and call the object distance p and the image distance q, as in Figure 36.23. Hence, we can write Equation (4) in the form

$$\frac{1}{p} + \frac{1}{q} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \textbf{(36.10)}$$

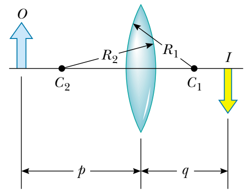


Figure 36.23 Simplified geometry for a thin lens.

This expression relates the image distance q of the image formed by a thin lens to the object distance p and to the thin-lens properties (index of refraction and radii of curvature). It is valid only for paraxial rays and only when the lens thickness is much less than R_1 and R_2.

The **focal length** f of a thin lens is the image distance that corresponds to an infinite object distance, just as with mirrors. Letting p approach ∞ and q approach f in Equation 36.10, we see that the inverse of the focal length for a thin lens is

Lens makers' equation

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \textbf{(36.11)}$$

This relationship is called the **lens makers' equation** because it can be used to determine the values of R_1 and R_2 that are needed for a given index of refraction and a desired focal length f. Conversely, if the index of refraction and the radii of curvature of a lens are given, this equation enables a calculation of the focal length. If the lens is immersed in something other than air, this same equation can be used, with n interpreted as the *ratio* of the index of refraction of the lens material to that of the surrounding fluid.

Quick Quiz 36.2

What is the focal length of a pane of window glass?

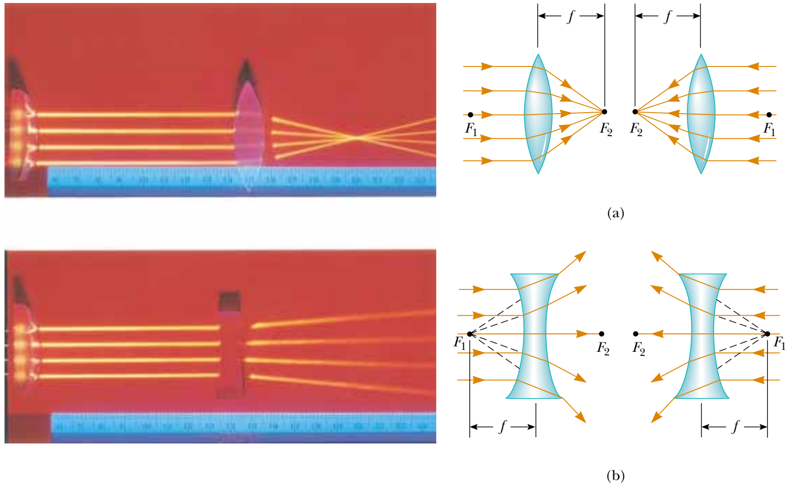


Figure 36.24 (*Left*) Effects of a converging lens (top) and a diverging lens (bottom) on parallel rays. (*Right*) The object and image focal points of (a) a converging lens and (b) a diverging lens.

Using Equation 36.11, we can write Equation 36.10 in a form identical to Equation 36.6 for mirrors:

Thin-lens equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \qquad (36.12)$$

This equation, called the **thin-lens equation,** can be used to relate the image distance and object distance for a thin lens.

Because light can travel in either direction through a lens, each lens has two focal points, one for light rays passing through in one direction and one for rays passing through in the other direction. This is illustrated in Figure 36.24 for a biconvex lens (two convex surfaces, resulting in a converging lens) and a biconcave lens (two concave surfaces, resulting in a diverging lens). Focal point F_1 is sometimes called the *object focal point,* and F_2 is called the *image focal point.*

Figure 36.25 is useful for obtaining the signs of p and q, and Table 36.3 gives the sign conventions for thin lenses. Note that these sign conventions are the same as those for refracting surfaces (see Table 36.2). Applying these rules to a biconvex lens, we see that when $p > f$, the quantities p, q, and R_1 are positive, and R_2 is negative. Therefore, p, q, and f are all positive when a converging lens forms a real image of an object. For a biconcave lens, p and R_2 are positive and q and R_1 are negative, with the result that f is negative.

Various lens shapes are shown in Figure 36.26. Note that a converging lens is thicker at the center than at the edge, whereas a diverging lens is thinner at the center than at the edge.

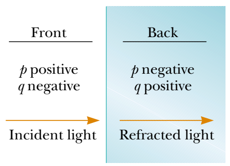


Figure 36.25 A diagram for obtaining the signs of p and q for a thin lens. (This diagram also applies to a refracting surface.)

TABLE 36.3 Sign Conventions for Thin Lenses

p is **positive** if object is in **front** of lens (real object).
p is **negative** if object is in **back** of lens (virtual object).
q is **positive** if image is in **back** of lens (real image).
q is **negative** if image is in **front** of lens (virtual image).
R_1 and R_2 are **positive** if center of curvature is in **back** of lens.
R_1 and R_2 are **negative** if center of curvature is in **front** of lens.
f is **positive** if the lens is **converging.**
f is **negative** if the lens is **diverging.**

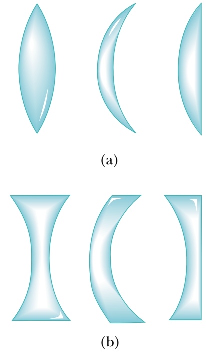


Figure 36.26 Various lens shapes. (a) Biconvex, convex–concave, and plano–convex. These are all converging lenses; they have a positive focal length and are thickest at the middle. (b) Biconcave, convex–concave, and plano–concave. These are all diverging lenses; they have a negative focal length and are thickest at the edges.

Magnification of Images

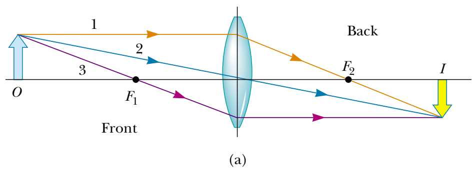
Consider a thin lens through which light rays from an object pass. As with mirrors (Eq. 36.2), the lateral magnification of the lens is defined as the ratio of the image height h' to the object height h:

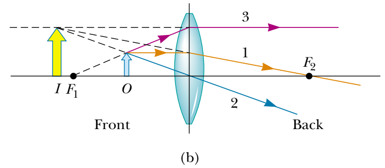
$$M = \frac{h'}{h} = -\frac{q}{p}$$

From this expression, it follows that when M is positive, the image is upright and on the same side of the lens as the object. When M is negative, the image is inverted and on the side of the lens opposite the object.

Ray Diagrams for Thin Lenses

Ray diagrams are convenient for locating the images formed by thin lenses or systems of lenses. They also help clarify our sign conventions. Figure 36.27 shows such diagrams for three single-lens situations. To locate the image of a converg-





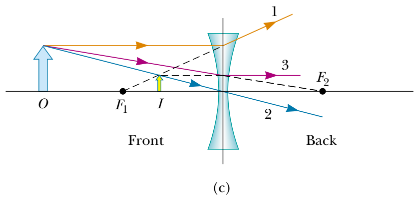


Figure 36.27 Ray diagrams for locating the image formed by a thin lens. (a) When the object is in front of and outside the object focal point F_1 of a converging lens, the image is real, inverted, and on the back side of the lens. (b) When the object is between F_1 and a converging lens, the image is virtual, upright, larger than the object, and on the front side of the lens. (c) When an object is anywhere in front of a diverging lens, the image is virtual, upright, smaller than the object, and on the front side of the lens.

ing lens (Fig. 36.27a and b), the following three rays are drawn from the top of the object:

- Ray 1 is drawn parallel to the principal axis. After being refracted by the lens, this ray passes through the focal point on the back side of the lens.
- Ray 2 is drawn through the center of the lens and continues in a straight line.
- Ray 3 is drawn through that focal point on the front side of the lens (or as if coming from the focal point if $p < f$) and emerges from the lens parallel to the principal axis.

To locate the image of a diverging lens (Fig. 36.27c), the following three rays are drawn from the top of the object:

- Ray 1 is drawn parallel to the principal axis. After being refracted by the lens, this ray emerges such that it appears to have passed through the focal point on the front side of the lens. (This apparent direction is indicated by the dashed line in Fig. 36.27c.)
- Ray 2 is drawn through the center of the lens and continues in a straight line.
- Ray 3 is drawn toward the focal point on the back side of the lens and emerges from the lens parallel to the optic axis.

Quick Quiz 36.3

In Figure 36.27a, the blue object arrow is replaced by one that is much taller than the lens. How many rays from the object will strike the lens?

For the converging lens in Figure 36.27a, where the object is to the left of the object focal point ($p > f_1$), the image is real and inverted. When the object is between the object focal point and the lens ($p < f_1$), as shown in Figure 36.27b, the image is virtual and upright. For a diverging lens (see Fig. 36.27c), the image is always virtual and upright, regardless of where the object is placed. These geometric constructions are reasonably accurate only if the distance between the rays and the principal axis is much less than the radii of the lens surfaces.

It is important to realize that refraction occurs only at the surfaces of the lens. A certain lens design takes advantage of this fact to produce the *Fresnel lens,* a powerful lens without great thickness. Because only the surface curvature is important in the refracting qualities of the lens, material in the middle of a Fresnel lens is removed, as shown in Figure 36.28. Because the edges of the curved segments cause some distortion, Fresnel lenses are usually used only in situations in which image quality is less important than reduction of weight.

The lines that are visible across the faces of most automobile headlights are the edges of curved segments. A headlight requires a short-focal-length lens to collimate light from the nearby filament into a parallel beam. If it were not for the Fresnel design, the glass would be very thick in the center and quite heavy. The weight of the glass would probably cause the thin edge where the lens is supported to break when subjected to the shocks and vibrations that are typical of travel on rough roads.

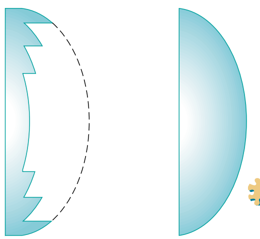


Figure 36.28 The Fresnel lens on the left has the same focal length as the thick lens on the right but is made of much less glass.

Quick Quiz 36.4

If you cover the top half of a lens, which of the following happens to the appearance of the image of an object? (a) The bottom half disappears; (b) the top half disappears; (c) the entire image is visible but has half the intensity; (d) no change occurs; (e) the entire image disappears.

Example 36.9 An Image Formed by a Diverging Lens

A diverging lens has a focal length of -20.0 cm. An object 2.00 cm tall is placed 30.0 cm in front of the lens. Locate the image.

Solution Using the thin-lens equation (Eq. 36.12) with $p = 30.0$ cm and $f = -20.0$ cm, we obtain

$$\frac{1}{30.0\text{ cm}} + \frac{1}{q} = \frac{1}{-20.0\text{ cm}}$$

$$q = \boxed{-12.0\text{ cm}}$$

The negative sign tells us that the image is in front of the lens and virtual, as indicated in Figure 36.27c.

Exercise Determine both the magnification and the height of the image.

Answer $M = 0.400$; $h' = 0.800$ cm.

Example 36.10 An Image Formed by a Converging Lens

A converging lens of focal length 10.0 cm forms an image of each of three objects placed (a) 30.0 cm, (b) 10.0 cm, and (c) 5.00 cm in front of the lens. In each case, find the image distance and describe the image.

Solution (a) The thin-lens equation can be used again:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\frac{1}{30.0\text{ cm}} + \frac{1}{q} = \frac{1}{10.0\text{ cm}}$$

$$q = \boxed{15.0\text{ cm}}$$

The positive sign indicates that the image is in back of the lens and real. The magnification is

$$M = -\frac{q}{p} = -\frac{15.0\text{ cm}}{30.0\text{ cm}} = \boxed{-0.500}$$

The image is reduced in size by one half, and the negative sign for M means that the image is inverted. The situation is like that pictured in Figure 36.27a.

(b) No calculation is necessary for this case because we know that, when the object is placed at the focal point, the image is formed at infinity. We can readily verify this by substituting $p = 10.0$ cm into the thin-lens equation.

(c) We now move inside the focal point, to an object distance of 5.00 cm:

$$\frac{1}{5.00\text{ cm}} + \frac{1}{q} = \frac{1}{10.0\text{ cm}}$$

$$q = \boxed{-10.0\text{ cm}}$$

$$M = -\frac{q}{p} = -\left(\frac{-10.0\text{ cm}}{5.00\text{ cm}}\right) = \boxed{2.00}$$

The negative image distance indicates that the image is in front of the lens and virtual. The image is enlarged, and the positive sign for M tells us that the image is upright, as shown in Figure 36.27b.

Example 36.11 A Lens Under Water

A converging glass lens ($n = 1.52$) has a focal length of 40.0 cm in air. Find its focal length when it is immersed in water, which has an index of refraction of 1.33.

Solution We can use the lens makers' equation (Eq. 36.11) in both cases, noting that R_1 and R_2 remain the same in air and water:

$$\frac{1}{f_{\text{air}}} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_{\text{water}}} = (n'-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where n' is the ratio of the index of refraction of glass to that of water: $n' = 1.52/1.33 = 1.14$. Dividing the first equation by the second gives

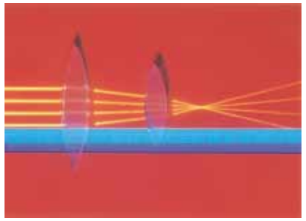
$$\frac{f_{\text{water}}}{f_{\text{air}}} = \frac{n-1}{n'-1} = \frac{1.52-1}{1.14-1} = 3.71$$

Because $f_{\text{air}} = 40.0$ cm, we find that

$$f_{\text{water}} = 3.71 f_{\text{air}} = 3.71(40.0\text{ cm}) = \boxed{148\text{ cm}}$$

The focal length of any glass lens is increased by a factor $(n-1)/(n'-1)$ when the lens is immersed in water.

Combination of Thin Lenses



Light from a distant object brought into focus by two converging lenses.

If two thin lenses are used to form an image, the system can be treated in the following manner. First, the image formed by the first lens is located as if the second lens were not present. Then a ray diagram is drawn for the second lens, with the image formed by the first lens now serving as the object for the second lens. The second image formed is the final image of the system. One configuration is particularly straightforward; that is, if the image formed by the first lens lies on the back side of the second lens, then that image is treated as a **virtual object** for the second lens (that is, p is negative). The same procedure can be extended to a system of three or more lenses. The overall magnification of a system of thin lenses equals the product of the magnifications of the separate lenses.

Let us consider the special case of a system of two lenses in contact. Suppose two thin lenses of focal lengths f_1 and f_2 are placed in contact with each other. If p is the object distance for the combination, application of the thin-lens equation (Eq. 36.12) to the first lens gives

$$\frac{1}{p} + \frac{1}{q_1} = \frac{1}{f_1}$$

where q_1 is the image distance for the first lens. Treating this image as the object for the second lens, we see that the object distance for the second lens must be $-q_1$ (negative because the object is virtual). Therefore, for the second lens,

$$\frac{1}{-q_1} + \frac{1}{q} = \frac{1}{f_2}$$

where q is the final image distance from the second lens. Adding these equations eliminates q_1 and gives

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f_1} + \frac{1}{f_2}$$

Focal length of two thin lenses in contact

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \tag{36.13}$$

Because the two thin lenses are touching, q is also the distance of the final image from the first lens. Therefore, **two thin lenses in contact with each other are equivalent to a single thin lens having a focal length given by Equation 36.13.**

EXAMPLE 36.12 Where Is the Final Image?

Even when the conditions just described do not apply, the lens equations yield image position and magnification. For example, two thin converging lenses of focal lengths $f_1 = 10.0$ cm and $f_2 = 20.0$ cm are separated by 20.0 cm, as illustrated in Figure 36.29. An object is placed 15.0 cm to the left of lens 1. Find the position of the final image and the magnification of the system.

Solution First we locate the image formed by lens 1 while ignoring lens 2:

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{f_1}$$

$$\frac{1}{15.0\text{ cm}} + \frac{1}{q_1} = \frac{1}{10.0\text{ cm}}$$

$$q_1 = 30.0\text{ cm}$$

where q_1 is measured from lens 1. A positive value for q_1 means that this first image is in back of lens 1.

Because q_1 is greater than the separation between the two lenses, this image formed by lens 1 lies 10.0 cm to the right of lens 2. We take this as the object distance for the second lens, so $p_2 = -10.0$ cm, where distances are now measured from lens 2:

$$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2}$$

$$\frac{1}{-10.0\text{ cm}} + \frac{1}{q_2} = \frac{1}{20.0\text{ cm}}$$

$$q_2 = \boxed{6.67\text{ cm}}$$

The final image lies 6.67 cm to the right of lens 2.

The individual magnifications of the images are

$$M_1 = -\frac{q_1}{p_1} = -\frac{30.0\text{ cm}}{15.0\text{ cm}} = -2.00$$

$$M_2 = -\frac{q_2}{p_2} = -\frac{6.67\text{ cm}}{-10.0\text{ cm}} = 0.667$$

The total magnification M is equal to the product $M_1M_2 = (-2.00)(0.667) = \boxed{-1.33.}$ The final image is real because q_2 is positive. The image is also inverted and enlarged.

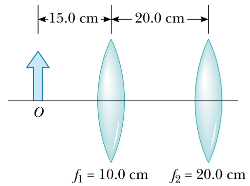


Figure 36.29 A combination of two converging lenses.

CONCEPTUAL EXAMPLE 36.13 Watch Your *p*'s and *q*'s!

Use a spreadsheet or a similar tool to create two graphs of image distance as a function of object distance—one for a lens for which the focal length is 10 cm and one for a lens for which the focal length is -10 cm.

Solution The graphs are shown in Figure 36.30. In each graph a gap occurs where $p = f$, which we shall discuss. Note the similarity in the shapes—a result of the fact that image and object distances for both lenses are related according to the same equation—the thin-lens equation.

The curve in the upper right portion of the $f = +10$ cm graph corresponds to an object on the *front* side of a lens, which we have drawn as the left side of the lens in our previous diagrams. When the object is at positive infinity, a real image forms at the focal point on the back side (the positive side) of the lens, $q = f$. (The incoming rays are parallel in this case.) As the object gets closer to the lens, the image moves farther from the lens, corresponding to the upward path of the curve. This continues until the object is located at the focal point on the near side of the lens. At this point, the rays leaving the lens are parallel, making the image infinitely far away. This is described in the graph by the asymptotic approach of the curve to the line $p = f = 10$ cm.

As the object moves inside the focal point, the image becomes virtual and located near $q = -\infty$. We are now following the curve in the lower left portion of Figure 36.30a. As the object moves closer to the lens, the virtual image also moves closer to the lens. As $p \to 0$, the image distance q also approaches 0. Now imagine that we bring the object to the back side of the lens, where $p < 0$. The object is now a virtual object, so it must have been formed by some other lens. For all locations of the virtual object, the image distance is positive and less than the focal length. The final image is real, and its position approaches the focal point as p gets more and more negative.

The $f = -10$ cm graph shows that a distant real object forms an image at the focal point on the front side of the lens. As the object approaches the lens, the image remains virtual and moves closer to the lens. But as we continue toward the left end of the p axis, the object becomes virtual. As the position of this virtual object approaches the focal point, the image recedes toward infinity. As we pass the focal point, the image shifts from a location at positive infinity to one at negative infinity. Finally, as the virtual object continues moving away from the lens, the image is virtual, starts moving in from negative infinity, and approaches the focal point.

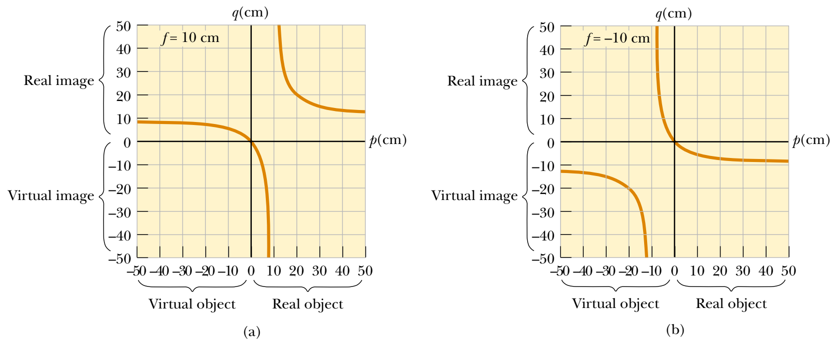


Figure 36.30 (a) Image position as a function of object position for a lens having a focal length of +10 cm. (b) Image position as a function of object position for a lens having a focal length of −10 cm.

Optional Section

36.5 LENS ABERRATIONS

One problem with lenses is imperfect images. The theory of mirrors and lenses that we have been using assumes that rays make small angles with the principal axis and that the lenses are thin. In this simple model, all rays leaving a point source focus at a single point, producing a sharp image. Clearly, this is not always true. When the approximations used in this theory do not hold, imperfect images are formed.

A precise analysis of image formation requires tracing each ray, using Snell's law at each refracting surface and the law of reflection at each reflecting surface. This procedure shows that the rays from a point object do not focus at a single point, with the result that the image is blurred. The departures of actual (imperfect) images from the ideal predicted by theory are called **aberrations.**

Spherical Aberrations

Spherical aberrations occur because the focal points of rays far from the principal axis of a spherical lens (or mirror) are different from the focal points of rays of the same wavelength passing near the axis. Figure 36.31 illustrates spherical aberration for parallel rays passing through a converging lens. Rays passing through points near the center of the lens are imaged farther from the lens than rays passing through points near the edges.

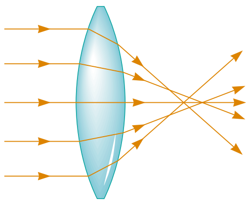
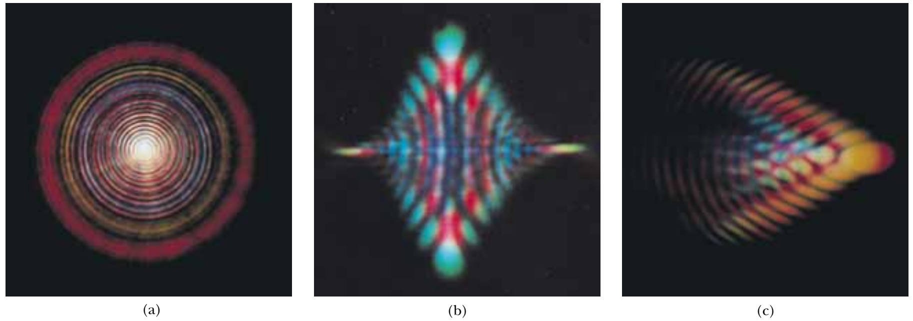


Figure 36.31 Spherical aberration caused by a converging lens. Does a diverging lens cause spherical aberration?

Many cameras have an adjustable aperture to control light intensity and reduce spherical aberration. (An aperture is an opening that controls the amount of light passing through the lens.) Sharper images are produced as the aperture size is reduced because with a small aperture only the central portion of the lens is exposed to the light; as a result, a greater percentage of the rays are paraxial. At the



Lens aberrations. (a) *Spherical aberration* occurs when light passing through the lens at different distances from the principal axis is focused at different points. (b) *Astigmatism* occurs for objects not located on the principal axis of the lens. (c) *Coma* occurs as light passing through the lens far from the principal axis and light passing near the center of the lens focus at different parts of the focal plane.

same time, however, less light passes through the lens. To compensate for this lower light intensity, a longer exposure time is used.

In the case of mirrors used for very distant objects, spherical aberration can be minimized through the use of a parabolic reflecting surface rather than a spherical surface. Parabolic surfaces are not used often, however, because those with high-quality optics are very expensive to make. Parallel light rays incident on a parabolic surface focus at a common point, regardless of their distance from the principal axis. Parabolic reflecting surfaces are used in many astronomical telescopes to enhance image quality.

Chromatic Aberrations

The fact that different wavelengths of light refracted by a lens focus at different points gives rise to chromatic aberrations. In Chapter 35 we described how the index of refraction of a material varies with wavelength. For instance, when white light passes through a lens, violet rays are refracted more than red rays (Fig. 36.32). From this we see that the focal length is greater for red light than for violet light. Other wavelengths (not shown in Fig. 36.32) have focal points intermediate between those of red and violet.

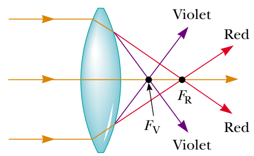


Figure 36.32 Chromatic aberration caused by a converging lens. Rays of different wavelengths focus at different points.

Chromatic aberration for a diverging lens also results in a shorter focal length for violet light than for red light, but on the front side of the lens. Chromatic aberration can be greatly reduced by combining a converging lens made of one type of glass and a diverging lens made of another type of glass.

Optional Section

36.6 THE CAMERA

The photographic **camera** is a simple optical instrument whose essential features are shown in Figure 36.33. It consists of a light-tight box, a converging lens that produces a real image, and a film behind the lens to receive the image. One focuses the camera by varying the distance between lens and film. This is accom-

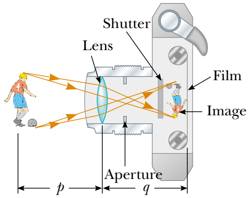


Figure 36.33 Cross-sectional view of a simple camera. Note that in reality, $p \gg q$.

plished with an adjustable bellows in older-style cameras and with some other mechanical arrangement in modern cameras. For proper focusing—which is necessary for the formation of sharp images—the lens-to-film distance depends on the object distance as well as on the focal length of the lens.

The shutter, positioned behind the lens, is a mechanical device that is opened for selected time intervals, called *exposure times*. One can photograph moving objects by using short exposure times, or photograph dark scenes (with low light levels) by using long exposure times. If this adjustment were not available, it would be impossible to take stop-action photographs. For example, a rapidly moving vehicle could move enough in the time that the shutter was open to produce a blurred image. Another major cause of blurred images is the movement of the camera while the shutter is open. To prevent such movement, either short exposure times or a tripod should be used, even for stationary objects. Typical shutter speeds (that is, exposure times) are 1/30, 1/60, 1/125, and 1/250 s. For handheld cameras, the use of slower speeds can result in blurred images (due to movement), but the use of faster speeds reduces the gathered light intensity. In practice, stationary objects are normally shot with an intermediate shutter speed of 1/60 s.

More expensive cameras have an aperture of adjustable diameter to further control the intensity of the light reaching the film. As noted earlier, when an aperture of small diameter is used, only light from the central portion of the lens reaches the film; in this way spherical aberration is reduced.

The intensity I of the light reaching the film is proportional to the area of the lens. Because this area is proportional to the square of the diameter D, we conclude that I is also proportional to D^2. Light intensity is a measure of the rate at which energy is received by the film per unit area of the image. Because the area of the image is proportional to q^2, and $q \approx f$ (when $p \gg f$, so p can be approximated as infinite), we conclude that the intensity is also proportional to $1/f^2$, and thus $I \propto D^2/f^2$. The brightness of the image formed on the film depends on the light intensity, so we see that the image brightness depends on both the focal length and the diameter of the lens.

The ratio f/D is called the ***f*-number** of a lens:

$$f\text{-number} \equiv \frac{f}{D} \qquad \textbf{(36.14)}$$

Hence, the intensity of light incident on the film can be expressed as

$$I \propto \frac{1}{(f/D)^2} \propto \frac{1}{(f\text{-number})^2} \qquad \textbf{(36.15)}$$

The *f*-number is often given as a description of the lens "speed." The lower the *f*-number, the wider the aperture and the higher the rate at which energy from the light exposes the film—thus, a lens with a low *f*-number is a "fast" lens. The conventional notation for an *f*-number is "$f/$" followed by the actual number. For example, "$f/4$" means an *f*-number of 4—it *does not* mean to divide f by 4! Extremely fast lenses, which have *f*-numbers as low as approximately $f/1.2$, are expensive because it is very difficult to keep aberrations acceptably small with light rays passing through a large area of the lens. Camera lens systems (that is, combinations of lenses with adjustable apertures) are often marked with multiple *f*-numbers, usually $f/2.8$, $f/4$, $f/5.6$, $f/8$, $f/11$, and $f/16$. Any one of these settings can be selected by adjusting the aperture, which changes the value of D. Increasing the setting from one *f*-number to the next higher value (for example, from $f/2.8$ to $f/4$) decreases the area of the aperture by a factor of two. The lowest *f*-number setting on a camera lens corresponds to a wide-open aperture and the use of the maximum possible lens area.

Simple cameras usually have a fixed focal length and a fixed aperture size, with an *f*-number of about $f/11$. This high value for the *f*-number allows for a large **depth of field,** meaning that objects at a wide range of distances from the lens form reasonably sharp images on the film. In other words, the camera does not have to be focused.

EXAMPLE 36.14 Finding the Correct Exposure Time

The lens of a certain 35-mm camera (where 35 mm is the width of the film strip) has a focal length of 55 mm and a speed (an *f*-number) of $f/1.8$. The correct exposure time for this speed under certain conditions is known to be (1/500) s. (a) Determine the diameter of the lens.

Solution From Equation 36.14, we find that

$$D = \frac{f}{f\text{-number}} = \frac{55\text{ mm}}{1.8} = \boxed{31\text{ mm}}$$

(b) Calculate the correct exposure time if the *f*-number is changed to $f/4$ under the same lighting conditions.

Solution The total light energy hitting the film is proportional to the product of the intensity and the exposure time. If I is the light intensity reaching the film, then in a time t the energy per unit area received by the film is proportional to It. Comparing the two situations, we require that $I_1t_1 = I_2t_2$, where t_1 is the correct exposure time for $f/1.8$ and t_2 is the correct exposure time for $f/4$. Using this result together with Equation 36.15, we find that

$$\frac{t_1}{(f_1\text{-number})^2} = \frac{t_2}{(f_2\text{-number})^2}$$

$$t_2 = \left(\frac{f_2\text{-number}}{f_1\text{-number}}\right)^2 t_1$$

$$= \left(\frac{4}{1.8}\right)^2\left(\frac{1}{500}\text{ s}\right) \approx \boxed{\frac{1}{100}\text{ s}}$$

As the aperture size is reduced, exposure time must increase.

Optional Section

36.7 THE EYE

Like a camera, a normal eye focuses light and produces a sharp image. However, the mechanisms by which the eye controls the amount of light admitted and adjusts to produce correctly focused images are far more complex, intricate, and effective than those in even the most sophisticated camera. In all respects, the eye is a physiological wonder.

Close-up photograph of the cornea of the human eye.

Figure 36.34 shows the essential parts of the human eye. Light entering the eye passes through a transparent structure called the *cornea*, behind which are a clear liquid (the *aqueous humor*), a variable aperture (the *pupil*, which is an opening in the *iris*), and the *crystalline lens*. Most of the refraction occurs at the outer surface of the eye, where the cornea is covered with a film of tears. Relatively little refraction occurs in the crystalline lens because the aqueous humor in contact with the lens has an average index of refraction close to that of the lens. The iris, which is the colored portion of the eye, is a muscular diaphragm that controls pupil size. The iris regulates the amount of light entering the eye by dilating the pupil in low-light conditions and contracting the pupil in high-light conditions. The *f*-number range of the eye is from about $f/2.8$ to $f/16$.

The cornea–lens system focuses light onto the back surface of the eye, the *retina*, which consists of millions of sensitive receptors called *rods* and *cones*. When stimulated by light, these receptors send impulses via the optic nerve to the brain,

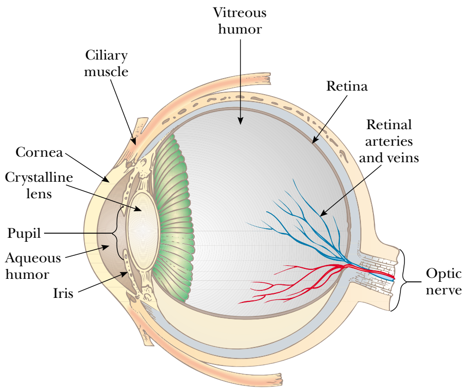


Figure 36.34 Essential parts of the eye.

where an image is perceived. By this process, a distinct image of an object is observed when the image falls on the retina.

The eye focuses on an object by varying the shape of the pliable crystalline lens through an amazing process called **accommodation.** An important component of accommodation is the *ciliary muscle,* which is situated in a circle around the rim of the lens. Thin filaments, called *zonules,* run from this muscle to the edge of the lens. When the eye is focused on a distant object, the ciliary muscle is relaxed, tightening the zonules that attach the muscle to the edge of the lens. The force of the zonules causes the lens to flatten, increasing its focal length. For an object distance of infinity, the focal length of the eye is equal to the fixed distance between lens and retina, about 1.7 cm. The eye focuses on nearby objects by tensing the ciliary muscle, which relaxes the zonules. This action allows the lens to bulge a bit, and its focal length decreases, resulting in the image being focused on the retina. All these lens adjustments take place so swiftly that we are not even aware of the change. In this respect, even the finest electronic camera is a toy compared with the eye.

QuickLab

Move this book toward your face until the letters just begin to blur. The distance from the book to your eyes is your near point.

Accommodation is limited in that objects that are very close to the eye produce blurred images. The **near point** is the closest distance for which the lens can accommodate to focus light on the retina. This distance usually increases with age and has an average value of 25 cm. Typically, at age 10 the near point of the eye is about 18 cm. It increases to about 25 cm at age 20, to 50 cm at age 40, and to 500 cm or greater at age 60. The **far point** of the eye represents the greatest distance for which the lens of the relaxed eye can focus light on the retina. A person with normal vision can see very distant objects, such as the Moon, and thus has a far point near infinity.

Recall that the light leaving the mirror in Figure 36.8 becomes white where it comes together but then diverges into separate colors again. Because nothing but air exists at the point where the rays cross (and hence nothing exists to cause the colors to separate again), seeing white light as a result of a combination of colors must be a visual illusion. In fact, this is the case. Only three types of color-sensitive

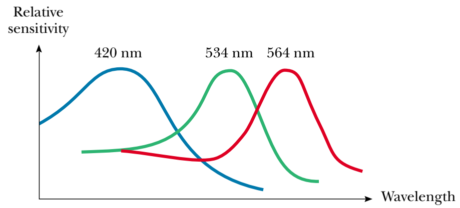


Figure 36.35 Approximate color sensitivity of the three types of cones in the retina.

cells are present in the retina; they are called red, green, and blue cones because of the peaks of the color ranges to which they respond (Fig. 36.35). If the red and green cones are stimulated simultaneously (as would be the case if yellow light were shining on them), the brain interprets what we see as yellow. If all three types of cones are stimulated by the separate colors red, blue, and green, as in Figure 36.8, we see white. If all three types of cones are stimulated by light that contains *all* colors, such as sunlight, we again see white light.

QuickLab

Pour a pile of salt or sugar into your palm. Compare its white appearance with the transparency of a single grain.

Color televisions take advantage of this visual illusion by having only red, green, and blue dots on the screen. With specific combinations of brightness in these three primary colors, our eyes can be made to see any color in the rainbow. Thus, the yellow lemon you see in a television commercial is not really yellow, it is red and green! The paper on which this page is printed is made of tiny, matted, translucent fibers that scatter light in all directions; the resultant mixture of colors appears white to the eye. Snow, clouds, and white hair are not really white. In fact, there is no such thing as a white pigment. The appearance of these things is a consequence of the scattering of light containing all colors, which we interpret as white.

Conditions of the Eye

When the eye suffers a mismatch between the focusing range of the lens–cornea system and the length of the eye, with the result that light rays reach the retina before they converge to form an image, as shown in Figure 36.36a, the condition is known as **farsightedness** (or *hyperopia*). A farsighted person can usually see faraway objects clearly but not nearby objects. Although the near point of a normal eye is approximately 25 cm, the near point of a farsighted person is much farther away. The eye of a farsighted person tries to focus by accommodation—that is, by shortening its focal length. This works for distant objects, but because the focal length of the farsighted eye is greater than normal, the light from nearby objects cannot be brought to a sharp focus before it reaches the retina, and it thus causes a blurred image. The refracting power in the cornea and lens is insufficient to focus the light from all but distant objects satisfactorily. The condition can be corrected by placing a converging lens in front of the eye, as shown in Figure 36.36b. The lens refracts the incoming rays more toward the principal axis before entering the eye, allowing them to converge and focus on the retina.

A person with **nearsightedness** (or *myopia*), another mismatch condition, can focus on nearby objects but not on faraway objects. In the case of *axial myopia,* the nearsightedness is caused by the lens being too far from the retina. In *refractive my-*

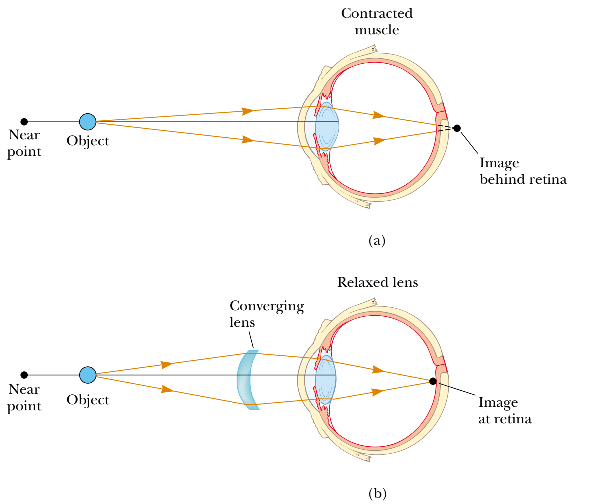


Figure 36.36 (a) When a farsighted eye looks at an object located between the near point and the eye, the image point is behind the retina, resulting in blurred vision. The eye muscle contracts to try to bring the object into focus. (b) Farsightedness is corrected with a converging lens.

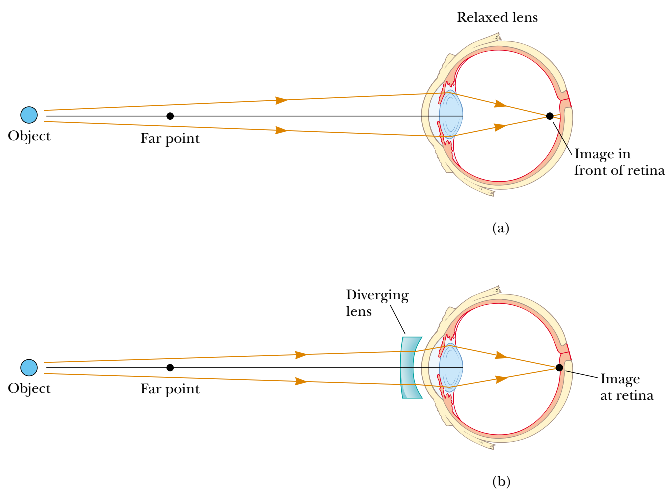


Figure 36.37 (a) When a nearsighted eye looks at an object that lies beyond the eye's far point, the image is formed in front of the retina, resulting in blurred vision. (b) Nearsightedness is corrected with a diverging lens.

opia, the lens–cornea system is too powerful for the length of the eye. The far point of the nearsighted eye is not infinity and may be less than 1 m. The maximum focal length of the nearsighted eye is insufficient to produce a sharp image on the retina, and rays from a distant object converge to a focus in front of the retina. They then continue past that point, diverging before they finally reach the retina and causing blurred vision (Fig. 36.37a). Nearsightedness can be corrected with a diverging lens, as shown in Figure 36.37b. The lens refracts the rays away from the principal axis before they enter the eye, allowing them to focus on the retina.

Quick Quiz 36.5

Which glasses in Figure 36.38 correct nearsightedness and which correct farsightedness?

(a) (b)

Figure 36.38

Beginning in middle age, most people lose some of their accommodation ability as the ciliary muscle weakens and the lens hardens. Unlike farsightedness, which is a mismatch between focusing power and eye length, **presbyopia** (literally, "old-age vision") is due to a reduction in accommodation ability. The cornea and lens do not have sufficient focusing power to bring nearby objects into focus on the retina. The symptoms are the same as those of farsightedness, and the condition can be corrected with converging lenses.

In the eye defect known as **astigmatism,** light from a point source produces a line image on the retina. This condition arises when either the cornea or the lens or both are not perfectly symmetric. Astigmatism can be corrected with lenses that have different curvatures in two mutually perpendicular directions.

Optometrists and ophthalmologists usually prescribe lenses[1] measured in **diopters:**

> The **power** P of a lens in diopters equals the inverse of the focal length in meters: $P = 1/f$.

For example, a converging lens of focal length $+20$ cm has a power of $+5.0$ diopters, and a diverging lens of focal length -40 cm has a power of -2.5 diopters.

EXAMPLE 36.15 A Case of Nearsightedness

A particular nearsighted person is unable to see objects clearly when they are beyond 2.5 m away (the far point of this particular eye). What should the focal length be in a lens prescribed to correct this problem?

Solution The purpose of the lens in this instance is to "move" an object from infinity to a distance where it can be seen clearly. This is accomplished by having the lens produce an image at the far point. From the thin-lens equation, we have

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\infty} + \frac{1}{-2.5\text{ m}} = \frac{1}{f}$$

$$f = \boxed{-2.5\text{ m}}$$

Why did we use a negative sign for the image distance? As you should have suspected, the lens must be a diverging lens (one with a negative focal length) to correct nearsightedness.

Exercise What is the power of this lens?

Answer -0.40 diopter.

Optional Section

36.8 THE SIMPLE MAGNIFIER

The simple magnifier consists of a single converging lens. As the name implies, this device increases the apparent size of an object.

Suppose an object is viewed at some distance p from the eye, as illustrated in Figure 36.39. The size of the image formed at the retina depends on the angle θ subtended by the object at the eye. As the object moves closer to the eye, θ increases and a larger image is observed. However, an average normal eye cannot focus on an object closer than about 25 cm, the near point (Fig. 36.40a). Therefore, θ is maximum at the near point.

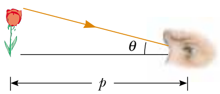


Figure 36.39 The size of the image formed on the retina depends on the angle θ subtended at the eye.

To further increase the apparent angular size of an object, a converging lens can be placed in front of the eye as in Figure 36.40b, with the object located at point O, just inside the focal point of the lens. At this location, the lens forms a virtual, upright, enlarged image. We define **angular magnification** m as the ratio of the angle subtended by an object with a lens in use (angle θ in Fig. 36.40b) to the angle subtended by the object placed at the near point with no lens in use (angle

[1] The word *lens* comes from *lentil,* the name of an Italian legume. (You may have eaten lentil soup.) Early eyeglasses were called "glass lentils" because the biconvex shape of their lenses resembled the shape of a lentil. The first lenses for farsightedness and presbyopia appeared around 1280; concave eyeglasses for correcting nearsightedness did not appear for more than 100 years after that.

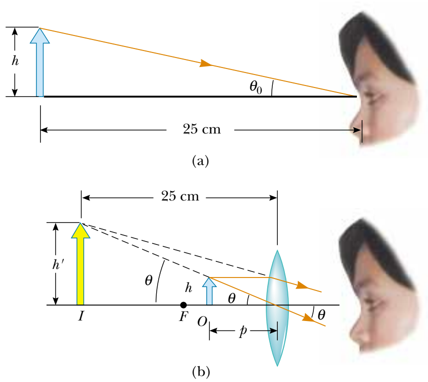


Figure 36.40 (a) An object placed at the near point of the eye ($p = 25$ cm) subtends an angle $\theta_0 \approx h/25$ at the eye. (b) An object placed near the focal point of a converging lens produces a magnified image that subtends an angle $\theta \approx h'/25$ at the eye.

θ_0 in Fig. 36.40a):

$$m \equiv \frac{\theta}{\theta_0} \tag{36.16}$$

Angular magnification with the object at the near point

The angular magnification is a maximum when the image is at the near point of the eye—that is, when $q = -25$ cm. The object distance corresponding to this image distance can be calculated from the thin-lens equation:

$$\frac{1}{p} + \frac{1}{-25\text{ cm}} = \frac{1}{f}$$

$$p = \frac{25f}{25 + f}$$

where f is the focal length of the magnifier in centimeters. If we make the small-angle approximations

$$\tan\theta_0 \approx \theta_0 \approx \frac{h}{25} \quad \text{and} \quad \tan\theta \approx \theta \approx \frac{h}{p} \tag{36.17}$$

Equation 36.16 becomes

$$m_{\text{max}} = \frac{\theta}{\theta_0} = \frac{h/p}{h/25} = \frac{25}{p} = \frac{25}{25f/(25+f)}$$

$$m_{\text{max}} = 1 + \frac{25\text{ cm}}{f} \tag{36.18}$$

Although the eye can focus on an image formed anywhere between the near point and infinity, it is most relaxed when the image is at infinity. For the image formed by the magnifying lens to appear at infinity, the object has to be at the focal point of the lens. In this case, Equations 36.17 become

$$\theta_0 \approx \frac{h}{25} \quad \text{and} \quad \theta \approx \frac{h}{f}$$

and the magnification is

$$m_{\text{min}} = \frac{\theta}{\theta_0} = \frac{25 \text{ cm}}{f} \qquad \textbf{(36.19)}$$

With a single lens, it is possible to obtain angular magnifications up to about 4 without serious aberrations. Magnifications up to about 20 can be achieved by using one or two additional lenses to correct for aberrations.

EXAMPLE 36.16 Maximum Magnification of a Lens

What is the maximum magnification that is possible with a lens having a focal length of 10 cm, and what is the magnification of this lens when the eye is relaxed?

Solution The maximum magnification occurs when the image is located at the near point of the eye. Under these circumstances, Equation 36.18 gives

$$m_{\text{max}} = 1 + \frac{25 \text{ cm}}{f} = 1 + \frac{25 \text{ cm}}{10 \text{ cm}} = 3.5$$

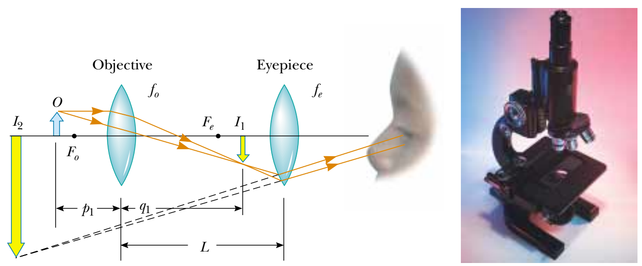
When the eye is relaxed, the image is at infinity. In this case, we use Equation 36.19:

$$m_{\text{min}} = \frac{25 \text{ cm}}{f} = \frac{25 \text{ cm}}{10 \text{ cm}} = 2.5$$

Optional Section

36.9 THE COMPOUND MICROSCOPE

A simple magnifier provides only limited assistance in inspecting minute details of an object. Greater magnification can be achieved by combining two lenses in a device called a **compound microscope,** a schematic diagram of which is shown in Figure 36.41a. It consists of one lens, the *objective,* that has a very short focal length



(a) (b)

Figure 36.41 (a) Diagram of a compound microscope, which consists of an objective lens and an eyepiece lens. (b) A compound microscope. The three-objective turret allows the user to choose from several powers of magnification. Combinations of eyepieces with different focal lengths and different objectives can produce a wide range of magnifications.

$f_o < 1$ cm and a second lens, the *eyepiece,* that has a focal length f_e of a few centimeters. The two lenses are separated by a distance L that is much greater than either f_o or f_e. The object, which is placed just outside the focal point of the objective, forms a real, inverted image at I_1, and this image is located at or close to the focal point of the eyepiece. The eyepiece, which serves as a simple magnifier, produces at I_2 a virtual, inverted image of I_1. The lateral magnification M_1 of the first image is $-q_1/p_1$. Note from Figure 36.41a that q_1 is approximately equal to L and that the object is very close to the focal point of the objective: $p_1 \approx f_o$. Thus, the lateral magnification by the objective is

$$M_1 \approx -\frac{L}{f_o}$$

The angular magnification by the eyepiece for an object (corresponding to the image at I_1) placed at the focal point of the eyepiece is, from Equation 36.19,

$$m_e = \frac{25 \text{ cm}}{f_e}$$

The overall magnification of the compound microscope is defined as the product of the lateral and angular magnifications:

$$M = M_1 m_e = -\frac{L}{f_o}\left(\frac{25 \text{ cm}}{f_e}\right) \qquad \textbf{(36.20)}$$

The negative sign indicates that the image is inverted.

The microscope has extended human vision to the point where we can view previously unknown details of incredibly small objects. The capabilities of this instrument have steadily increased with improved techniques for precision grinding of lenses. An often-asked question about microscopes is: "If one were extremely patient and careful, would it be possible to construct a microscope that would enable the human eye to see an atom?" The answer is no, as long as light is used to illuminate the object. The reason is that, for an object under an optical microscope (one that uses visible light) to be seen, the object must be at least as large as a wavelength of light. Because the diameter of any atom is many times smaller than the wavelengths of visible light, the mysteries of the atom must be probed using other types of "microscopes."

The ability to use other types of waves to "see" objects also depends on wavelength. We can illustrate this with water waves in a bathtub. Suppose you vibrate your hand in the water until waves having a wavelength of about 15 cm are moving along the surface. If you hold a small object, such as a toothpick, so that it lies in the path of the waves, it does not appreciably disturb the waves; they continue along their path "oblivious" to it. Now suppose you hold a larger object, such as a toy sailboat, in the path of the 15-cm waves. In this case, the waves are considerably disturbed by the object. Because the toothpick was smaller than the wavelength of the waves, the waves did not "see" it (the intensity of the scattered waves was low). Because it is about the same size as the wavelength of the waves, however, the boat creates a disturbance. In other words, the object acts as the source of scattered waves that appear to come from it.

Light waves behave in this same general way. The ability of an optical microscope to view an object depends on the size of the object relative to the wavelength of the light used to observe it. Hence, we can never observe atoms with an optical microscope[2] because their dimensions are small (≈ 0.1 nm) relative to the wavelength of the light (≈ 500 nm).

Optional Section

36.10 THE TELESCOPE

14.1 & 14.9

Two fundamentally different types of **telescopes** exist; both are designed to aid in viewing distant objects, such as the planets in our Solar System. The **refracting telescope** uses a combination of lenses to form an image, and the **reflecting telescope** uses a curved mirror and a lens.

The lens combination shown in Figure 36.42a is that of a refracting telescope. Like the compound microscope, this telescope has an objective and an eyepiece. The two lenses are arranged so that the objective forms a real, inverted image of the distant object very near the focal point of the eyepiece. Because the object is essentially at infinity, this point at which I_1 forms is the focal point of the objective. Hence, the two lenses are separated by a distance $f_o + f_e$, which corresponds to the length of the telescope tube. The eyepiece then forms, at I_2, an enlarged, inverted image of the image at I_1.

The angular magnification of the telescope is given by θ/θ_o, where θ_o is the angle subtended by the object at the objective and θ is the angle subtended by the final image at the viewer's eye. Consider Figure 36.42a, in which the object is a very great distance to the left of the figure. The angle θ_o (to the *left* of the objective) subtended by the object at the objective is the same as the angle (to the *right* of the objective) subtended by the first image at the objective. Thus,

$$\tan \theta_o \approx \theta_o \approx -\frac{h'}{f_o}$$

where the negative sign indicates that the image is inverted.

The angle θ subtended by the final image at the eye is the same as the angle that a ray coming from the tip of I_1 and traveling parallel to the principal axis makes with the principal axis after it passes through the lens. Thus,

$$\tan \theta \approx \theta \approx \frac{h'}{f_e}$$

We have not used a negative sign in this equation because the final image is not inverted; the object creating this final image I_2 is I_1, and both it and I_2 point in the same direction. To see why the adjacent side of the triangle containing angle θ is f_e and not $2f_e$, note that we must use only the bent length of the refracted ray. Hence, the angular magnification of the telescope can be expressed as

$$m = \frac{\theta}{\theta_o} = \frac{h'/f_e}{-h'/f_o} = -\frac{f_o}{f_e} \quad \textbf{(36.21)}$$

and we see that the angular magnification of a telescope equals the ratio of the objective focal length to the eyepiece focal length. The negative sign indicates that the image is inverted.

Quick Quiz 36.6

Why isn't the lateral magnification given by Equation 36.1 a useful concept for telescopes?

[2] Single-molecule near-field optic studies are routinely performed with visible light having wavelengths of about 500 nm. The technique uses very small apertures to produce images having resolution as small as 10 nm.

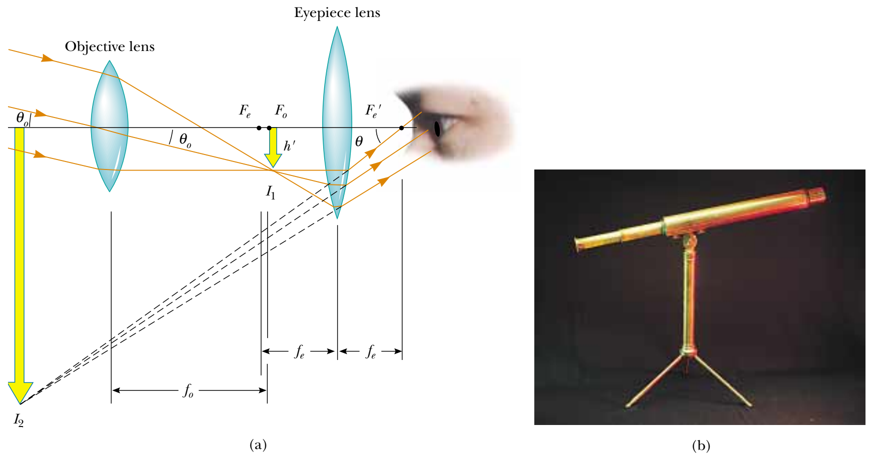


Figure 36.42 (a) Lens arrangement in a refracting telescope, with the object at infinity. (b) A refracting telescope.

When we look through a telescope at such relatively nearby objects as the Moon and the planets, magnification is important. However, stars are so far away that they always appear as small points of light no matter how great the magnification. A large research telescope that is used to study very distant objects must have a great diameter to gather as much light as possible. It is difficult and expensive to manufacture large lenses for refracting telescopes. Another difficulty with large lenses is that their weight leads to sagging, which is an additional source of aberration. These problems can be partially overcome by replacing the objective with a concave mirror, which results in a reflecting telescope. Because light is reflected from the mirror and does not pass through a lens, the mirror can have rigid supports on the back side. Such supports eliminate the problem of sagging.

Figure 36.43 shows the design for a typical reflecting telescope. Incoming light rays pass down the barrel of the telescope and are reflected by a parabolic mirror at the base. These rays converge toward point A in the figure, where an image would be formed. However, before this image is formed, a small, flat mirror M reflects the light toward an opening in the side of the tube that passes into an eyepiece. This particular design is said to have a Newtonian focus because Newton developed it. Note that in the reflecting telescope the light never passes through glass (except through the small eyepiece). As a result, problems associated with chromatic aberration are virtually eliminated.

The largest reflecting telescopes in the world are at the Keck Observatory on Mauna Kea, Hawaii. The site includes two telescopes with diameters of 10 m, each containing 36 hexagonally shaped, computer-controlled mirrors that work together to form a large reflecting surface. In contrast, the largest refracting telescope in the world, at the Yerkes Observatory in Williams Bay, Wisconsin, has a diameter of only 1 m.

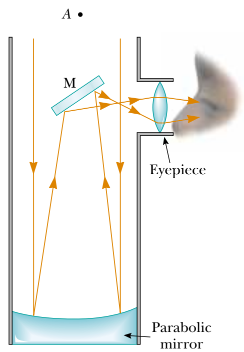


Figure 36.43 A Newtonian-focus reflecting telescope.

web

For more information on the Keck telescopes, visit **http://www2.keck.hawaii.edu:3636/**

SUMMARY

The **lateral magnification** M of a mirror or lens is defined as the ratio of the image height h' to the object height h:

$$M = \frac{h'}{h} \tag{36.1}$$

In the paraxial ray approximation, the object distance p and image distance q for a spherical mirror of radius R are related by the **mirror equation:**

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R} = \frac{1}{f} \tag{36.4, 36.6}$$

where $f = R/2$ is the **focal length** of the mirror.

An image can be formed by refraction from a spherical surface of radius R. The object and image distances for refraction from such a surface are related by

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \tag{36.8}$$

where the light is incident in the medium for which the index of refraction is n_1 and is refracted in the medium for which the index of refraction is n_2.

The inverse of the **focal length** f of a thin lens surrounded by air is given by the **lens makers' equation:**

$$\frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{36.11}$$

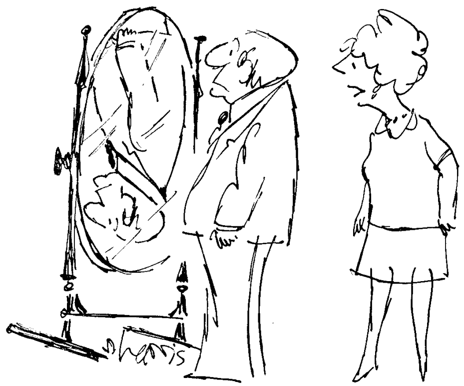
Converging lenses have positive focal lengths, and **diverging lenses** have negative focal lengths.

For a thin lens, and in the paraxial ray approximation, the object and image distances are related by the **thin-lens equation:**

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \tag{36.12}$$

QUESTIONS

1. What is wrong with the caption of the cartoon shown in Figure Q36.1?
2. Using a simple ray diagram, such as the one shown in Figure 36.2, show that a flat mirror whose top is at eye level need not be as long as you are for you to see your entire body in it.
3. Consider a concave spherical mirror with a real object. Is the image always inverted? Is the image always real? Give conditions for your answers.
4. Repeat the preceding question for a convex spherical mirror.
5. Why does a clear stream of water, such as a creek, always appear to be shallower than it actually is? By how much is its depth apparently reduced?
6. Consider the image formed by a thin converging lens. Under what conditions is the image (a) inverted, (b) upright, (c) real, (d) virtual, (e) larger than the object, and (f) smaller than the object?
7. Repeat Question 6 for a thin diverging lens.
8. Use the lens makers' equation to verify the sign of the focal length of each of the lenses in Figure 36.26.



"Most mirrors reverse left and right. This one reverses top and bottom."
Figure Q36.1

9. If a cylinder of solid glass or clear plastic is placed above the words LEAD OXIDE and viewed from the side as shown in Figure Q36.9, the LEAD appears inverted but the OXIDE does not. Explain.

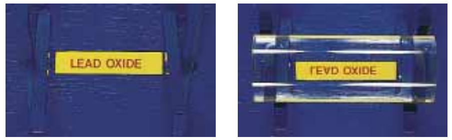


Figure Q36.9

10. If the camera "sees" a movie actor's reflection in a mirror, what does the actor see in the mirror?
11. Explain why a mirror cannot give rise to chromatic aberration.
12. Why do some automobile mirrors have printed on them the statement "Objects in mirror are closer than they appear"? (See Fig. Q36.12.)

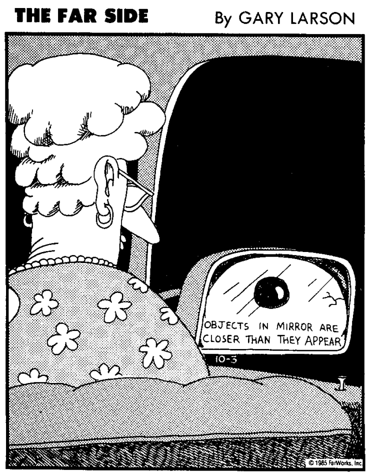


Figure Q36.12

13. Why do some emergency vehicles have the symbol ƎƆNA⅃UᗺMA written on the front?
14. Explain why a fish in a spherical goldfish bowl appears larger than it really is.
15. Lenses used in eyeglasses, whether converging or diverging, are always designed such that the middle of the lens curves away from the eye, like the center lenses of Figure 36.26a and b. Why?
16. A mirage is formed when the air gets gradually cooler with increasing altitude. What might happen if the air grew gradually warmer with altitude? This often happens over bodies of water or snow-covered ground; the effect is called *looming*.
17. Consider a spherical concave mirror, with an object positioned to the left of the mirror beyond the focal point. Using ray diagrams, show that the image moves to the left as the object approaches the focal point.
18. In a Jules Verne novel, a piece of ice is shaped into a magnifying lens to focus sunlight to start a fire. Is this possible?
19. The *f*-number of a camera is the focal length of the lens divided by its aperture (or diameter). How can the *f*-number of the lens be changed? How does changing this number affect the required exposure time?
20. A solar furnace can be constructed through the use of a concave mirror to reflect and focus sunlight into a furnace enclosure. What factors in the design of the reflecting mirror would guarantee very high temperatures?
21. One method for determining the position of an image, either real or virtual, is by means of *parallax*. If a finger or another object is placed at the position of the image, as shown in Figure Q36.21, and the finger and the image are viewed simultaneously (the image is viewed through the lens if it is virtual), the finger and image have the same parallax; that is, if the image is viewed from different positions, it will appear to move along with the finger. Use this method to locate the image formed by a lens. Explain why the method works.

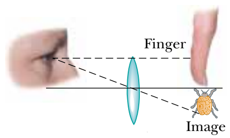


Figure Q36.21

22. Figure Q36.22 shows a lithograph by M. C. Escher titled *Hand with Reflection Sphere (Self-Portrait in Spherical Mirror).* Escher had this to say about the work: "The picture shows a spherical mirror, resting on a left hand. But as a print is the reverse of the original drawing on stone, it was my right hand that you see depicted. (Being left-handed, I needed my left hand to make the drawing.) Such a globe reflection collects almost one's whole surroundings in one disk-shaped image. The whole room, four walls, the

floor, and the ceiling, everything, albeit distorted, is compressed into that one small circle. Your own head, or more exactly the point between your eyes, is the absolute center. No matter how you turn or twist yourself, you can't get out of that central point. You are immovably the focus, the unshakable core, of your world." Comment on the accuracy of Escher's description.

23. You can make a corner reflector by placing three flat mirrors in the corner of a room where the ceiling meets the walls. Show that no matter where you are in the room, you can see yourself reflected in the mirrors—upside down.

Figure Q36.22

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
□ = paired numerical/symbolic problems

Section 36.1 Images Formed by Flat Mirrors

1. Does your bathroom mirror show you older or younger than you actually are? Compute an order-of-magnitude estimate for the age difference, based on data that you specify.

2. In a church choir loft, two parallel walls are 5.30 m apart. The singers stand against the north wall. The organist faces the south wall, sitting 0.800 m away from it. To enable her to see the choir, a flat mirror 0.600 m wide is mounted on the south wall, straight in front of her. What width of the north wall can she see? *Hint:* Draw a top-view diagram to justify your answer.

3. Determine the minimum height of a vertical flat mirror in which a person 5′10″ in height can see his or her full image. (A ray diagram would be helpful.)

4. Two flat mirrors have their reflecting surfaces facing each other, with an edge of one mirror in contact with an edge of the other, so that the angle between the mirrors is α. When an object is placed between the mirrors, a number of images are formed. In general, if the angle α is such that $n\alpha = 360°$, where n is an integer, the number of images formed is $n - 1$. Graphically, find all the image positions for the case $n = 6$ when a point object is between the mirrors (but not on the angle bisector).

5. A person walks into a room with two flat mirrors on opposite walls, which produce multiple images. When the person is 5.00 ft from the mirror on the left wall and 10.0 ft from the mirror on the right wall, find the distances from that person to the first three images seen in the mirror on the left.

Section 36.2 Images Formed by Spherical Mirrors

6. A concave spherical mirror has a radius of curvature of 20.0 cm. Find the location of the image for object distances of (a) 40.0 cm, (b) 20.0 cm, and (c) 10.0 cm. For each case, state whether the image is real or virtual and upright or inverted, and find the magnification.

7. At an intersection of hospital hallways, a convex mirror is mounted high on a wall to help people avoid collisions. The mirror has a radius of curvature of 0.550 m. Locate and describe the image of a patient 10.0 m from the mirror. Determine the magnification.

8. A large church has a niche in one wall. On the floor plan it appears as a semicircular indentation of radius 2.50 m. A worshiper stands on the center line of the niche, 2.00 m out from its deepest point, and whispers a prayer. Where is the sound concentrated after reflection from the back wall of the niche?

WEB **9.** A spherical convex mirror has a radius of curvature of 40.0 cm. Determine the position of the virtual image and the magnification (a) for an object distance of 30.0 cm and (b) for an object distance of 60.0 cm. (c) Are the images upright or inverted?

10. The height of the real image formed by a concave mirror is four times the object height when the object is 30.0 cm in front of the mirror. (a) What is the radius of curvature of the mirror? (b) Use a ray diagram to locate this image.

11. A concave mirror has a radius of curvature of 60.0 cm. Calculate the image position and magnification of an object placed in front of the mirror (a) at a distance of 90.0 cm and (b) at a distance of 20.0 cm. (c) In each case, draw ray diagrams to obtain the image characteristics.

12. A concave mirror has a focal length of 40.0 cm. Determine the object position for which the resulting image is upright and four times the size of the object.

13. A spherical mirror is to be used to form, on a screen 5.00 m from the object, an image five times the size of the object. (a) Describe the type of mirror required. (b) Where should the mirror be positioned relative to the object?

14. A rectangle 10.0 cm × 20.0 cm is placed so that its right edge is 40.0 cm to the left of a concave spherical mirror, as in Figure P36.14. The radius of curvature of the mirror is 20.0 cm. (a) Draw the image formed by this mirror. (b) What is the area of the image?

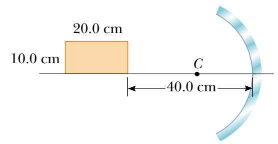


Figure P36.14

15. A dedicated sports-car enthusiast polishes the inside and outside surfaces of a hubcap that is a section of a sphere. When she looks into one side of the hubcap, she sees an image of her face 30.0 cm in back of the hubcap. She then flips the hubcap over and sees another image of her face 10.0 cm in back of the hubcap. (a) How far is her face from the hubcap? (b) What is the radius of curvature of the hubcap?

16. An object is 15.0 cm from the surface of a reflective spherical Christmas-tree ornament 6.00 cm in diameter. What are the magnification and position of the image?

17. A ball is dropped from rest 3.00 m directly above the vertex of a concave mirror that has a radius of 1.00 m and lies in a horizontal plane. (a) Describe the motion of the ball's image in the mirror. (b) At what time do the ball and its image coincide?

Section 36.3 Images Formed by Refraction

18. A flint-glass plate ($n = 1.66$) rests on the bottom of an aquarium tank. The plate is 8.00 cm thick (vertical dimension) and covered with water ($n = 1.33$) to a depth of 12.0 cm. Calculate the apparent thickness of the plate as viewed from above the water. (Assume nearly normal incidence.)

19. A cubical block of ice 50.0 cm on a side is placed on a level floor over a speck of dust. Find the location of the image of the speck if the index of refraction of ice is 1.309.

20. A simple model of the human eye ignores its lens entirely. Most of what the eye does to light happens at the transparent cornea. Assume that this outer surface has a 6.00-mm radius of curvature, and assume that the eyeball contains just one fluid with an index of refraction of 1.40. Prove that a very distant object will be imaged on the retina, 21.0 mm behind the cornea. Describe the image.

21. A glass sphere ($n = 1.50$) with a radius of 15.0 cm has a tiny air bubble 5.00 cm above its center. The sphere is viewed looking down along the extended radius containing the bubble. What is the apparent depth of the bubble below the surface of the sphere?

22. A transparent sphere of unknown composition is observed to form an image of the Sun on the surface of the sphere opposite the Sun. What is the refractive index of the sphere material?

23. One end of a long glass rod ($n = 1.50$) is formed into a convex surface of radius 6.00 cm. An object is positioned in air along the axis of the rod. Find the image positions corresponding to object distances of (a) 20.0 cm, (b) 10.0 cm, and (c) 3.00 cm from the end of the rod.

24. A goldfish is swimming at 2.00 cm/s toward the front wall of a rectangular aquarium. What is the apparent speed of the fish as measured by an observer looking in from outside the front wall of the tank? The index of refraction of water is 1.33.

25. A goldfish is swimming inside a spherical plastic bowl of water, with an index of refraction of 1.33. If the goldfish is 10.0 cm from the wall of the 15.0-cm-radius bowl, where does it appear to an observer outside the bowl?

Section 36.4 Thin Lenses

26. A contact lens is made of plastic with an index of refraction of 1.50. The lens has an outer radius of curvature of +2.00 cm and an inner radius of curvature of +2.50 cm. What is the focal length of the lens?

WEB **27.** The left face of a biconvex lens has a radius of curvature of magnitude 12.0 cm, and the right face has a radius of curvature of magnitude 18.0 cm. The index of refraction of the glass is 1.44. (a) Calculate the focal length of the lens. (b) Calculate the focal length if the radii of curvature of the two faces are interchanged.

28. A converging lens has a focal length of 20.0 cm. Locate the image for object distances of (a) 40.0 cm, (b) 20.0 cm, and (c) 10.0 cm. For each case, state whether the image is real or virtual and upright or inverted. Find the magnification in each case.

29. A thin lens has a focal length of 25.0 cm. Locate and describe the image when the object is placed (a) 26.0 cm and (b) 24.0 cm in front of the lens.

30. An object positioned 32.0 cm in front of a lens forms an image on a screen 8.00 cm behind the lens. (a) Find the focal length of the lens. (b) Determine the magnification. (c) Is the lens converging or diverging?

WEB 31. The nickel's image in Figure P36.31 has twice the diameter of the nickel and is 2.84 cm from the lens. Determine the focal length of the lens.

Figure P36.31

32. A magnifying glass is a converging lens of focal length 15.0 cm. At what distance from a postage stamp should you hold this lens to get a magnification of $+2.00$?

33. A transparent photographic slide is placed in front of a converging lens with a focal length of 2.44 cm. The lens forms an image of the slide 12.9 cm from the slide. How far is the lens from the slide if the image is (a) real? (b) virtual?

34. A person looks at a gem with a jeweler's loupe—a converging lens that has a focal length of 12.5 cm. The loupe forms a virtual image 30.0 cm from the lens. (a) Determine the magnification. Is the image upright or inverted? (b) Construct a ray diagram for this arrangement.

35. Suppose an object has thickness dp so that it extends from object distance p to $p + dp$. Prove that the thickness dq of its image is given by $(-q^2/p^2)\,dp$, so the longitudinal magnification $dq/dp = -M^2$, where M is the lateral magnification.

36. The projection lens in a certain slide projector is a single thin lens. A slide 24.0 mm high is to be projected so that its image fills a screen 1.80 m high. The slide-to-screen distance is 3.00 m. (a) Determine the focal length of the projection lens. (b) How far from the slide should the lens of the projector be placed to form the image on the screen?

37. An object is positioned 20.0 cm to the left of a diverging lens with focal length $f = -32.0$ cm. Determine (a) the location and (b) the magnification of the image. (c) Construct a ray diagram for this arrangement.

38. Figure P36.38 shows a thin glass ($n = 1.50$) converging lens for which the radii of curvature are $R_1 = 15.0$ cm and $R_2 = -12.0$ cm. To the left of the lens is a cube with a face area of 100 cm^2. The base of the cube is on the axis of the lens, and the right face is 20.0 cm to the left of the lens. (a) Determine the focal length of the lens. (b) Draw the image of the square face formed by the lens. What type of geometric figure is this? (c) Determine the area of the image.

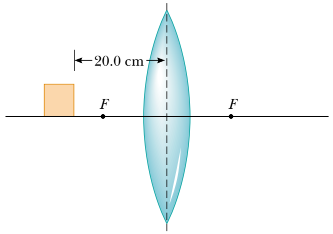


Figure P36.38

39. An object is 5.00 m to the left of a flat screen. A converging lens for which the focal length is $f = 0.800$ m is placed between object and screen. (a) Show that two lens positions exist that form images on the screen, and determine how far these positions are from the object. (b) How do the two images differ from each other?

40. An object is at a distance d to the left of a flat screen. A converging lens with focal length $f < d/4$ is placed between object and screen. (a) Show that two lens positions exist that form an image on the screen, and determine how far these positions are from the object. (b) How do the two images differ from each other?

41. Figure 36.33 diagrams a cross-section of a camera. It has a single lens with a focal length of 65.0 mm, which is to form an image on the film at the back of the camera. Suppose the position of the lens has been adjusted to focus the image of a distant object. How far and in what direction must the lens be moved to form a sharp image of an object that is 2.00 m away?

(Optional)
Section 36.5 Lens Aberrations

42. The magnitudes of the radii of curvature are 32.5 cm and 42.5 cm for the two faces of a biconcave lens. The glass has index 1.53 for violet light and 1.51 for red light. For a very distant object, locate and describe (a) the image formed by violet light and (b) the image formed by red light.

43. Two rays traveling parallel to the principal axis strike a large plano–convex lens having a refractive index of 1.60 (Fig. P36.43). If the convex face is spherical, a ray near the edge does not pass through the focal point (spherical aberration occurs). If this face has a radius of curvature of magnitude 20.0 cm and the two rays are $h_1 = 0.500$ cm and $h_2 = 12.0$ cm from the principal axis, find the difference in the positions where they cross the principal axis.

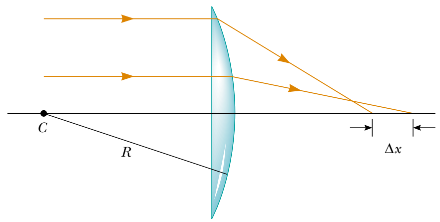


Figure P36.43

(Optional)
Section 36.7 The Eye

44. The accommodation limits for Nearsighted Nick's eyes are 18.0 cm and 80.0 cm. When he wears his glasses, he can see faraway objects clearly. At what minimum distance can he see objects clearly?

45. A nearsighted person cannot see objects clearly beyond 25.0 cm (her far point). If she has no astigmatism and contact lenses are prescribed for her, what power and type of lens are required to correct her vision?

46. A person sees clearly when he wears eyeglasses that have a power of -4.00 diopters and sit 2.00 cm in front of his eyes. If he wants to switch to contact lenses, which are placed directly on the eyes, what lens power should be prescribed?

(Optional)
Section 36.8 The Simple Magnifier
Section 36.9 The Compound Microscope
Section 36.10 The Telescope

47. A philatelist examines the printing detail on a stamp, using a biconvex lens with a focal length of 10.0 cm as a simple magnifier. The lens is held close to the eye, and the lens-to-object distance is adjusted so that the virtual image is formed at the normal near point (25.0 cm). Calculate the magnification.

48. A lens that has a focal length of 5.00 cm is used as a magnifying glass. (a) Where should the object be placed to obtain maximum magnification? (b) What is the magnification?

49. The distance between the eyepiece and the objective lens in a certain compound microscope is 23.0 cm. The focal length of the eyepiece is 2.50 cm, and that of the objective is 0.400 cm. What is the overall magnification of the microscope?

50. The desired overall magnification of a compound microscope is 140×. The objective alone produces a lateral magnification of 12.0×. Determine the required focal length of the eyepiece.

51. The Yerkes refracting telescope has a 1.00-m-diameter objective lens with a focal length of 20.0 m. Assume that it is used with an eyepiece that has a focal length of 2.50 cm. (a) Determine the magnification of the planet Mars as seen through this telescope. (b) Are the Martian polar caps seen right side up or upside down?

52. Astronomers often take photographs with the objective lens or the mirror of a telescope alone, without an eyepiece. (a) Show that the image size h' for this telescope is given by $h' = fh/(f - p)$, where h is the object size, f is the objective focal length, and p is the object distance. (b) Simplify the expression in part (a) for the case in which the object distance is much greater than objective focal length. (c) The "wingspan" of the International Space Station is 108.6 m, the overall width of its solar-panel configuration. Find the width of the image formed by a telescope objective of focal length 4.00 m when the station is orbiting at an altitude of 407 km.

53. Galileo devised a simple terrestrial telescope that produces an upright image. It consists of a converging objective lens and a diverging eyepiece at opposite ends of the telescope tube. For distant objects, the tube length is the objective focal length less the absolute value of the eyepiece focal length. (a) Does the user of the telescope see a real or virtual image? (b) Where is the final image? (c) If a telescope is to be constructed with a tube 10.0 cm long and a magnification of 3.00, what are the focal lengths of the objective and eyepiece?

54. A certain telescope has an objective mirror with an aperture diameter of 200 mm and a focal length of 2 000 mm. It captures the image of a nebula on photographic film at its prime focus with an exposure time of 1.50 min. To produce the same light energy per unit area on the film, what is the required exposure time to photograph the same nebula with a smaller telescope, which has an objective lens with an aperture diameter of 60.0 mm and a focal length of 900 mm?

ADDITIONAL PROBLEMS

55. The distance between an object and its upright image is 20.0 cm. If the magnification is 0.500, what is the focal length of the lens that is being used to form the image?

56. The distance between an object and its upright image is d. If the magnification is M, what is the focal length of the lens that is being used to form the image?

57. The lens and mirror in Figure P36.57 have focal lengths of $+80.0$ cm and -50.0 cm, respectively. An object is

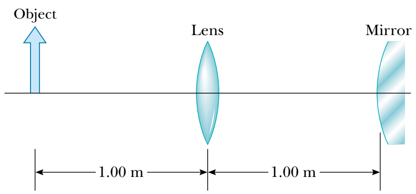


Figure P36.57

placed 1.00 m to the left of the lens, as shown. Locate the final image, which is formed by light that has gone through the lens twice. State whether the image is upright or inverted, and determine the overall magnification.

58. Your friend needs glasses with diverging lenses of focal length -65.0 cm for both eyes. You tell him he looks good when he does not squint, but he is worried about how thick the lenses will be. If the radius of curvature of the first surface is $R_1 = 50.0$ cm and the high-index plastic has a refractive index of 1.66, (a) find the required radius of curvature of the second surface. (b) Assume that the lens is ground from a disk 4.00 cm in diameter and 0.100 cm thick at the center. Find the thickness of the plastic at the edge of the lens, measured parallel to the axis. *Hint:* Draw a large cross-sectional diagram.

59. The object in Figure P36.59 is midway between the lens and the mirror. The mirror's radius of curvature is 20.0 cm, and the lens has a focal length of -16.7 cm. Considering only the light that leaves the object and travels first toward the mirror, locate the final image formed by this system. Is this image real or virtual? Is it upright or inverted? What is the overall magnification?

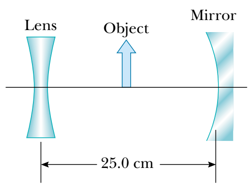


Figure P36.59

60. An object placed 10.0 cm from a concave spherical mirror produces a real image 8.00 cm from the mirror. If the object is moved to a new position 20.0 cm from the mirror, what is the position of the image? Is the latter image real or virtual?

WEB **61.** A parallel beam of light enters a glass hemisphere perpendicular to the flat face, as shown in Figure P36.61. The radius is $|R| = 6.00$ cm, and the index of refraction is $n = 1.560$. Determine the point at which the beam is focused. (Assume paraxial rays.)

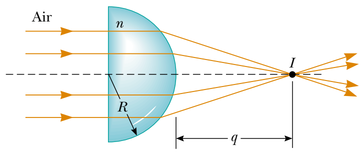


Figure P36.61

62. Review Problem. A spherical lightbulb with a diameter of 3.20 cm radiates light equally in all directions, with a power of 4.50 W. (a) Find the light intensity at the surface of the bulb. (b) Find the light intensity 7.20 m from the center of the bulb. (c) At this 7.20-m distance, a lens is set up with its axis pointing toward the bulb. The lens has a circular face with a diameter of 15.0 cm and a focal length of 35.0 cm. Find the diameter of the image of the bulb. (d) Find the light intensity at the image.

63. An object is placed 12.0 cm to the left of a diverging lens with a focal length of -6.00 cm. A converging lens with a focal length of 12.0 cm is placed a distance d to the right of the diverging lens. Find the distance d that corresponds to a final image at infinity. Draw a ray diagram for this case.

64. Assume that the intensity of sunlight is 1.00 kW/m^2 at a particular location. A highly reflecting concave mirror is to be pointed toward the Sun to produce a power of at least 350 W at the image. (a) Find the required radius R_a of the circular face area of the mirror. (b) Now suppose the light intensity is to be at least 120 kW/m^2 at the image. Find the required relationship between R_a and the radius of curvature R of the mirror. The disk of the Sun subtends an angle of 0.533° at the Earth.

WEB **65.** The disk of the Sun subtends an angle of 0.533° at the Earth. What are the position and diameter of the solar image formed by a concave spherical mirror with a radius of curvature of 3.00 m?

66. Figure P36.66 shows a thin converging lens for which the radii are $R_1 = 9.00$ cm and $R_2 = -11.0$ cm. The lens is in front of a concave spherical mirror of radius $R = 8.00$ cm. (a) If its focal points F_1 and F_2 are 5.00 cm from the vertex of the lens, determine its index of refraction. (b) If the lens and mirror are 20.0 cm apart and an object is placed 8.00 cm to the left of the lens, determine the position of the final image and its magnification as seen by the eye in the figure. (c) Is the final image inverted or upright? Explain.

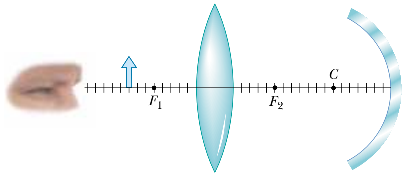


Figure P36.66

67. In a darkened room, a burning candle is placed 1.50 m from a white wall. A lens is placed between candle and wall at a location that causes a larger, inverted image to form on the wall. When the lens is moved 90.0 cm toward the wall, another image of the candle is formed. Find (a) the two object distances that produce the specified images and (b) the focal length of the lens. (c) Characterize the second image.

68. A thin lens of focal length f lies on a horizontal front-surfaced flat mirror. How far above the lens should an object be held if its image is to coincide with the object?

69. A compound microscope has an objective of focal length 0.300 cm and an eyepiece of focal length 2.50 cm. If an object is 3.40 mm from the objective, what is the magnification? (*Hint:* Use the lens equation for the objective.)

70. Two converging lenses with focal lengths of 10.0 cm and 20.0 cm are positioned 50.0 cm apart, as shown in Figure P36.70. The final image is to be located between the lenses, at the position indicated. (a) How far to the left of the first lens should the object be? (b) What is the overall magnification? (c) Is the final image upright or inverted?

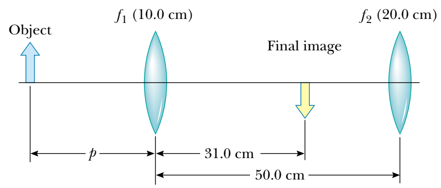


Figure P36.70

71. A cataract-impaired lens in an eye may be surgically removed and replaced by a manufactured lens. The focal length required for the new lens is determined by the lens-to-retina distance, which is measured by a sonar-like device, and by the requirement that the implant provide for correct distant vision. (a) If the distance from lens to retina is 22.4 mm, calculate the power of the implanted lens in diopters. (b) Since no accommodation occurs and the implant allows for correct distant vision, a corrective lens for close work or reading must be used. Assume a reading distance of 33.0 cm, and calculate the power of the lens in the reading glasses.

72. A floating strawberry illusion consists of two parabolic mirrors, each with a focal length of 7.50 cm, facing each other so that their centers are 7.50 cm apart (Fig. P36.72). If a strawberry is placed on the lower mirror, an image of the strawberry is formed at the small opening at the center of the top mirror. Show that the final image is formed at that location, and describe its characteristics. (*Note:* A very startling effect is to shine a flashlight beam on these images. Even at a glancing angle, the incoming light beam is seemingly reflected off the images! Do you understand why?)

Figure P36.72

73. An object 2.00 cm high is placed 40.0 cm to the left of a converging lens with a focal length of 30.0 cm. A diverging lens with a focal length of -20.0 cm is placed 110 cm to the right of the converging lens. (a) Determine the final position and magnification of the final image. (b) Is the image upright or inverted? (c) Repeat parts (a) and (b) for the case in which the second lens is a converging lens with a focal length of $+20.0$ cm.

Answers to Quick Quizzes

36.1 At *C*. A ray traced from the stone to the mirror and then to observer 2 looks like this:

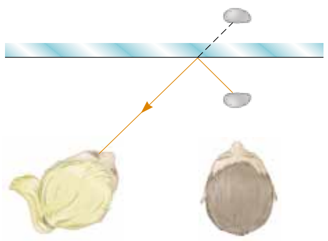


Figure QQA36.1

36.2 The focal length is infinite. Because the flat surfaces of the pane have infinite radii of curvature, Equation 36.11 indicates that the focal length is also infinite. Parallel rays striking the pane focus at infinity, which means that they remain parallel after passing through the glass.

36.3 An infinite number. In general, an infinite number of rays leave each point of any object and travel outward in all directions. (The three principal rays that we use to locate an image make up a selected subset of the infinite number of rays.) When an object is taller than a lens, we merely extend the plane containing the lens, as shown in Figure QQA36.2.

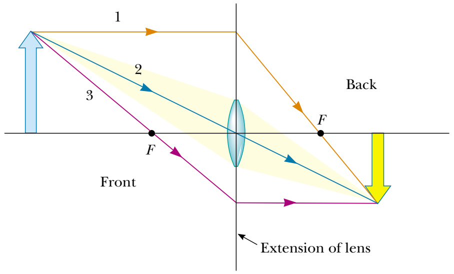
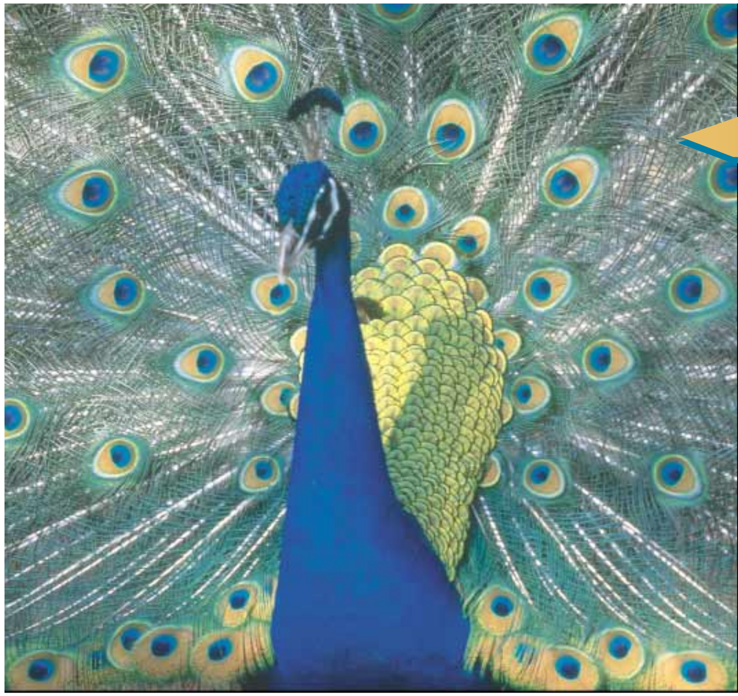


Figure QQA36.2

36.4 (c) The entire image is visible but has half the intensity. Each point on the object is a source of rays that travel in all directions. Thus, light from all parts of the object goes through all parts of the lens and forms an image. If you block part of the lens, you are blocking some of the rays, but the remaining ones still come from all parts of the object.

36.5 The eyeglasses on the left are diverging lenses, which correct for nearsightedness. If you look carefully at the edge of the person's face through the lens, you will see that everything viewed through these glasses is reduced in size. The eyeglasses on the right are converging lenses, which correct for farsightedness. These lenses make everything that is viewed through them look larger.

36.6 The lateral magnification of a telescope is not well defined. For viewing with the eye relaxed, the user may slightly adjust the position of the eyepiece to place the final image I_2 in Figure 36.42a at infinity. Then, its height and its lateral magnification also are infinite. The angular magnification of a telescope as we define it is the factor by which the telescope increases in the diameter—on the retina of the viewer's eye—of the real image of an extended object.



PUZZLER

The brilliant colors seen in peacock feathers are not caused by pigments in the feathers. If they are not produced by pigments, how *are* these beautiful colors created? *(Terry Qing/FPG International)*

chapter 37

Interference of Light Waves

Chapter Outline

In the preceding chapter on geometric optics, we used light rays to examine what happens when light passes through a lens or reflects from a mirror. Here in Chapter 37 and in the next chapter, we are concerned with *wave optics,* the study of interference, diffraction, and polarization of light. These phenomena cannot be adequately explained with the ray optics used in Chapter 36. We now learn how treating light as waves rather than as rays leads to a satisfying description of such phenomena.

37.1 CONDITIONS FOR INTERFERENCE

In Chapter 18, we found that the adding together of two mechanical waves can be constructive or destructive. In constructive interference, the amplitude of the resultant wave is greater than that of either individual wave, whereas in destructive interference, the resultant amplitude is less than that of either individual wave. Light waves also interfere with each other. Fundamentally, all interference associated with light waves arises when the electromagnetic fields that constitute the individual waves combine.

If two lightbulbs are placed side by side, no interference effects are observed because the light waves from one bulb are emitted independently of those from the other bulb. The emissions from the two lightbulbs do not maintain a constant phase relationship with each other over time. Light waves from an ordinary source such as a lightbulb undergo random changes about once every 10^{-8} s. Therefore, the conditions for constructive interference, destructive interference, or some intermediate state last for lengths of time of the order of 10^{-8} s. Because the eye cannot follow such short-term changes, no interference effects are observed. (In 1993 interference from two separate light sources was photographed in an extremely fast exposure. Nonetheless, we do not ordinarily see interference effects because of the rapidly changing phase relationship between the light waves.) Such light sources are said to be **incoherent.**

Interference effects in light waves are not easy to observe because of the short wavelengths involved (from 4×10^{-7} m to 7×10^{-7} m). For sustained interference in light waves to be observed, the following conditions must be met:

Conditions for interference

- The sources must be **coherent**—that is, they must maintain a constant phase with respect to each other.
- The sources should be **monochromatic**—that is, of a single wavelength.

We now describe the characteristics of coherent sources. As we saw when we studied mechanical waves, two sources (producing two traveling waves) are needed to create interference. In order to produce a stable interference pattern, **the individual waves must maintain a constant phase relationship with one another.** As an example, the sound waves emitted by two side-by-side loudspeakers driven by a single amplifier can interfere with each other because the two speakers are coherent—that is, they respond to the amplifier in the same way at the same time.

A common method for producing two coherent light sources is to use one monochromatic source to illuminate a barrier containing two small openings (usually in the shape of slits). The light emerging from the two slits is coherent because a single source produces the original light beam and the two slits serve only to separate the original beam into two parts (which, after all, is what was done to the sound signal from the side-by-side loudspeakers). Any random change in the light emitted by the source occurs in both beams at the same time, and as a result interference effects can be observed when the light from the two slits arrives at a viewing screen.

37.2 YOUNG'S DOUBLE-SLIT EXPERIMENT

Interference in light waves from two sources was first demonstrated by Thomas Young in 1801. A schematic diagram of the apparatus that Young used is shown in Figure 37.1a. Light is incident on a first barrier in which there is a slit S_0. The waves emerging from this slit arrive at a second barrier that contains two parallel slits S_1 and S_2. These two slits serve as a pair of coherent light sources because waves emerging from them originate from the same wave front and therefore maintain a constant phase relationship. The light from S_1 and S_2 produces on a viewing screen a visible pattern of bright and dark parallel bands called **fringes** (Fig. 37.1b). When the light from S_1 and that from S_2 both arrive at a point on the screen such that constructive interference occurs at that location, a bright fringe appears. When the light from the two slits combines destructively at any location on the screen, a dark fringe results. Figure 37.2 is a photograph of an interference pattern produced by two coherent vibrating sources in a water tank.

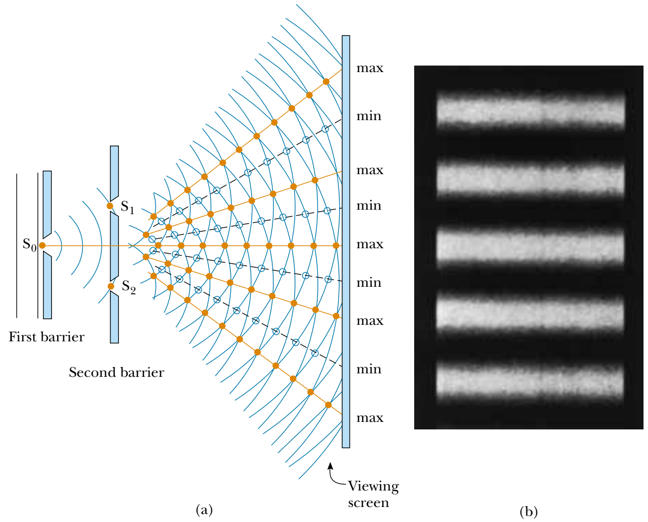


Figure 37.1 (a) Schematic diagram of Young's double-slit experiment. Slits S_1 and S_2 behave as coherent sources of light waves that produce an interference pattern on the viewing screen (drawing not to scale). (b) An enlargement of the center of a fringe pattern formed on the viewing screen with many slits could look like this.

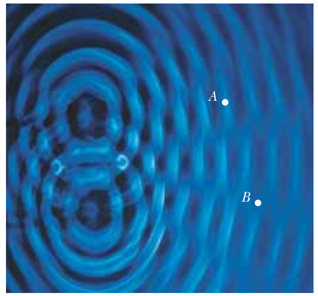


Figure 37.2 An interference pattern involving water waves is produced by two vibrating sources at the water's surface. The pattern is analogous to that observed in Young's double-slit experiment. Note the regions of constructive (*A*) and destructive (*B*) interference.

Quick Quiz 37.1

If you were to blow smoke into the space between the second barrier and the viewing screen of Figure 37.1a, what would you see?

QuickLab

Look through the fabric of an umbrella at a distant streetlight. Can you explain what you see? (The fringe pattern in Figure 37.1b is from rectangular slits. The fabric of the umbrella creates a two-dimensional set of square holes.)

Quick Quiz 37.2

Figure 37.2 is an overhead view of a shallow water tank. If you wanted to use a small ruler to measure the water's depth, would this be easier to do at location A or at location B?

Figure 37.3 shows some of the ways in which two waves can combine at the screen. In Figure 37.3a, the two waves, which leave the two slits in phase, strike the screen at the central point P. Because both waves travel the same distance, they arrive at P in phase. As a result, constructive interference occurs at this location, and a bright fringe is observed. In Figure 37.3b, the two waves also start in phase, but in this case the upper wave has to travel one wavelength farther than the lower wave to reach point Q. Because the upper wave falls behind the lower one by exactly one wavelength, they still arrive in phase at Q, and so a second bright fringe appears at this location. At point R in Figure 37.3c, however, midway between points P and Q, the upper wave has fallen half a wavelength behind the lower wave. This means that a trough of the lower wave overlaps a crest of the upper wave; this gives rise to destructive interference at point R. For this reason, a dark fringe is observed at this location.

We can describe Young's experiment quantitatively with the help of Figure 37.4. The viewing screen is located a perpendicular distance L from the double-slitted barrier. S_1 and S_2 are separated by a distance d, and the source is monochromatic. To reach any arbitrary point P, a wave from the lower slit travels farther than a wave from the upper slit by a distance $d \sin\theta$. This distance is called the **path difference** δ (lowercase Greek delta). If we assume that r_1 and r_2 are parallel, which is approximately true because L is much greater than d, then δ is given by

Path difference

$$\delta = r_2 - r_1 = d \sin\theta \qquad \textbf{(37.1)}$$

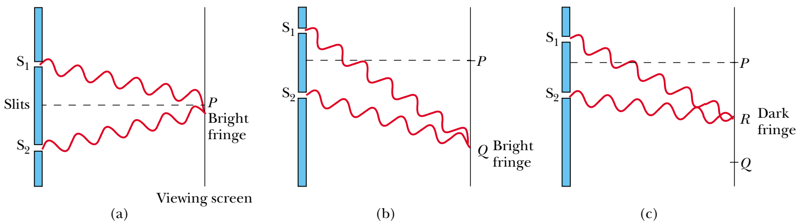


Figure 37.3 (a) Constructive interference occurs at point P when the waves combine. (b) Constructive interference also occurs at point Q. (c) Destructive interference occurs at R when the two waves combine because the upper wave falls half a wavelength behind the lower wave (all figures not to scale).

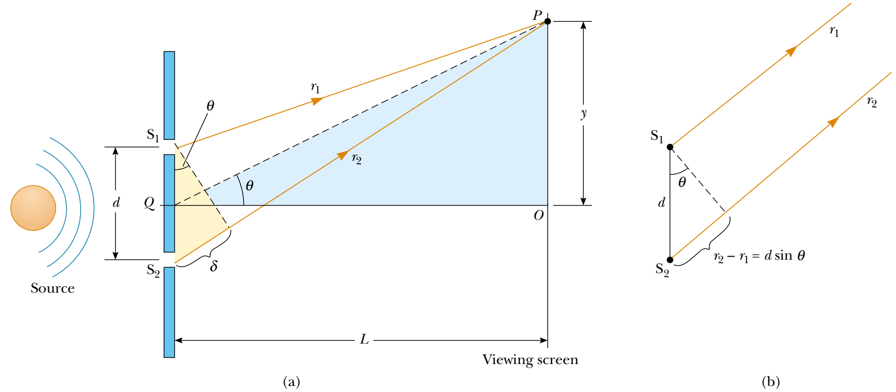


Figure 37.4 (a) Geometric construction for describing Young's double-slit experiment (not to scale). (b) When we assume that r_1 is parallel to r_2, the path difference between the two rays is $r_2 - r_1 = d \sin\theta$. For this approximation to be valid, it is essential that $L \gg d$.

The value of δ determines whether the two waves are in phase when they arrive at point P. If δ is either zero or some integer multiple of the wavelength, then the two waves are in phase at point P and constructive interference results. Therefore, the condition for bright fringes, or **constructive interference,** at point P is

Conditions for constructive interference

$$\delta = d \sin\theta = m\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \qquad \textbf{(37.2)}$$

The number m is called the **order number.** The central bright fringe at $\theta = 0$ ($m = 0$) is called the *zeroth-order maximum.* The first maximum on either side, where $m = \pm 1$, is called the *first-order maximum,* and so forth.

When δ is an odd multiple of $\lambda/2$, the two waves arriving at point P are 180° out of phase and give rise to destructive interference. Therefore, the condition for dark fringes, or **destructive interference,** at point P is

Conditions for destructive interference

$$d \sin\theta = (m + \tfrac{1}{2})\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \qquad \textbf{(37.3)}$$

It is useful to obtain expressions for the positions of the bright and dark fringes measured vertically from O to P. In addition to our assumption that $L \gg d$, we assume that $d \gg \lambda$. These can be valid assumptions because in practice L is often of the order of 1 m, d a fraction of a millimeter, and λ a fraction of a micrometer for visible light. Under these conditions, θ is small; thus, we can use the approximation $\sin\theta \approx \tan\theta$. Then, from triangle OPQ in Figure 37.4, we see that

$$y = L \tan\theta \approx L \sin\theta \qquad \textbf{(37.4)}$$

Solving Equation 37.2 for $\sin\theta$ and substituting the result into Equation 37.4, we see that the positions of the bright fringes measured from O are given by the expression

$$y_{\text{bright}} = \frac{\lambda L}{d}\, m \qquad \textbf{(37.5)}$$

Using Equations 37.3 and 37.4, we find that the dark fringes are located at

$$y_{\text{dark}} = \frac{\lambda L}{d}\left(m + \tfrac{1}{2}\right) \tag{37.6}$$

As we demonstrate in Example 37.1, Young's double-slit experiment provides a method for measuring the wavelength of light. In fact, Young used this technique to do just that. Additionally, the experiment gave the wave model of light a great deal of credibility. It was inconceivable that particles of light coming through the slits could cancel each other in a way that would explain the dark fringes.

EXAMPLE 37.1 Measuring the Wavelength of a Light Source

A viewing screen is separated from a double-slit source by 1.2 m. The distance between the two slits is 0.030 mm. The second-order bright fringe ($m = 2$) is 4.5 cm from the center line. (a) Determine the wavelength of the light.

Solution We can use Equation 37.5, with $m = 2$, $y_2 = 4.5 \times 10^{-2}$ m, $L = 1.2$ m, and $d = 3.0 \times 10^{-5}$ m:

$$\lambda = \frac{dy_2}{mL} = \frac{(3.0 \times 10^{-5}\text{ m})(4.5 \times 10^{-2}\text{ m})}{2(1.2\text{ m})}$$

$$= 5.6 \times 10^{-7}\text{ m} = \boxed{560\text{ nm}}$$

(b) Calculate the distance between adjacent bright fringes.

Solution From Equation 37.5 and the results of part (a), we obtain

$$y_{m+1} - y_m = \frac{\lambda L(m+1)}{d} - \frac{\lambda L m}{d}$$

$$= \frac{\lambda L}{d} = \frac{(5.6 \times 10^{-7}\text{ m})(1.2\text{ m})}{3.0 \times 10^{-5}\text{ m}}$$

$$= 2.2 \times 10^{-2}\text{ m} = \boxed{2.2\text{ cm}}$$

Note that the spacing between all fringes is equal.

EXAMPLE 37.2 Separating Double-Slit Fringes of Two Wavelengths

A light source emits visible light of two wavelengths: $\lambda = 430$ nm and $\lambda' = 510$ nm. The source is used in a double-slit interference experiment in which $L = 1.5$ m and $d = 0.025$ mm. Find the separation distance between the third-order bright fringes.

Solution Using Equation 37.5, with $m = 3$, we find that the fringe positions corresponding to these two wavelengths are

$$y_3 = \frac{\lambda L}{d} m = 3\frac{\lambda L}{d} = 7.74 \times 10^{-2}\text{ m}$$

$$y_3' = \frac{\lambda' L}{d} m = 3\frac{\lambda' L}{d} = 9.18 \times 10^{-2}\text{ m}$$

Hence, the separation distance between the two fringes is

$$\Delta y = y_3' - y_3 = 9.18 \times 10^{-2}\text{ m} - 7.74 \times 10^{-2}\text{ m}$$

$$= 1.4 \times 10^{-2}\text{ m} = \boxed{1.4\text{ cm}}$$

37.3 INTENSITY DISTRIBUTION OF THE DOUBLE-SLIT INTERFERENCE PATTERN

Note that the edges of the bright fringes in Figure 37.1b are fuzzy. So far we have discussed the locations of only the centers of the bright and dark fringes on a distant screen. We now direct our attention to the intensity of the light at other points between the positions of maximum constructive and destructive interference. In other words, we now calculate the distribution of light intensity associated with the double-slit interference pattern.

Again, suppose that the two slits represent coherent sources of sinusoidal waves such that the two waves from the slits have the same angular frequency ω and a constant phase difference ϕ. The total magnitude of the electric field at point P on the screen in Figure 37.5 is the vector superposition of the two waves. Assuming that the two waves have the same amplitude E_0, we can write the magnitude of the electric field at point P due to each wave separately as

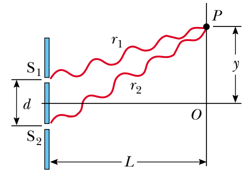


Figure 37.5 Construction for analyzing the double-slit interference pattern. A bright fringe, or intensity maximum, is observed at O.

$$E_1 = E_0 \sin \omega t \quad \text{and} \quad E_2 = E_0 \sin(\omega t + \phi) \tag{37.7}$$

Although the waves are in phase at the slits, *their phase difference ϕ at point P depends on the path difference* $\delta = r_2 - r_1 = d \sin\theta$. Because a path difference of λ (constructive interference) corresponds to a phase difference of 2π rad, we obtain the ratio

$$\frac{\delta}{\lambda} = \frac{\phi}{2\pi}$$

Phase difference

$$\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d \sin\theta \tag{37.8}$$

This equation tells us precisely how the phase difference ϕ depends on the angle θ in Figure 37.4.

Using the superposition principle and Equation 37.7, we can obtain the magnitude of the resultant electric field at point P:

$$E_P = E_1 + E_2 = E_0[\sin \omega t + \sin(\omega t + \phi)] \tag{37.9}$$

To simplify this expression, we use the trigonometric identity

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Taking $A = \omega t + \phi$ and $B = \omega t$, we can write Equation 37.9 in the form

$$E_P = 2E_0 \cos\left(\frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right) \tag{37.10}$$

This result indicates that the electric field at point P has the same frequency ω as the light at the slits, but that the amplitude of the field is multiplied by the factor $2\cos(\phi/2)$. To check the consistency of this result, note that if $\phi = 0, 2\pi, 4\pi, \ldots$, then the electric field at point P is $2E_0$, corresponding to the condition for constructive interference. These values of ϕ are consistent with Equation 37.2 for constructive interference. Likewise, if $\phi = \pi, 3\pi, 5\pi, \ldots$, then the magnitude of the electric field at point P is zero; this is consistent with Equation 37.3 for destructive interference.

Finally, to obtain an expression for the light intensity at point P, recall from Section 34.3 that *the intensity of a wave is proportional to the square of the resultant electric field magnitude at that point* (Eq. 34.20). Using Equation 37.10, we can therefore express the light intensity at point P as

$$I \propto E_P^2 = 4E_0^2 \cos^2\left(\frac{\phi}{2}\right)\sin^2\left(\omega t + \frac{\phi}{2}\right)$$

Most light-detecting instruments measure time-averaged light intensity, and the time-averaged value of $\sin^2(\omega t + \phi/2)$ over one cycle is $\frac{1}{2}$. Therefore, we can write the average light intensity at point P as

$$I = I_{\max} \cos^2\left(\frac{\phi}{2}\right) \tag{37.11}$$

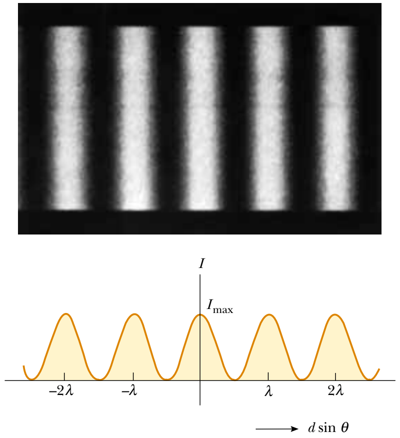


Figure 37.6 Light intensity versus $d \sin \theta$ for a double-slit interference pattern when the screen is far from the slits ($L \gg d$).

where I_{max} is the maximum intensity on the screen and the expression represents the time average. Substituting the value for ϕ given by Equation 37.8 into this expression, we find that

$$I = I_{max} \cos^2\left(\frac{\pi d \sin \theta}{\lambda}\right) \qquad \textbf{(37.12)}$$

Alternatively, because $\sin \theta \approx y/L$ for small values of θ in Figure 37.4, we can write Equation 37.12 in the form

$$I = I_{max} \cos^2\left(\frac{\pi d}{\lambda L}\, y\right) \qquad \textbf{(37.13)}$$

Constructive interference, which produces light intensity maxima, occurs when the quantity $\pi d y/\lambda L$ is an integral multiple of π, corresponding to $y = (\lambda L/d)m$. This is consistent with Equation 37.5.

A plot of light intensity versus $d \sin \theta$ is given in Figure 37.6. Note that the interference pattern consists of equally spaced fringes of equal intensity. Remember, however, that this result is valid only if the slit-to-screen distance L is much greater than the slit separation, and only for small values of θ.

We have seen that the interference phenomena arising from two sources depend on the relative phase of the waves at a given point. Furthermore, the phase difference at a given point depends on the path difference between the two waves. **The resultant light intensity at a point is proportional to the square of the resultant electric field at that point.** That is, the light intensity is proportional to $(E_1 + E_2)^2$. It would be incorrect to calculate the light intensity by adding the intensities of the individual waves. This procedure would give $E_1{}^2 + E_2{}^2$, which of course is not the same as $(E_1 + E_2)^2$. Note, however, that $(E_1 + E_2)^2$ has the same *average* value as $E_1{}^2 + E_2{}^2$ when the time average is taken over all values of the

phase difference between E_1 and E_2. Hence, the law of conservation of energy is not violated.

37.4 PHASOR ADDITION OF WAVES

In the preceding section, we combined two waves algebraically to obtain the resultant wave amplitude at some point on a screen. Unfortunately, this analytical procedure becomes cumbersome when we must add several wave amplitudes. Because we shall eventually be interested in combining a large number of waves, we now describe a graphical procedure for this purpose.

Let us again consider a sinusoidal wave whose electric field component is given by

$$E_1 = E_0 \sin \omega t$$

where E_0 is the wave amplitude and ω is the angular frequency. This wave can be represented graphically by a phasor of magnitude E_0 rotating about the origin counterclockwise with an angular frequency ω, as shown in Figure 37.7a. Note that the phasor makes an angle ωt with the horizontal axis. The projection of the phasor on the vertical axis represents E_1, the magnitude of the wave disturbance at some time t. Hence, as the phasor rotates in a circle, the projection E_1 oscillates along the vertical axis about the origin.

Now consider a second sinusoidal wave whose electric field component is given by

$$E_2 = E_0 \sin(\omega t + \phi)$$

This wave has the same amplitude and frequency as E_1, but its phase is ϕ with respect to E_1. The phasor representing E_2 is shown in Figure 37.7b. We can obtain the resultant wave, which is the sum of E_1 and E_2, graphically by redrawing the phasors as shown in Figure 37.7c, in which the tail of the second phasor is placed at the tip of the first. As with vector addition, the resultant phasor $\mathbf{E}_R$ runs from the tail of the first phasor to the tip of the second. Furthermore, $\mathbf{E}_R$ rotates along with the two individual phasors at the same angular frequency ω. The projection of $\mathbf{E}_R$ along the vertical axis equals the sum of the projections of the two other phasors: $E_P = E_1 + E_2$.

It is convenient to construct the phasors at $t = 0$ as in Figure 37.8. From the geometry of one of the right triangles, we see that

$$\cos \alpha = \frac{E_R/2}{E_0}$$

which gives

$$E_R = 2E_0 \cos \alpha$$

Because the sum of the two opposite interior angles equals the exterior angle ϕ, we see that $\alpha = \phi/2$; thus,

$$E_R = 2E_0 \cos\left(\frac{\phi}{2}\right)$$

Hence, the projection of the phasor $\mathbf{E}_R$ along the vertical axis at any time t is

$$E_P = E_R \sin\left(\omega t + \frac{\phi}{2}\right) = 2E_0 \cos(\phi/2) \sin\left(\omega t + \frac{\phi}{2}\right)$$

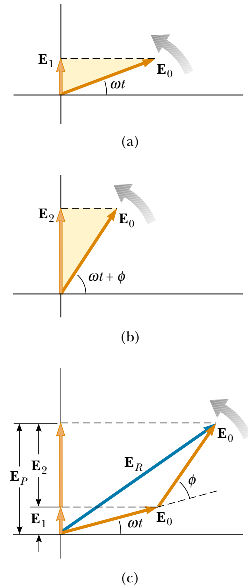


Figure 37.7 (a) Phasor diagram for the wave disturbance $E_1 = E_0 \sin \omega t$. The phasor is a vector of length E_0 rotating counterclockwise. (b) Phasor diagram for the wave $E_2 = E_0 \sin(\omega t + \phi)$. (c) The disturbance $\mathbf{E}_R$ is the resultant phasor formed from the phasors of parts (a) and (b).

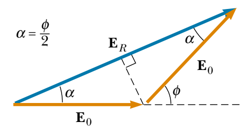


Figure 37.8 A reconstruction of the resultant phasor $\mathbf{E}_R$. From the geometry, note that $\alpha = \phi/2$.

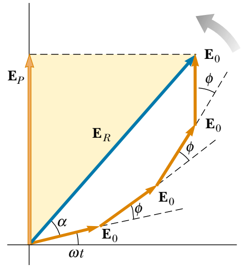


Figure 37.9 The phasor $\mathbf{E}_R$ is the resultant of four phasors of equal amplitude E_0. The phase of $\mathbf{E}_R$ with respect to the first phasor is α.

This is consistent with the result obtained algebraically, Equation 37.10. The resultant phasor has an amplitude $2E_0 \cos(\phi/2)$ and makes an angle $\phi/2$ with the first phasor. Furthermore, the average light intensity at point P, which varies as $E_P{}^2$, is proportional to $\cos^2(\phi/2)$, as described in Equation 37.11.

We can now describe how to obtain the resultant of several waves that have the same frequency:

- Represent the waves by phasors, as shown in Figure 37.9, remembering to maintain the proper phase relationship between one phasor and the next.
- The resultant phasor $\mathbf{E}_R$ is the vector sum of the individual phasors. At each instant, the projection of $\mathbf{E}_R$ along the vertical axis represents the time variation of the resultant wave. The phase angle α of the resultant wave is the angle between $\mathbf{E}_R$ and the first phasor. From Figure 37.9, drawn for four phasors, we see that the phasor of the resultant wave is given by the expression $E_P = E_R \sin(\omega t + \alpha)$.

Phasor Diagrams for Two Coherent Sources

As an example of the phasor method, consider the interference pattern produced by two coherent sources. Figure 37.10 represents the phasor diagrams for various values of the phase difference ϕ and the corresponding values of the path difference δ, which are obtained from Equation 37.8. The light intensity at a point is a maximum when $\mathbf{E}_R$ is a maximum; this occurs at $\phi = 0, 2\pi, 4\pi, \ldots$. The light intensity at some point is zero when $\mathbf{E}_R$ is zero; this occurs at $\phi = \pi, 3\pi, 5\pi, \ldots$. These results are in complete agreement with the analytical procedure described in the preceding section.

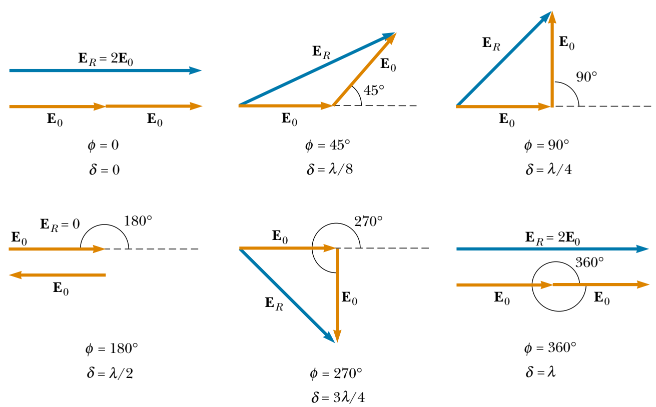


Figure 37.10 Phasor diagrams for a double-slit interference pattern. The resultant phasor $\mathbf{E}_R$ is a maximum when $\phi = 0, 2\pi, 4\pi, \ldots$ and is zero when $\phi = \pi, 3\pi, 5\pi, \ldots$.

Three-Slit Interference Pattern

Using phasor diagrams, let us analyze the interference pattern caused by three equally spaced slits. We can express the electric field components at a point P on the screen caused by waves from the individual slits as

$$E_1 = E_0 \sin \omega t$$

$$E_2 = E_0 \sin(\omega t + \phi)$$

$$E_3 = E_0 \sin(\omega t + 2\phi)$$

where ϕ is the phase difference between waves from adjacent slits. We can obtain the resultant magnitude of the electric field at point P from the phasor diagram in Figure 37.11.

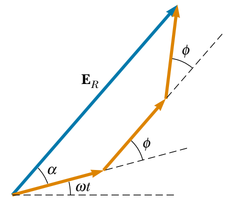


Figure 37.11 Phasor diagram for three equally spaced slits.

The phasor diagrams for various values of ϕ are shown in Figure 37.12. Note that the resultant magnitude of the electric field at P has a maximum value of $3E_0$, a condition that occurs when $\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$. These points are called *primary maxima.* Such primary maxima occur whenever the three phasors are aligned as shown in Figure 37.12a. We also find secondary maxima of amplitude E_0 occurring between the primary maxima at points where $\phi = \pm\pi, \pm 3\pi, \ldots$. For these points, the wave from one slit exactly cancels that from another slit (Fig. 37.12d). This means that only light from the third slit contributes to the resultant, which consequently has a total amplitude of E_0. Total destructive interference occurs whenever the three phasors form a closed triangle, as shown in Figure 37.12c. These points where $E_R = 0$ correspond to $\phi = \pm 2\pi/3, \pm 4\pi/3, \ldots$. You should be able to construct other phasor diagrams for values of ϕ greater than π.

Figure 37.13 shows multiple-slit interference patterns for a number of configurations. For three slits, note that the primary maxima are nine times more intense than the secondary maxima as measured by the height of the curve. This is because the intensity varies as $E_R{}^2$. For N slits, the intensity of the primary maxima is N^2 times greater than that due to a single slit. As the number of slits increases, the primary maxima increase in intensity and become narrower, while the secondary maxima decrease in intensity relative to the primary maxima. Figure 37.13 also shows that as the number of slits increases, the number of secondary maxima also increases. In fact, the number of secondary maxima is always $N - 2$, where N is the number of slits.

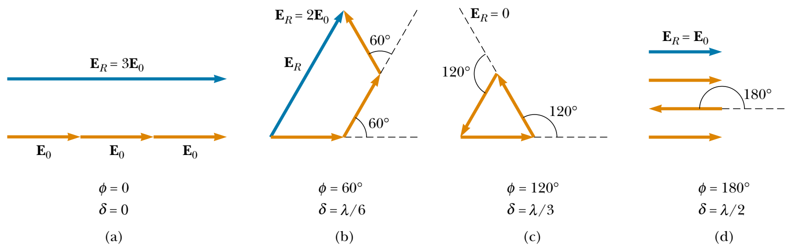


Figure 37.12 Phasor diagrams for three equally spaced slits at various values of ϕ. Note from (a) that there are primary maxima of amplitude $3E_0$ and from (d) that there are secondary maxima of amplitude E_0.

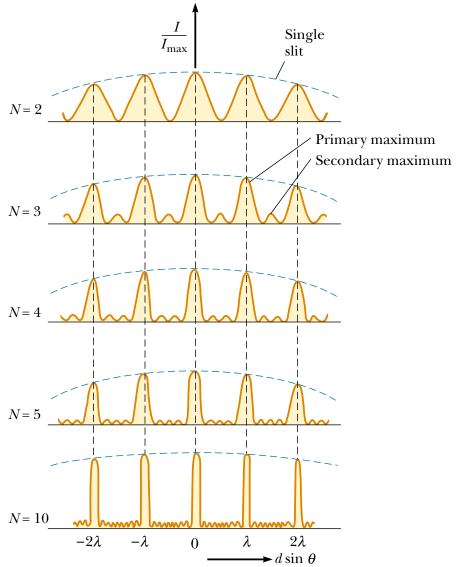


Figure 37.13 Multiple-slit interference patterns. As *N,* the number of slits, is increased, the primary maxima (the tallest peaks in each graph) become narrower but remain fixed in position, and the number of secondary maxima increases. For any value of *N,* the decrease in intensity in maxima to the left and right of the central maximum, indicated by the blue dashed arcs, is due to diffraction, which is discussed in Chapter 38.

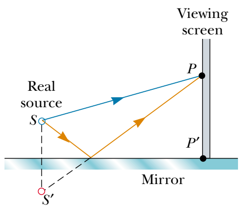


Figure 37.14 Lloyd's mirror. An interference pattern is produced at point *P* on the screen as a result of the combination of the direct ray (blue) and the reflected ray (red). The reflected ray undergoes a phase change of 180°.

Quick Quiz 37.3

Using Figure 37.13 as a model, sketch the interference pattern from six slits.

37.5 CHANGE OF PHASE DUE TO REFLECTION

Young's method for producing two coherent light sources involves illuminating a pair of slits with a single source. Another simple, yet ingenious, arrangement for producing an interference pattern with a single light source is known as *Lloyd's mirror* (Fig. 37.14). A light source is placed at point S close to a mirror, and a viewing screen is positioned some distance away at right angles to the mirror. Light waves can reach point P on the screen either by the direct path SP or by the path involving reflection from the mirror. The reflected ray can be treated as a ray originating from a virtual source at point S'. As a result, we can think of this arrangement as a double-slit source with the distance between

points S and S' comparable to length d in Figure 37.4. Hence, at observation points far from the source ($L \gg d$), we expect waves from points S and S' to form an interference pattern just like the one we see from two real coherent sources. An interference pattern is indeed observed. However, the positions of the dark and bright fringes are reversed relative to the pattern created by two real coherent sources (Young's experiment). This is because the coherent sources at points S and S' differ in phase by 180°, a phase change produced by reflection.

To illustrate this further, consider point P', the point where the mirror intersects the screen. This point is equidistant from points S and S'. If path difference alone were responsible for the phase difference, we would see a bright fringe at point P' (because the path difference is zero for this point), corresponding to the central bright fringe of the two-slit interference pattern. Instead, we observe a dark fringe at point P' because of the 180° phase change produced by reflection. In general,

> an electromagnetic wave undergoes a phase change of 180° upon reflection from a medium that has a higher index of refraction than the one in which the wave is traveling.

It is useful to draw an analogy between reflected light waves and the reflections of a transverse wave pulse on a stretched string (see Section 16.6). The reflected pulse on a string undergoes a phase change of 180° when reflected from the boundary of a denser medium, but no phase change occurs when the pulse is reflected from the boundary of a less dense medium. Similarly, an electromagnetic wave undergoes a 180° phase change when reflected from a boundary leading to an optically denser medium, but no phase change occurs when the wave is reflected from a boundary leading to a less dense medium. These rules, summarized in Figure 37.15, can be deduced from Maxwell's equations, but the treatment is beyond the scope of this text.

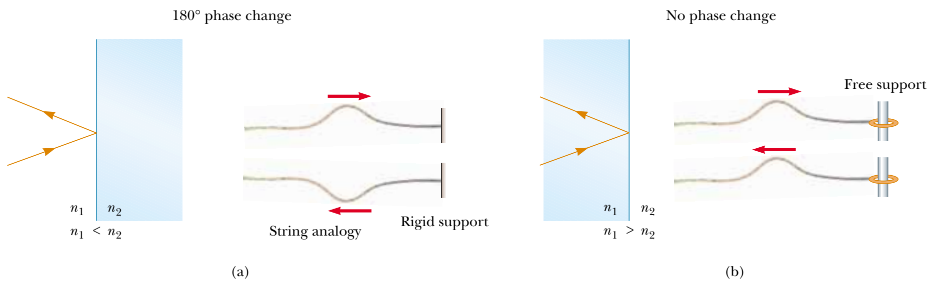


Figure 37.15 (a) For $n_1 < n_2$, a light ray traveling in medium 1 when reflected from the surface of medium 2 undergoes a 180° phase change. The same thing happens with a reflected pulse traveling along a string fixed at one end. (b) For $n_1 > n_2$, a light ray traveling in medium 1 undergoes no phase change when reflected from the surface of medium 2. The same is true of a reflected wave pulse on a string whose supported end is free to move.

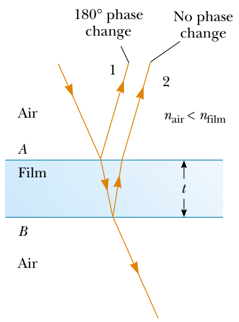


Figure 37.16 Interference in light reflected from a thin film is due to a combination of rays reflected from the upper and lower surfaces of the film.

37.6 INTERFERENCE IN THIN FILMS

Interference effects are commonly observed in thin films, such as thin layers of oil on water or the thin surface of a soap bubble. The varied colors observed when white light is incident on such films result from the interference of waves reflected from the two surfaces of the film.

Consider a film of uniform thickness t and index of refraction n, as shown in Figure 37.16. Let us assume that the light rays traveling in air are nearly normal to the two surfaces of the film. To determine whether the reflected rays interfere constructively or destructively, we first note the following facts:

- A wave traveling from a medium of index of refraction n_1 toward a medium of index of refraction n_2 undergoes a 180° phase change upon reflection when $n_2 > n_1$ and undergoes no phase change if $n_2 < n_1$.
- The wavelength of light λ_n in a medium whose refraction index is n (see Section 35.5) is

$$\lambda_n = \frac{\lambda}{n} \qquad \textbf{(37.14)}$$

where λ is the wavelength of the light in free space.

Let us apply these rules to the film of Figure 37.16, where $n_{film} > n_{air}$. Reflected ray 1, which is reflected from the upper surface (A), undergoes a phase change of 180° with respect to the incident wave. Reflected ray 2, which is reflected from the lower film surface (B), undergoes no phase change because it is reflected from a medium (air) that has a lower index of refraction. Therefore, ray 1 is 180° out of phase with ray 2, which is equivalent to a path difference of $\lambda_n/2$.

Interference in soap bubbles. The colors are due to interference between light rays reflected from the front and back surfaces of the thin film of soap making up the bubble. The color depends on the thickness of the film, ranging from black where the film is thinnest to red where it is thickest.

The brilliant colors in a peacock's feathers are due to interference. The multilayer structure of the feathers causes constructive interference for certain colors, such as blue and green. The colors change as you view a peacock's feathers from different angles. Iridescent colors of butterflies and hummingbirds are the result of similar interference effects.

However, we must also consider that ray 2 travels an extra distance $2t$ before the waves recombine in the air above surface A. If $2t = \lambda_n/2$, then rays 1 and 2 recombine in phase, and the result is constructive interference. In general, the condition for constructive interference in such situations is

$$2t = (m + \tfrac{1}{2})\lambda_n \qquad m = 0, 1, 2, \ldots \qquad \textbf{(37.15)}$$

This condition takes into account two factors: (1) the difference in path length for the two rays (the term $m\lambda_n$) and (2) the 180° phase change upon reflection (the term $\lambda_n/2$). Because $\lambda_n = \lambda/n$, we can write Equation 37.15 as

Conditions for constructive interference in thin films

$$2nt = (m + \tfrac{1}{2})\lambda \qquad m = 0, 1, 2, \ldots \qquad \textbf{(37.16)}$$

If the extra distance $2t$ traveled by ray 2 corresponds to a multiple of λ_n, then the two waves combine out of phase, and the result is destructive interference. The general equation for destructive interference is

Conditions for destructive interference in thin films

$$2nt = m\lambda \qquad m = 0, 1, 2, \ldots \qquad \textbf{(37.17)}$$

The foregoing conditions for constructive and destructive interference are valid when the medium above the top surface of the film is the same as the medium below the bottom surface. The medium surrounding the film may have a refractive index less than or greater than that of the film. In either case, the rays reflected from the two surfaces are out of phase by 180°. If the film is placed between two different media, one with $n < n_{film}$ and the other with $n > n_{film}$, then the conditions for constructive and destructive interference are reversed. In this case, either there is a phase change of 180° for both ray 1 reflecting from surface A and ray 2 reflecting from surface B, or there is no phase change for either ray; hence, the net change in relative phase due to the reflections is zero.

Quick Quiz 37.4

In Figure 37.17, where does the oil film thickness vary the least?

Figure 37.17 A thin film of oil floating on water displays interference, as shown by the pattern of colors produced when white light is incident on the film. Variations in film thickness produce the interesting color pattern. The razor blade gives one an idea of the size of the colored bands.

Newton's Rings

Another method for observing interference in light waves is to place a planoconvex lens on top of a flat glass surface, as shown in Figure 37.18a. With this arrangement, the air film between the glass surfaces varies in thickness from zero at the point of contact to some value t at point P. If the radius of curvature R of the lens is much greater than the distance r, and if the system is viewed from above using light of a single wavelength λ, a pattern of light and dark rings is observed, as shown in Figure 37.18b. These circular fringes, discovered by Newton, are called **Newton's rings.**

The interference effect is due to the combination of ray 1, reflected from the flat plate, with ray 2, reflected from the curved surface of the lens. Ray 1 undergoes a phase change of 180° upon reflection (because it is reflected from a medium of higher refractive index), whereas ray 2 undergoes no phase change (because it is reflected from a medium of lower refractive index). Hence, the conditions for constructive and destructive interference are given by Equations 37.16 and 37.17, respectively, with $n = 1$ because the film is air.

The contact point at O is dark, as seen in Figure 37.18b, because ray 1 undergoes a 180° phase change upon external reflection (from the flat surface); in con-

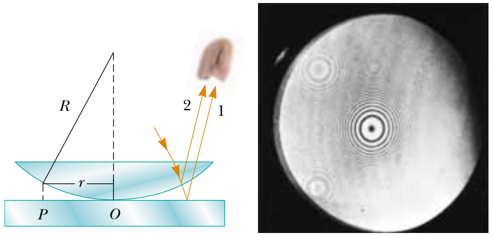


Figure 37.18 (a) The combination of rays reflected from the flat plate and the curved lens surface gives rise to an interference pattern known as Newton's rings. (b) Photograph of Newton's rings.

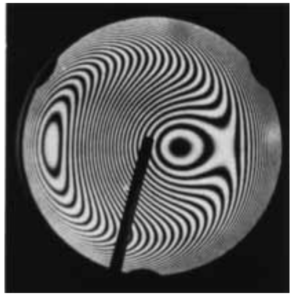


Figure 37.19 This asymmetrical interference pattern indicates imperfections in the lens of a Newton's-rings apparatus.

trast, ray 2 undergoes no phase change upon internal reflection (from the curved surface).

Using the geometry shown in Figure 37.18a, we can obtain expressions for the radii of the bright and dark bands in terms of the radius of curvature R and wavelength λ. For example, the dark rings have radii given by the expression $r \approx \sqrt{m\lambda R/n}$. The details are left as a problem for you to solve (see Problem 67). We can obtain the wavelength of the light causing the interference pattern by measuring the radii of the rings, provided R is known. Conversely, we can use a known wavelength to obtain R.

One important use of Newton's rings is in the testing of optical lenses. A circular pattern like that pictured in Figure 37.18b is obtained only when the lens is ground to a perfectly symmetric curvature. Variations from such symmetry might produce a pattern like that shown in Figure 37.19. These variations indicate how the lens must be reground and repolished to remove the imperfections.

Problem-Solving Hints

Thin-Film Interference

You should keep the following ideas in mind when you work thin-film interference problems:

- Identify the thin film causing the interference.
- The type of interference that occurs is determined by the phase relationship between the portion of the wave reflected at the upper surface of the film and the portion reflected at the lower surface.
- Phase differences between the two portions of the wave have two causes: (1) differences in the distances traveled by the two portions and (2) phase changes that may occur upon reflection.
- When the distance traveled and phase changes upon reflection are both taken into account, the interference is constructive if the equivalent path difference between the two waves is an integral multiple of λ, and it is destructive if the path difference is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, and so forth.

QuickLab

Observe the colors appearing to swirl on the surface of a soap bubble. What do you see just before a bubble bursts? Why?

EXAMPLE 37.3 Interference in a Soap Film

Calculate the minimum thickness of a soap-bubble film ($n = 1.33$) that results in constructive interference in the reflected light if the film is illuminated with light whose wavelength in free space is $\lambda = 600$ nm.

Solution The minimum film thickness for constructive interference in the reflected light corresponds to $m = 0$ in Equation 37.16. This gives $2nt = \lambda/2$, or

$$t = \frac{\lambda}{4n} = \frac{600 \text{ nm}}{4(1.33)} = 113 \text{ nm}$$

Exercise What other film thicknesses produce constructive interference?

Answer 338 nm, 564 nm, 789 nm, and so on.

EXAMPLE 37.4 Nonreflective Coatings for Solar Cells

Solar cells—devices that generate electricity when exposed to sunlight—are often coated with a transparent, thin film of silicon monoxide (SiO, $n = 1.45$) to minimize reflective losses from the surface. Suppose that a silicon solar cell ($n = 3.5$) is coated with a thin film of silicon monoxide for this purpose (Fig. 37.20). Determine the minimum film thickness that produces the least reflection at a wavelength of 550 nm, near the center of the visible spectrum.

Solution The reflected light is a minimum when rays 1 and 2 in Figure 37.20 meet the condition of destructive interference. Note that both rays undergo a 180° phase change upon reflection—ray 1 from the upper SiO surface and ray 2 from the lower SiO surface. The net change in phase due to reflection is therefore zero, and the condition for a reflection minimum requires a path difference of $\lambda_n/2$. Hence, $2t = \lambda/2n$, and the required thickness is

$$t = \frac{\lambda}{4n} = \frac{550 \text{ nm}}{4(1.45)} = 94.8 \text{ nm}$$

A typical uncoated solar cell has reflective losses as high as 30%; a SiO coating can reduce this value to about 10%. This significant decrease in reflective losses increases the cell's efficiency because less reflection means that more sunlight enters the silicon to create charge carriers in the cell. No coating can ever be made perfectly nonreflecting because the required thickness is wavelength-dependent and the incident light covers a wide range of wavelengths.

Glass lenses used in cameras and other optical instruments are usually coated with a transparent thin film to reduce or eliminate unwanted reflection and enhance the transmission of light through the lenses.

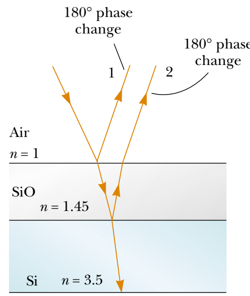


Figure 37.20 Reflective losses from a silicon solar cell are minimized by coating the surface of the cell with a thin film of silicon monoxide.

This camera lens has several coatings (of different thicknesses) that minimize reflection of light waves having wavelengths near the center of the visible spectrum. As a result, the little light that is reflected by the lens has a greater proportion of the far ends of the spectrum and appears reddish-violet.

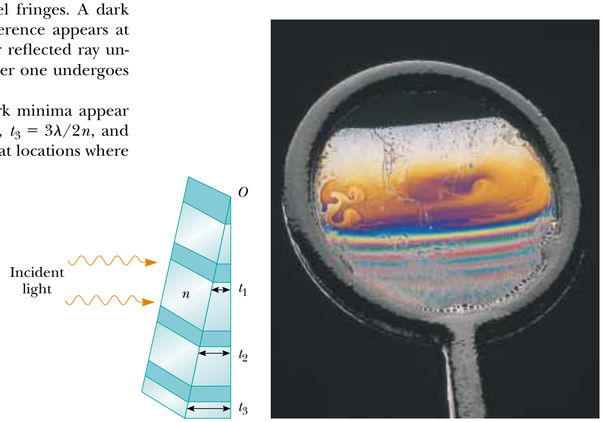
EXAMPLE 37.5 Interference in a Wedge-Shaped Film

A thin, wedge-shaped film of refractive index n is illuminated with monochromatic light of wavelength λ, as illustrated in Figure 37.21a. Describe the interference pattern observed for this case.

Solution The interference pattern, because it is created by a thin film of variable thickness surrounded by air, is a series of alternating bright and dark parallel fringes. A dark fringe corresponding to destructive interference appears at point O, the apex, because here the upper reflected ray undergoes a 180° phase change while the lower one undergoes no phase change.

According to Equation 37.17, other dark minima appear when $2nt = m\lambda$; thus, $t_1 = \lambda/2n$, $t_2 = \lambda/n$, $t_3 = 3\lambda/2n$, and so on. Similarly, the bright maxima appear at locations where the thickness satisfies Equation 37.16, $2nt = (m + \frac{1}{2})\lambda$, corresponding to thicknesses of $\lambda/4n$, $3\lambda/4n$, $5\lambda/4n$, and so on.

If white light is used, bands of different colors are observed at different points, corresponding to the different wavelengths of light (see Fig. 37.21b). This is why we see different colors in soap bubbles.



(a) (b)

Figure 37.21 (a) Interference bands in reflected light can be observed by illuminating a wedge-shaped film with monochromatic light. The darker areas correspond to regions where rays tend to cancel each other because of interference effects. (b) Interference in a vertical film of variable thickness. The top of the film appears darkest where the film is thinnest.

Optional Section

37.7 THE MICHELSON INTERFEROMETER

The **interferometer,** invented by the American physicist A. A. Michelson (1852–1931), splits a light beam into two parts and then recombines the parts to form an interference pattern. The device can be used to measure wavelengths or other lengths with great precision.

A schematic diagram of the interferometer is shown in Figure 37.22. A ray of light from a monochromatic source is split into two rays by mirror M, which is inclined at 45° to the incident light beam. Mirror M, called a *beam splitter,* transmits half the light incident on it and reflects the rest. One ray is reflected from M vertically upward toward mirror M_1, and the second ray is transmitted horizontally through M toward mirror M_2. Hence, the two rays travel separate paths L_1 and L_2. After reflecting from M_1 and M_2, the two rays eventually recombine at M to produce an interference pattern, which can be viewed through a telescope. The glass plate P, equal in thickness to mirror M, is placed in the path of the horizontal ray to ensure that the two returning rays travel the same thickness of glass.

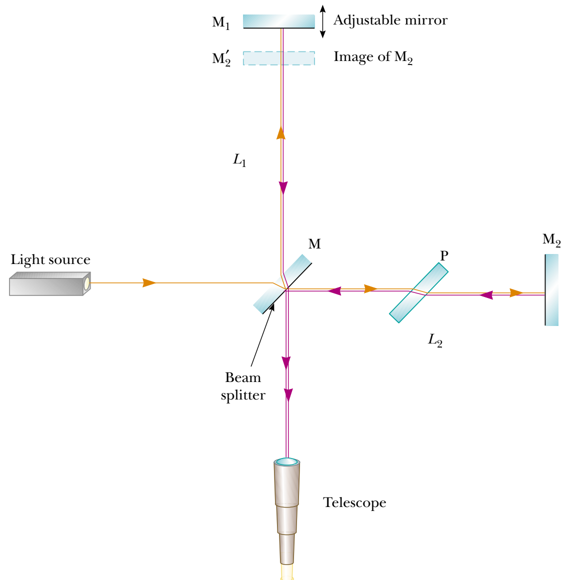


Figure 37.22 Diagram of the Michelson interferometer. A single ray of light is split into two rays by mirror M, which is called a beam splitter. The path difference between the two rays is varied with the adjustable mirror M_1. As M_1 is moved toward M, an interference pattern moves across the field of view.

The interference condition for the two rays is determined by their path length differences. When the two rays are viewed as shown, the image of M_2 produced by the mirror M is at M_2', which is nearly parallel to M_1. (Because M_1 and M_2 are not exactly perpendicular to each other, the image M_2' is at a slight angle to M_1.) Hence, the space between M_2' and M_1 is the equivalent of a wedge-shaped air film. The effective thickness of the air film is varied by moving mirror M_1 parallel to itself with a finely threaded screw adjustment. Under these conditions, the interference pattern is a series of bright and dark parallel fringes as described in Example 37.5. As M_1 is moved, the fringe pattern shifts. For example, if a dark fringe appears in the field of view (corresponding to destructive interference) and M_1 is then moved a distance $\lambda/4$ toward M, the path difference changes by $\lambda/2$ (twice the separation between M_1 and M_2'). What was a dark fringe now becomes a bright fringe. As M_1 is moved an additional distance $\lambda/4$ toward M, the bright fringe becomes a dark fringe. Thus, the fringe pattern shifts by one-half fringe each time M_1 is moved a distance $\lambda/4$. The wavelength of light is then measured by counting the number of fringe shifts for a given displacement of M_1. If the wavelength is accurately known (as with a laser beam), mirror displacements can be measured to within a fraction of the wavelength.

SUMMARY

Interference in light waves occurs whenever two or more waves overlap at a given point. A sustained interference pattern is observed if (1) the sources are coherent and (2) the sources have identical wavelengths.

In Young's double-slit experiment, two slits S_1 and S_2 separated by a distance d are illuminated by a single-wavelength light source. An interference pattern consisting of bright and dark fringes is observed on a viewing screen. The condition for bright fringes **(constructive interference)** is

$$d \sin\theta = m\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \tag{37.2}$$

The condition for dark fringes **(destructive interference)** is

$$d \sin\theta = (m + \tfrac{1}{2})\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \tag{37.3}$$

The number m is called the **order number** of the fringe.

The **intensity** at a point in the double-slit interference pattern is

$$I = I_{max} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \tag{37.12}$$

where I_{max} is the maximum intensity on the screen and the expression represents the time average.

A wave traveling from a medium of index of refraction n_1 toward a medium of index of refraction n_2 undergoes a 180° phase change upon reflection when $n_2 > n_1$ and undergoes no phase change when $n_2 < n_1$.

The condition for constructive interference in a film of thickness t and refractive index n surrounded by air is

$$2nt = (m + \tfrac{1}{2})\lambda \qquad m = 0, 1, 2, \ldots \tag{37.16}$$

where λ is the wavelength of the light in free space.

Similarly, the condition for destructive interference in a thin film is

$$2nt = m\lambda \qquad m = 0, 1, 2, \ldots \tag{37.17}$$

QUESTIONS

1. What is the necessary condition on the path length difference between two waves that interfere (a) constructively and (b) destructively?
2. Explain why two flashlights held close together do not produce an interference pattern on a distant screen.
3. If Young's double-slit experiment were performed under water, how would the observed interference pattern be affected?
4. In Young's double-slit experiment, why do we use monochromatic light? If white light is used, how would the pattern change?
5. Consider a dark fringe in an interference pattern, at which almost no light is arriving. Light from both slits is arriving at this point, but the waves are canceling. Where does the energy go?
6. An oil film on water appears brightest at the outer regions, where it is thinnest. From this information, what can you say about the index of refraction of oil relative to that of water?
7. In our discussion of thin-film interference, we looked at light *reflecting* from a thin film. Consider one light ray, the direct ray, that transmits through the film without reflecting. Consider a second ray, the reflected ray, that transmits through the first surface, reflects from the second, reflects again from the first, and then transmits out into the air, parallel to the direct ray. For normal incidence, how thick must the film be, in terms of the wavelength of light, for the outgoing rays to interfere destructively? Is it the same thickness as for reflected destructive interference?
8. Suppose that you are watching television connected to an antenna rather than a cable system. If an airplane flies near your location, you may notice wavering ghost images in the television picture. What might cause this?
9. If we are to observe interference in a thin film, why must the film not be very thick (on the order of a few wavelengths)?
10. A lens with outer radius of curvature R and index of refraction n rests on a flat glass plate, and the combination is illuminated with white light from above. Is there a dark spot or a light spot at the center of the lens? What does it mean if the observed rings are noncircular?
11. Why is the lens on a high-quality camera coated with a thin film?
12. Why is it so much easier to perform interference experiments with a laser than with an ordinary light source?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 37.1 Conditions for Interference
Section 37.2 Young's Double-Slit Experiment

1. A laser beam (λ = 632.8 nm) is incident on two slits 0.200 mm apart. How far apart are the bright interference fringes on a screen 5.00 m away from the slits?
2. A Young's interference experiment is performed with monochromatic light. The separation between the slits is 0.500 mm, and the interference pattern on a screen 3.30 m away shows the first maximum 3.40 mm from the center of the pattern. What is the wavelength?
3. WEB Two radio antennas separated by 300 m as shown in Figure P37.3 simultaneously broadcast identical signals at the same wavelength. A radio in a car traveling due north receives the signals. (a) If the car is at the position of the second maximum, what is the wavelength of the signals? (b) How much farther must the car travel to encounter the next minimum in reception? (*Note:* Do not use the small-angle approximation in this problem.)

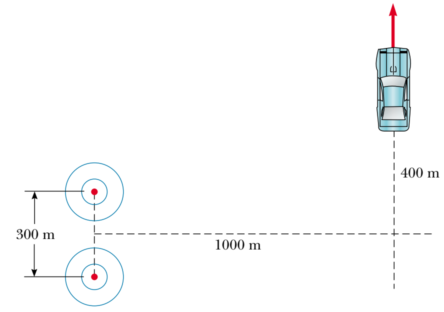


Figure P37.3

4. In a location where the speed of sound is 354 m/s, a 2 000-Hz sound wave impinges on two slits 30.0 cm apart. (a) At what angle is the first maximum located? (b) If the sound wave is replaced by 3.00-cm microwaves, what slit separation gives the same angle for the first maximum? (c) If the slit separation is 1.00 μm, what frequency of light gives the same first maximum angle?
5. WEB Young's double-slit experiment is performed with 589-nm light and a slit-to-screen distance of 2.00 m. The tenth interference minimum is observed 7.26 mm from the central maximum. Determine the spacing of the slits.
6. The two speakers of a boom box are 35.0 cm apart. A single oscillator makes the speakers vibrate in phase at a frequency of 2.00 kHz. At what angles, measured from the perpendicular bisector of the line joining the speakers, would a distant observer hear maximum sound intensity? minimum sound intensity? (Take the speed of sound as 340 m/s.)
7. A pair of narrow, parallel slits separated by 0.250 mm are illuminated by green light (λ = 546.1 nm). The interference pattern is observed on a screen 1.20 m away from the plane of the slits. Calculate the distance (a) from the central maximum to the first bright region on either side of the central maximum and (b) between the first and second dark bands.
8. Light with a wavelength of 442 nm passes through a double-slit system that has a slit separation d = 0.400 mm. Determine how far away a screen must be placed so that a dark fringe appears directly opposite both slits, with just one bright fringe between them.
9. A riverside warehouse has two open doors, as illustrated in Figure P37.9. Its walls are lined with sound-absorbing material. A boat on the river sounds its horn. To person A, the sound is loud and clear. To person B, the sound is barely audible. The principal wavelength of the sound waves is 3.00 m. Assuming that person B is at the position of the first minimum, determine the distance between the doors, center to center.

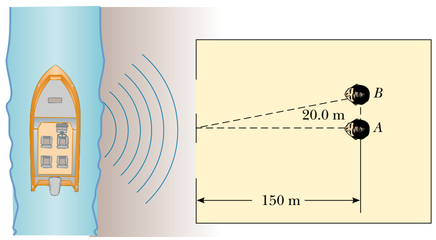


Figure P37.9

10. Two slits are separated by 0.320 mm. A beam of 500-nm light strikes the slits, producing an interference pattern. Determine the number of maxima observed in the angular range $-30.0° < \theta < 30.0°$.

11. In Figure 37.4 let $L = 1.20$ m and $d = 0.120$ mm, and assume that the slit system is illuminated with monochromatic 500-nm light. Calculate the phase difference between the two wavefronts arriving at point P when (a) $\theta = 0.500°$ and (b) $y = 5.00$ mm. (c) What is the value of θ for which the phase difference is 0.333 rad? (d) What is the value of θ for which the path difference is $\lambda/4$?

12. Coherent light rays of wavelength λ strike a pair of slits separated by distance d at an angle of θ_1, as shown in Figure P37.12. If an interference maximum is formed at an angle of θ_2 a great distance from the slits, show that $d(\sin\theta_2 - \sin\theta_1) = m\lambda$, where m is an integer.

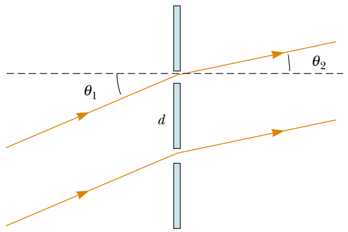


Figure P37.12

13. In the double-slit arrangement of Figure 37.4, $d = 0.150$ mm, $L = 140$ cm, $\lambda = 643$ nm, and $y = 1.80$ cm. (a) What is the path difference δ for the rays from the two slits arriving at point P? (b) Express this path difference in terms of λ. (c) Does point P correspond to a maximum, a minimum, or an intermediate condition?

Section 37.3 Intensity Distribution of the Double-Slit Interference Pattern

14. The intensity on the screen at a certain point in a double-slit interference pattern is 64.0% of the maximum value. (a) What minimum phase difference (in radians) between sources produces this result? (b) Express this phase difference as a path difference for 486.1-nm light.

WEB **15.** In Figure 37.4, let $L = 120$ cm and $d = 0.250$ cm. The slits are illuminated with coherent 600-nm light. Calculate the distance y above the central maximum for which the average intensity on the screen is 75.0% of the maximum.

16. Two slits are separated by 0.180 mm. An interference pattern is formed on a screen 80.0 cm away by 656.3-nm light. Calculate the fraction of the maximum intensity 0.600 cm above the central maximum.

17. Two narrow parallel slits separated by 0.850 mm are illuminated by 600-nm light, and the viewing screen is 2.80 m away from the slits. (a) What is the phase difference between the two interfering waves on a screen at a point 2.50 mm from the central bright fringe? (b) What is the ratio of the intensity at this point to the intensity at the center of a bright fringe?

18. Monochromatic coherent light of amplitude E_0 and angular frequency ω passes through three parallel slits each separated by a distance d from its neighbor. (a) Show that the time-averaged intensity as a function of the angle θ is

$$I(\theta) = I_{\max}\left[1 + 2\cos\left(\frac{2\pi d\sin\theta}{\lambda}\right)\right]^2$$

(b) Determine the ratio of the intensities of the primary and secondary maxima.

Section 37.4 Phasor Addition of Waves

19. Marie Cornu invented phasors in about 1880. This problem helps you to see their utility. Find the amplitude and phase constant of the sum of two waves represented by the expressions

$$E_1 = (12.0 \text{ kN/C}) \sin(15x - 4.5t)$$

and

$$E_2 = (12.0 \text{ kN/C}) \sin(15x - 4.5t + 70°)$$

(a) by using a trigonometric identity (see Appendix B) and (b) by representing the waves by phasors. (c) Find the amplitude and phase constant of the sum of the three waves represented by

$$E_1 = (12.0 \text{ kN/C}) \sin(15x - 4.5t + 70°)$$

$$E_2 = (15.5 \text{ kN/C}) \sin(15x - 4.5t - 80°)$$

and

$$E_3 = (17.0 \text{ kN/C}) \sin(15x - 4.5t + 160°)$$

20. The electric fields from three coherent sources are described by $E_1 = E_0 \sin\omega t$, $E_2 = E_0\sin(\omega t + \phi)$, and $E_3 = E_0 \sin(\omega t + 2\phi)$. Let the resultant field be represented by $E_P = E_R \sin(\omega t + \alpha)$. Use phasors to find E_R and α when (a) $\phi = 20.0°$, (b) $\phi = 60.0°$, and (c) $\phi = 120°$. (d) Repeat when $\phi = (3\pi/2)$ rad.

WEB **21.** Determine the resultant of the two waves $E_1 = 6.0\sin(100\pi t)$ and $E_2 = 8.0\sin(100\pi t + \pi/2)$.

22. Suppose that the slit openings in a Young's double-slit experiment have different sizes so that the electric fields and the intensities from each slit are different. If $E_1 = E_{01}\sin(\omega t)$ and $E_2 = E_{02}\sin(\omega t + \phi)$, show that the resultant electric field is $E = E_0 \sin(\omega t + \theta)$, where

$$E_0 = \sqrt{E_{01}{}^2 + E_{02}{}^2 + 2E_{01}E_{02}\cos\phi}$$

and

$$\sin\theta = \frac{E_{02}\sin\phi}{E_0}$$

23. Use phasors to find the resultant (magnitude and phase angle) of two fields represented by $E_1 = 12\sin\omega t$ and $E_2 = 18\sin(\omega t + 60°)$. (Note that in this case the amplitudes of the two fields are unequal.)

24. Two coherent waves are described by the expressions

$$E_1 = E_0\sin\left(\frac{2\pi x_1}{\lambda} - 2\pi f t + \frac{\pi}{6}\right)$$

$$E_2 = E_0\sin\left(\frac{2\pi x_2}{\lambda} - 2\pi f t + \frac{\pi}{8}\right)$$

Determine the relationship between x_1 and x_2 that produces constructive interference when the two waves are superposed.

25. When illuminated, four equally spaced parallel slits act as multiple coherent sources, each differing in phase from the adjacent one by an angle ϕ. Use a phasor diagram to determine the smallest value of ϕ for which the resultant of the four waves (assumed to be of equal amplitude) is zero.

26. Sketch a phasor diagram to illustrate the resultant of $E_1 = E_{01}\sin\omega t$ and $E_2 = E_{02}\sin(\omega t + \phi)$, where $E_{02} = 1.50E_{01}$ and $\pi/6 \le \phi \le \pi/3$. Use the sketch and the law of cosines to show that, for two coherent waves, the resultant intensity can be written in the form $I_R = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$.

27. Consider N coherent sources described by $E_1 = E_0\sin(\omega t + \phi)$, $E_2 = E_0\sin(\omega t + 2\phi)$, $E_3 = E_0\sin(\omega t + 3\phi)$, . . . , $E_N = E_0\sin(\omega t + N\phi)$. Find the minimum value of ϕ for which $E_R = E_1 + E_2 + E_3 + \ldots + E_N$ is zero.

Section 37.5 Change of Phase Due to Reflection
Section 37.6 Interference in Thin Films

28. A soap bubble ($n = 1.33$) is floating in air. If the thickness of the bubble wall is 115 nm, what is the wavelength of the light that is most strongly reflected?

29. An oil film ($n = 1.45$) floating on water is illuminated by white light at normal incidence. The film is 280 nm thick. Find (a) the dominant observed color in the reflected light and (b) the dominant color in the transmitted light. Explain your reasoning.

30. A thin film of oil ($n = 1.25$) is located on a smooth, wet pavement. When viewed perpendicular to the pavement, the film appears to be predominantly red (640 nm) and has no blue color (512 nm). How thick is the oil film?

31. A possible means for making an airplane invisible to radar is to coat the plane with an antireflective polymer. If radar waves have a wavelength of 3.00 cm and the index of refraction of the polymer is $n = 1.50$, how thick would you make the coating?

32. A material having an index of refraction of 1.30 is used to coat a piece of glass ($n = 1.50$). What should be the minimum thickness of this film if it is to minimize reflection of 500-nm light?

33. A film of MgF_2 ($n = 1.38$) having a thickness of 1.00×10^{-5} cm is used to coat a camera lens. Are any wavelengths in the visible spectrum intensified in the reflected light?

34. Astronomers observe the chromosphere of the Sun with a filter that passes the red hydrogen spectral line of wavelength 656.3 nm, called the H_α line. The filter consists of a transparent dielectric of thickness d held between two partially aluminized glass plates. The filter is held at a constant temperature. (a) Find the minimum value of d that produces maximum transmission of perpendicular H_α light, if the dielectric has an index of refraction of 1.378. (b) Assume that the temperature of the filter increases above its normal value and that its index of refraction does not change significantly. What happens to the transmitted wavelength? (c) The dielectric will also pass what near-visible wavelength? One of the glass plates is colored red to absorb this light.

35. A beam of 580-nm light passes through two closely spaced glass plates, as shown in Figure P37.35. For what minimum nonzero value of the plate separation d is the transmitted light bright?

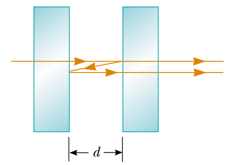


Figure P37.35

36. When a liquid is introduced into the air space between the lens and the plate in a Newton's-rings apparatus, the diameter of the tenth ring changes from 1.50 to 1.31 cm. Find the index of refraction of the liquid.

WEB **37.** An air wedge is formed between two glass plates separated at one edge by a very fine wire, as shown in Figure P37.37. When the wedge is illuminated from above by 600-nm light, 30 dark fringes are observed. Calculate the radius of the wire.

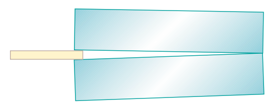


Figure P37.37 Problems 37 and 38.

38. Two rectangular flat glass plates ($n = 1.52$) are in contact along one end and separated along the other end by a sheet of paper 4.00×10^{-3} cm thick (see Fig. P37.37). The top plate is illuminated by monochromatic light ($\lambda = 546.1$ nm). Calculate the number of dark parallel bands crossing the top plate (include the dark band at zero thickness along the edge of contact between the two plates).

39. Two glass plates 10.0 cm long are in contact at one end and separated at the other end by a thread 0.050 0 mm in diameter. Light containing the two wavelengths 400 nm and 600 nm is incident perpendicularly. At what distance from the contact point is the next dark fringe?

(Optional)
Section 37.7 The Michelson Interferometer

40. Light of wavelength 550.5 nm is used to calibrate a Michelson interferometer, and mirror M_1 is moved 0.180 mm. How many dark fringes are counted?

41. Mirror M_1 in Figure 37.22 is displaced a distance ΔL. During this displacement, 250 fringe reversals (formation of successive dark or bright bands) are counted. The light being used has a wavelength of 632.8 nm. Calculate the displacement ΔL.

42. Monochromatic light is beamed into a Michelson interferometer. The movable mirror is displaced 0.382 mm; this causes the interferometer pattern to reproduce itself 1 700 times. Determine the wavelength and the color of the light.

43. One leg of a Michelson interferometer contains an evacuated cylinder 3.00 cm long having glass plates on each end. A gas is slowly leaked into the cylinder until a pressure of 1 atm is reached. If 35 bright fringes pass on the screen when light of wavelength 633 nm is used, what is the index of refraction of the gas?

44. One leg of a Michelson interferometer contains an evacuated cylinder of length L having glass plates on each end. A gas is slowly leaked into the cylinder until a pressure of 1 atm is reached. If N bright fringes pass on the screen when light of wavelength λ is used, what is the index of refraction of the gas?

ADDITIONAL PROBLEMS

45. One radio transmitter A operating at 60.0 MHz is 10.0 m from another similar transmitter B that is 180° out of phase with transmitter A. How far must an observer move from transmitter A toward transmitter B along the line connecting A and B to reach the nearest point where the two beams are in phase?

46. Raise your hand and hold it flat. Think of the space between your index finger and your middle finger as one slit, and think of the space between middle finger and ring finger as a second slit. (a) Consider the interference resulting from sending coherent visible light perpendicularly through this pair of openings. Compute an order-of-magnitude estimate for the angle between adjacent zones of constructive interference. (b) To make the angles in the interference pattern easy to measure with a plastic protractor, you should use an electromagnetic wave with frequency of what order of magnitude? How is this wave classified on the electromagnetic spectrum?

47. In a Young's double-slit experiment using light of wavelength λ, a thin piece of Plexiglas having index of refraction n covers one of the slits. If the center point on the screen is a dark spot instead of a bright spot, what is the minimum thickness of the Plexiglas?

48. **Review Problem.** A flat piece of glass is held stationary and horizontal above the flat top end of a 10.0-cm-long vertical metal rod that has its lower end rigidly fixed. The thin film of air between the rod and glass is observed to be bright by reflected light when it is illuminated by light of wavelength 500 nm. As the temperature is slowly increased by 25.0°C, the film changes from bright to dark and back to bright 200 times. What is the coefficient of linear expansion of the metal?

49. A certain crude oil has an index of refraction of 1.25. A ship dumps 1.00 m^3 of this oil into the ocean, and the oil spreads into a thin uniform slick. If the film produces a first-order maximum of light of wavelength 500 nm normally incident on it, how much surface area of the ocean does the oil slick cover? Assume that the index of refraction of the ocean water is 1.34.

50. Interference effects are produced at point P on a screen as a result of direct rays from a 500-nm source and reflected rays off the mirror, as shown in Figure P37.50. If the source is 100 m to the left of the screen and 1.00 cm above the mirror, find the distance y (in millimeters) to the first dark band above the mirror.

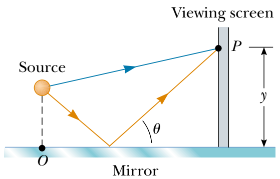


Figure P37.50

51. Astronomers observed a 60.0-MHz radio source both directly and by reflection from the sea. If the receiving dish is 20.0 m above sea level, what is the angle of the radio source above the horizon at first maximum?

52. The waves from a radio station can reach a home receiver by two paths. One is a straight-line path from transmitter to home, a distance of 30.0 km. The second path is by reflection from the ionosphere (a layer of ionized air molecules high in the atmosphere). Assume that this reflection takes place at a point midway between the receiver and the transmitter. The wavelength broadcast by the radio station is 350 m. Find the minimum height of the ionospheric layer that produces destructive interference between the direct and reflected beams. (Assume that no phase changes occur on reflection.)

53. Measurements are made of the intensity distribution in a Young's interference pattern (see Fig. 37.6). At a particular value of y, it is found that $I/I_{max} = 0.810$ when 600-nm light is used. What wavelength of light should be used if the relative intensity at the same location is to be reduced to 64.0%?

54. In a Young's interference experiment, the two slits are separated by 0.150 mm, and the incident light includes light of wavelengths $\lambda_1 = 540$ nm and $\lambda_2 = 450$ nm. The overlapping interference patterns are formed on a screen 1.40 m from the slits. Calculate the minimum distance from the center of the screen to the point where a bright line of the λ_1 light coincides with a bright line of the λ_2 light.

55. An air wedge is formed between two glass plates in contact along one edge and slightly separated at the opposite edge. When the plates are illuminated with monochromatic light from above, the reflected light has 85 dark fringes. Calculate the number of dark fringes that would appear if water ($n = 1.33$) were to replace the air between the plates.

56. Our discussion of the techniques for determining constructive and destructive interference by reflection from a thin film in air has been confined to rays striking the film at nearly normal incidence. Assume that a ray is incident at an angle of 30.0° (relative to the normal) on a film with an index of refraction of 1.38. Calculate the minimum thickness for constructive interference if the light is sodium light with a wavelength of 590 nm.

57. The condition for constructive interference by reflection from a thin film in air as developed in Section 37.6 assumes nearly normal incidence. Show that if the light is incident on the film at a nonzero angle ϕ_1 (relative to the normal), then the condition for constructive interference is $2nt \cos\theta_2 = (m + \frac{1}{2})\lambda$, where θ_2 is the angle of refraction.

58. (a) Both sides of a uniform film that has index of refraction n and thickness d are in contact with air. For normal incidence of light, an intensity minimum is observed in the reflected light at λ_2, and an intensity maximum is observed at λ_1, where $\lambda_1 > \lambda_2$. If no intensity minima are observed between λ_1 and λ_2, show that the integer m in Equations 37.16 and 37.17 is given by $m = \lambda_1/2(\lambda_1 - \lambda_2)$. (b) Determine the thickness of the film if $n = 1.40$, $\lambda_1 = 500$ nm, and $\lambda_2 = 370$ nm.

59. Figure P37.59 shows a radio wave transmitter and a receiver separated by a distance d and located a distance h above the ground. The receiver can receive signals both directly from the transmitter and indirectly from signals that reflect off the ground. Assume that the ground is level between the transmitter and receiver and that a 180° phase shift occurs upon reflection. Determine the longest wavelengths that interfere (a) constructively and (b) destructively.

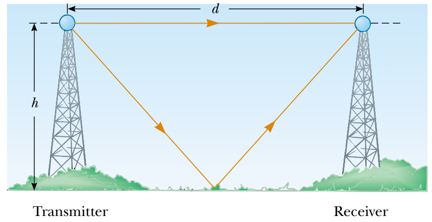


Figure P37.59

60. Consider the double-slit arrangement shown in Figure P37.60, where the separation d is 0.300 mm and the distance L is 1.00 m. A sheet of transparent plastic ($n = 1.50$) 0.050 0 mm thick (about the thickness of this page) is placed over the upper slit. As a result, the central maximum of the interference pattern moves upward a distance y'. Find y'.

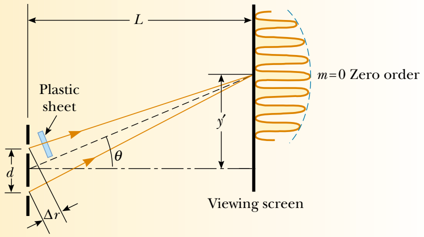


Figure P37.60 Problems 60 and 61.

61. Consider the double-slit arrangement shown in Figure P37.60, where the slit separation is d and the slit to screen distance is L. A sheet of transparent plastic having an index of refraction n and thickness t is placed over the upper slit. As a result, the central maximum of the interference pattern moves upward a distance y'. Find y'.

62. Waves broadcast by a 1 500-kHz radio station arrive at a home receiver by two paths. One is a direct path, and the other is from reflection off an airplane directly above the receiver. The airplane is approximately 100 m above the receiver, and the direct distance from station to home is 20.0 km. What is the precise height of the airplane if destructive interference is occurring? (Assume that no phase change occurs on reflection.)

63. In a Newton's-rings experiment, a plano-convex glass ($n = 1.52$) lens having a diameter of 10.0 cm is placed on a flat plate, as shown in Figure 37.18a. When 650-nm light is incident normally, 55 bright rings are observed, with the last ring right on the edge of the lens. (a) What is the radius of curvature of the convex surface of the lens? (b) What is the focal length of the lens?

64. A piece of transparent material having an index of refraction n is cut into the shape of a wedge, as shown in Figure P37.64. The angle of the wedge is small, and monochromatic light of wavelength λ is normally incident from above. If the height of the wedge is h and the width is ℓ, show that bright fringes occur at the positions $x = \lambda\ell(m + \frac{1}{2})/2hn$ and that dark fringes occur at the positions $x = \lambda\ell m/2hn$, where $m = 0, 1, 2, \ldots$ and x is measured as shown.

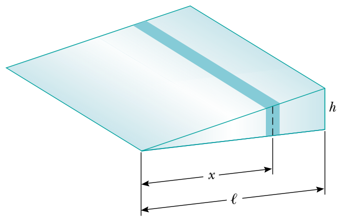


Figure P37.64

65. Use phasor addition to find the resultant amplitude and phase constant when the following three harmonic functions are combined: $E_1 = \sin(\omega t + \pi/6)$, $E_2 = 3.0 \sin(\omega t + 7\pi/2)$, $E_3 = 6.0 \sin(\omega t + 4\pi/3)$.

66. A plano-convex lens having a radius of curvature of $r = 4.00$ m is placed on a concave reflecting surface whose radius of curvature is $R = 12.0$ m, as shown in Figure P37.66. Determine the radius of the 100th bright ring if 500-nm light is incident normal to the flat surface of the lens.

67. A plano-convex lens has index of refraction n. The curved side of the lens has radius of curvature R and rests on a flat glass surface of the same index of refraction, with a film of index n_{film} between them. The lens is illuminated from above by light of wavelength λ. Show that the dark Newton's rings have radii given approximately by

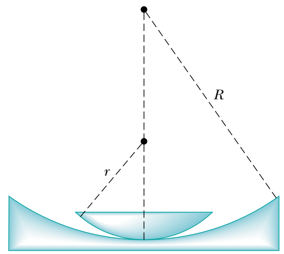


Figure P37.66

$$r \cong \sqrt{m\lambda R/n_{film}}$$

where m is an integer and r is much less than R.

68. A soap film ($n = 1.33$) is contained within a rectangular wire frame. The frame is held vertically so that the film drains downward and becomes thicker at the bottom than at the top, where the thickness is essentially zero. The film is viewed in white light with near-normal incidence, and the first violet ($\lambda = 420$ nm) interference band is observed 3.00 cm from the top edge of the film. (a) Locate the first red ($\lambda = 680$ nm) interference band. (b) Determine the film thickness at the positions of the violet and red bands. (c) What is the wedge angle of the film?

69. Interference fringes are produced using Lloyd's mirror and a 606-nm source, as shown in Figure 37.14. Fringes 1.20 mm apart are formed on a screen 2.00 m from the real source S. Find the vertical distance h of the source above the reflecting surface.

70. Slit 1 of a double slit is wider than slit 2, so that the light from slit 1 has an amplitude 3.00 times that of the light from slit 2. Show that Equation 37.11 is replaced by the equation $I = (4I_{max}/9)(1 + 3\cos^2\phi/2)$ for this situation.

Answers to Quick Quizzes

37.1 Bands of light along the orange lines interspersed with dark bands running along the dashed black lines.

37.2 At location *B*. At *A*, which is on a line of constructive interference, the water surface undulates so much that you probably could not determine the depth. Because *B* is on a line of destructive interference, the water level does not change, and you should be able to read the ruler easily.

37.3 The graph is shown in Figure QQA37.1. The width of the primary maxima is slightly narrower than the $N = 5$ primary width but wider than the $N = 10$ primary width. Because $N = 6$, the secondary maxima are $\frac{1}{36}$ as intense as the primary maxima.

37.4 The greater the variation in thickness, the narrower the bands of color (like the lines on a topographic map). The widest bands are the gold ones along the left edge of the photograph and at the bottom right corner of the razor blade. Thus, the thickness of the oil film changes most slowly with position in these areas.

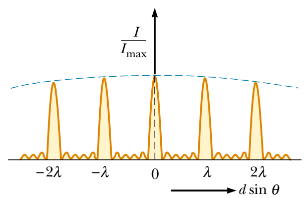


Figure QQA37.1

PUZZLER

At sunset, the sky is ablaze with brilliant reds, pinks, and oranges. Yet, we wouldn't be able to see this sunset were it not for the fact that someone else is simultaneously seeing a blue sky. What causes the beautiful colors of a sunset, and why must the sky be blue somewhere else for us to enjoy one? *(© W. A. Banaszewski/Visuals Unlimited)*

chapter 38

Diffraction and Polarization

Chapter Outline

When light waves pass through a small aperture, an interference pattern is observed rather than a sharp spot of light. This behavior indicates that light, once it has passed through the aperture, spreads beyond the narrow path defined by the aperture into regions that would be in shadow if light traveled in straight lines. Other waves, such as sound waves and water waves, also have this property of spreading when passing through apertures or by sharp edges. This phenomenon, known as diffraction, can be described only with a wave model for light.

In Chapter 34, we learned that electromagnetic waves are transverse. That is, the electric and magnetic field vectors are perpendicular to the direction of wave propagation. In this chapter, we see that under certain conditions these transverse waves can be polarized in various ways.

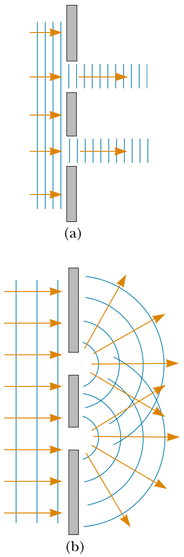


Figure 38.1 (a) If light waves did not spread out after passing through the slits, no interference would occur. (b) The light waves from the two slits overlap as they spread out, filling what we expect to be shadowed regions with light and producing interference fringes.

38.1 INTRODUCTION TO DIFFRACTION

In Section 37.2 we learned that an interference pattern is observed on a viewing screen when two slits are illuminated by a single-wavelength light source. If the light traveled only in its original direction after passing through the slits, as shown in Figure 38.1a, the waves would not overlap and no interference pattern would be seen. Instead, Huygens's principle requires that the waves spread out from the slits as shown in Figure 38.1b. In other words, the light deviates from a straight-line path and enters the region that would otherwise be shadowed. As noted in Section 35.1, this divergence of light from its initial line of travel is called **diffraction.**

In general, diffraction occurs when waves pass through small openings, around obstacles, or past sharp edges, as shown in Figure 38.2. When an opaque object is placed between a point source of light and a screen, no sharp boundary exists on the screen between a shadowed region and an illuminated region. The illuminated region above the shadow of the object contains alternating light and dark fringes. Such a display is called a **diffraction pattern.**

Figure 38.3 shows a diffraction pattern associated with the shadow of a penny. A bright spot occurs at the center, and circular fringes extend outward from the shadow's edge. We can explain the central bright spot only by using the wave theory of light, which predicts constructive interference at this point. From the viewpoint of geometric optics (in which light is viewed as rays traveling in straight lines), we expect the center of the shadow to be dark because that part of the viewing screen is completely shielded by the penny.

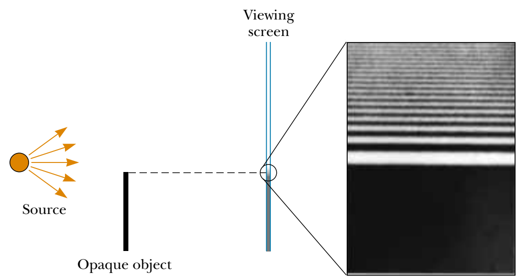


Figure 38.2 Light from a small source passes by the edge of an opaque object. We might expect no light to appear on the screen below the position of the edge of the object. In reality, light bends around the top edge of the object and enters this region. Because of these effects, a diffraction pattern consisting of bright and dark fringes appears in the region above the edge of the object.

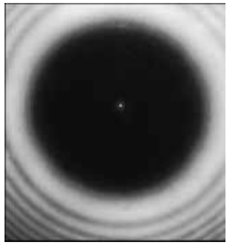


Figure 38.3 Diffraction pattern created by the illumination of a penny, with the penny positioned midway between screen and light source.

It is interesting to point out an historical incident that occurred shortly before the central bright spot was first observed. One of the supporters of geometric optics, Simeon Poisson, argued that if Augustin Fresnel's wave theory of light were valid, then a central bright spot should be observed in the shadow of a circular object illuminated by a point source of light. To Poisson's astonishment, the spot was observed by Dominique Arago shortly thereafter. Thus, Poisson's prediction reinforced the wave theory rather than disproving it.

In this chapter we restrict our attention to **Fraunhofer diffraction,** which occurs, for example, when all the rays passing through a narrow slit are approximately parallel to one another. This can be achieved experimentally either by placing the screen far from the opening used to create the diffraction or by using a converging lens to focus the rays once they pass through the opening, as shown in Figure 38.4a. A bright fringe is observed along the axis at $\theta = 0$, with alternating dark and bright fringes occurring on either side of the central bright one. Figure 38.4b is a photograph of a single-slit Fraunhofer diffraction pattern.

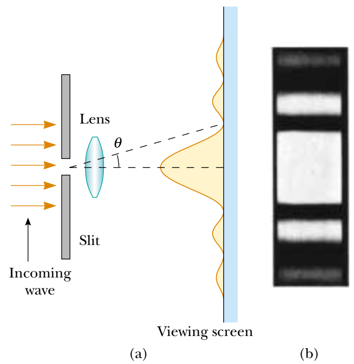


Figure 38.4 (a) Fraunhofer diffraction pattern of a single slit. The pattern consists of a central bright fringe flanked by much weaker maxima alternating with dark fringes (drawing not to scale). (b) Photograph of a single-slit Fraunhofer diffraction pattern.

38.2 DIFFRACTION FROM NARROW SLITS

Until now, we have assumed that slits are point sources of light. In this section, we abandon that assumption and see how the finite width of slits is the basis for understanding Fraunhofer diffraction.

We can deduce some important features of this phenomenon by examining waves coming from various portions of the slit, as shown in Figure 38.5. According to Huygens's principle, **each portion of the slit acts as a source of light waves.** Hence, light from one portion of the slit can interfere with light from another portion, and the resultant light intensity on a viewing screen depends on the direction θ.

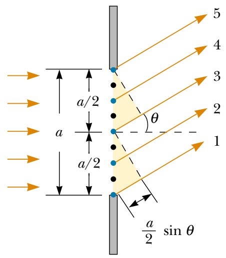


Figure 38.5 Diffraction of light by a narrow slit of width a. Each portion of the slit acts as a point source of light waves. The path difference between rays 1 and 3 or between rays 2 and 4 is $(a/2)\sin\theta$ (drawing not to scale).

To analyze the diffraction pattern, it is convenient to divide the slit into two halves, as shown in Figure 38.5. Keeping in mind that all the waves are in phase as they leave the slit, consider rays 1 and 3. As these two rays travel toward a viewing screen far to the right of the figure, ray 1 travels farther than ray 3 by an amount equal to the path difference $(a/2)\sin\theta$, where a is the width of the slit. Similarly, the path difference between rays 2 and 4 is also $(a/2)\sin\theta$. If this path difference is exactly half a wavelength (corresponding to a phase difference of 180°), then the two waves cancel each other and destructive interference results. This is true for any two rays that originate at points separated by half the slit width because the phase difference between two such points is 180°. Therefore, waves from the upper half of the slit interfere destructively with waves from the lower half when

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

or when

$$\sin\theta = \frac{\lambda}{a}$$

If we divide the slit into four equal parts and use similar reasoning, we find that the viewing screen is also dark when

$$\sin\theta = \frac{2\lambda}{a}$$

Likewise, we can divide the slit into six equal parts and show that darkness occurs on the screen when

$$\sin\theta = \frac{3\lambda}{a}$$

Therefore, the general condition for destructive interference is

Condition for destructive interference

$$\sin\theta = m\frac{\lambda}{a} \qquad m = \pm 1, \pm 2, \pm 3, \ldots \tag{38.1}$$

This equation gives the values of θ for which the diffraction pattern has zero light intensity—that is, when a dark fringe is formed. However, it tells us nothing about the variation in light intensity along the screen. The general features of the intensity distribution are shown in Figure 38.6. A broad central bright fringe is observed; this fringe is flanked by much weaker bright fringes alternating with dark fringes. The various dark fringes occur at the values of θ that satisfy Equation 38.1. Each bright-fringe peak lies approximately halfway between its bordering dark-fringe minima. Note that the central bright maximum is twice as wide as the secondary maxima.

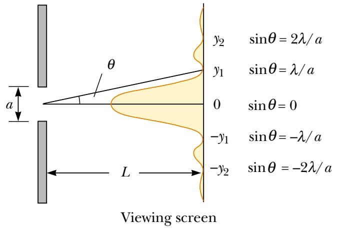
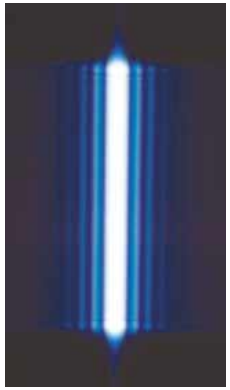


Figure 38.6 Intensity distribution for a Fraunhofer diffraction pattern from a single slit of width a. The positions of two minima on each side of the central maximum are labeled (drawing not to scale).



The diffraction pattern that appears on a screen when light passes through a narrow vertical slit. The pattern consists of a broad central bright fringe and a series of less intense and narrower side bright fringes.

Quick Quiz 38.1

If the door to an adjoining room is slightly ajar, why is it that you can hear sounds from the room but cannot see much of what is happening in the room?

EXAMPLE 38.1 Where Are the Dark Fringes?

Light of wavelength 580 nm is incident on a slit having a width of 0.300 mm. The viewing screen is 2.00 m from the slit. Find the positions of the first dark fringes and the width of the central bright fringe.

Solution The two dark fringes that flank the central bright fringe correspond to $m = \pm 1$ in Equation 38.1. Hence, we find that

$$\sin\theta = \pm\frac{\lambda}{a} = \pm\frac{5.80 \times 10^{-7}\ \text{m}}{0.300 \times 10^{-3}\ \text{m}} = \pm 1.93 \times 10^{-3}$$

From the triangle in Figure 38.6, note that $\tan\theta = y_1/L$. Because θ is very small, we can use the approximation $\sin\theta \approx \tan\theta$; thus, $\sin\theta \approx y_1/L$. Therefore, the positions of the first minima measured from the central axis are given by

$$y_1 \approx L\sin\theta = \pm L\frac{\lambda}{a} = \boxed{\pm 3.87 \times 10^{-3}\ \text{m}}$$

The positive and negative signs correspond to the dark fringes on either side of the central bright fringe. Hence, the width of the central bright fringe is equal to $2|y_1| = 7.74 \times 10^{-3}\ \text{m} = 7.74\ \text{mm}$. Note that this value is much greater than the width of the slit. However, as the slit width is increased, the diffraction pattern narrows, corresponding to smaller values of θ. In fact, for large values of a, the various maxima and minima are so closely spaced that only a large central bright area resembling the geometric image of the slit is observed. This is of great importance in the design of lenses used in telescopes, microscopes, and other optical instruments.

Exercise Determine the width of the first-order $(m = 1)$ bright fringe.

Answer 3.87 mm.

Intensity of Single-Slit Diffraction Patterns

We can use phasors to determine the light intensity distribution for a single-slit diffraction pattern. Imagine a slit divided into a large number of small zones, each of width Δy as shown in Figure 38.7. Each zone acts as a source of coherent radiation,

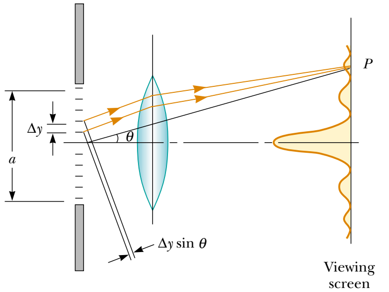


Figure 38.7 Fraunhofer diffraction by a single slit. The light intensity at point P is the resultant of all the incremental electric field magnitudes from zones of width Δy.

QuickLab

Make a V with your index and middle fingers. Hold your hand up very close to your eye so that you are looking between your two fingers toward a bright area. Now bring the fingers together until there is only a very tiny slit between them. You should be able to see a series of parallel lines. Although the lines appear to be located in the narrow space between your fingers, what you are actually seeing is a diffraction pattern cast upon your retina.

and each contributes an incremental electric field of magnitude ΔE at some point P on the screen. We obtain the total electric field magnitude E at point P by summing the contributions from all the zones. The light intensity at point P is proportional to the square of the magnitude of the electric field (see Section 37.3).

The incremental electric field magnitudes between adjacent zones are out of phase with one another by an amount $\Delta\beta$, where the phase difference $\Delta\beta$ is related to the path difference $\Delta y \sin\theta$ between adjacent zones by the expression

$$\Delta\beta = \frac{2\pi}{\lambda}\Delta y \sin\theta \qquad \textbf{(38.2)}$$

To find the magnitude of the total electric field on the screen at any angle θ, we sum the incremental magnitudes ΔE due to each zone. For small values of θ, we can assume that all the ΔE values are the same. It is convenient to use phasor diagrams for various angles, as shown in Figure 38.8. When $\theta = 0$, all phasors are aligned as shown in Figure 38.8a because all the waves from the various zones are in phase. In this case, the total electric field at the center of the screen is $E_0 = N\Delta E$, where N is the number of zones. The resultant magnitude E_R at some small angle θ is shown in Figure 38.8b, where each phasor differs in phase from an adjacent one by an amount $\Delta\beta$. In this case, E_R is the vector sum of the incremental

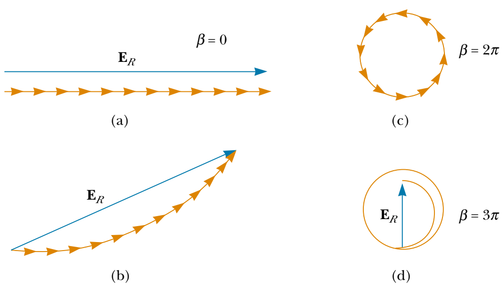


Figure 38.8 Phasor diagrams for obtaining the various maxima and minima of a single-slit diffraction pattern.

magnitudes and hence is given by the length of the chord. Therefore, $E_R < E_0$. The total phase difference β between waves from the top and bottom portions of the slit is

$$\beta = N\Delta\beta = \frac{2\pi}{\lambda}N\Delta y\sin\theta = \frac{2\pi}{\lambda}a\sin\theta \qquad \textbf{(38.3)}$$

where $a = N\Delta y$ is the width of the slit.

As θ increases, the chain of phasors eventually forms the closed path shown in Figure 38.8c. At this point, the vector sum is zero, and so $E_R = 0$, corresponding to the first minimum on the screen. Noting that $\beta = N\Delta\beta = 2\pi$ in this situation, we see from Equation 38.3 that

$$2\pi = \frac{2\pi}{\lambda}a\sin\theta$$

$$\sin\theta = \frac{\lambda}{a}$$

That is, the first minimum in the diffraction pattern occurs where $\sin\theta = \lambda/a$; this is in agreement with Equation 38.1.

At greater values of θ, the spiral chain of phasors tightens. For example, Figure 38.8d represents the situation corresponding to the second maximum, which occurs when $\beta = 360° + 180° = 540°$ (3π rad). The second minimum (two complete circles, not shown) corresponds to $\beta = 720°$ (4π rad), which satisfies the condition $\sin\theta = 2\lambda/a$.

We can obtain the total electric field magnitude E_R and light intensity I at any point P on the screen in Figure 38.7 by considering the limiting case in which Δy becomes infinitesimal (dy) and N approaches ∞. In this limit, the phasor chains in Figure 38.8 become the red curve of Figure 38.9. The arc length of the curve is E_0 because it is the sum of the magnitudes of the phasors (which is the total electric field magnitude at the center of the screen). From this figure, we see that at some angle θ, the resultant electric field magnitude E_R on the screen is equal to the chord length. From the triangle containing the angle $\beta/2$, we see that

$$\sin\frac{\beta}{2} = \frac{E_R/2}{R}$$

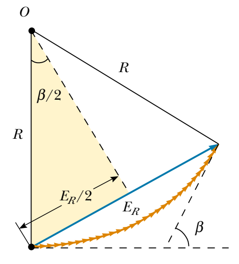


Figure 38.9 Phasor diagram for a large number of coherent sources. All the ends of the phasors lie on the circular red arc of radius R. The resultant electric field magnitude E_R equals the length of the chord.

where R is the radius of curvature. But the arc length E_0 is equal to the product $R\beta$, where β is measured in radians. Combining this information with the previous expression gives

$$E_R = 2R\sin\frac{\beta}{2} = 2\left(\frac{E_0}{\beta}\right)\sin\frac{\beta}{2} = E_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]$$

Because the resultant light intensity I at point P on the screen is proportional to the square of the magnitude E_R, we find that

Intensity of a single-slit Fraunhofer diffraction pattern

$$I = I_{\max}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 \qquad \textbf{(38.4)}$$

where $I_{\max}$ is the intensity at $\theta = 0$ (the central maximum). Substituting the expression for β (Eq. 38.3) into Equation 38.4, we have

$$I = I_{\max}\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2 \qquad \textbf{(38.5)}$$

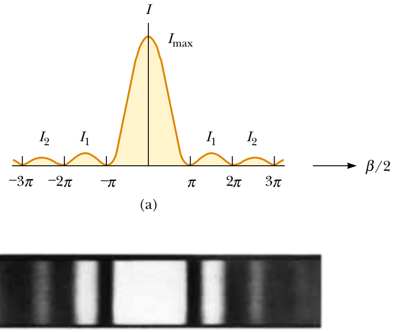


Figure 38.10 (a) A plot of light intensity I versus $\beta/2$ for the single-slit Fraunhofer diffraction pattern. (b) Photograph of a single-slit Fraunhofer diffraction pattern.

From this result, we see that minima occur when

$$\frac{\pi a \sin\theta}{\lambda} = m\pi$$

or

Condition for intensity minima

$$\sin\theta = m\frac{\lambda}{a} \qquad m = \pm 1, \pm 2, \pm 3, \ldots$$

in agreement with Equation 38.1.

Figure 38.10a represents a plot of Equation 38.5, and Figure 38.10b is a photograph of a single-slit Fraunhofer diffraction pattern. Note that most of the light intensity is concentrated in the central bright fringe.

EXAMPLE 38.2 Relative Intensities of the Maxima

Find the ratio of the intensities of the secondary maxima to the intensity of the central maximum for the single-slit Fraunhofer diffraction pattern.

Solution To a good approximation, the secondary maxima lie midway between the zero points. From Figure 38.10a, we see that this corresponds to $\beta/2$ values of $3\pi/2$, $5\pi/2$, $7\pi/2$, ... Substituting these values into Equation 38.4 gives for the first two ratios

$$\frac{I_1}{I_{max}} = \left[\frac{\sin(3\pi/2)}{(3\pi/2)}\right]^2 = \frac{1}{9\pi^2/4} = 0.045$$

$$\frac{I_2}{I_{max}} = \left[\frac{\sin(5\pi/2)}{5\pi/2}\right]^2 = \frac{1}{25\pi^2/4} = 0.016$$

That is, the first secondary maxima (the ones adjacent to the central maximum) have an intensity of 4.5% that of the central maximum, and the next secondary maxima have an intensity of 1.6% that of the central maximum.

Exercise Determine the intensity, relative to the central maximum, of the secondary maxima corresponding to $m = \pm 3$.

Answer 0.008 3.

Intensity of Two-Slit Diffraction Patterns

When more than one slit is present, we must consider not only diffraction due to the individual slits but also the interference of the waves coming from different slits. You may have noticed the curved dashed line in Figure 37.13, which indicates a decrease in intensity of the interference maxima as θ increases. This decrease is

due to diffraction. To determine the effects of both interference and diffraction, we simply combine Equation 37.12 and Equation 38.5:

$$I = I_{max}\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2 \qquad \textbf{(38.6)}$$

Although this formula looks complicated, it merely represents the diffraction pattern (the factor in brackets) acting as an "envelope" for a two-slit interference pattern (the cosine-squared factor), as shown in Figure 38.11.

Equation 37.2 indicates the conditions for interference maxima as $d\sin\theta = m\lambda$, where d is the distance between the two slits. Equation 38.1 specifies that the first diffraction minimum occurs when $a\sin\theta = \lambda$, where a is the slit width. Dividing Equation 37.2 by Equation 38.1 (with $m = 1$) allows us to determine which interference maximum coincides with the first diffraction minimum:

$$\frac{d\sin\theta}{a\sin\theta} = \frac{m\lambda}{\lambda}$$

$$\frac{d}{a} = m \qquad \textbf{(38.7)}$$

In Figure 38.11, $d/a = 18\ \mu\text{m}/3.0\ \mu\text{m} = 6$. Thus, the sixth interference maximum (if we count the central maximum as $m = 0$) is aligned with the first diffraction minimum and cannot be seen.

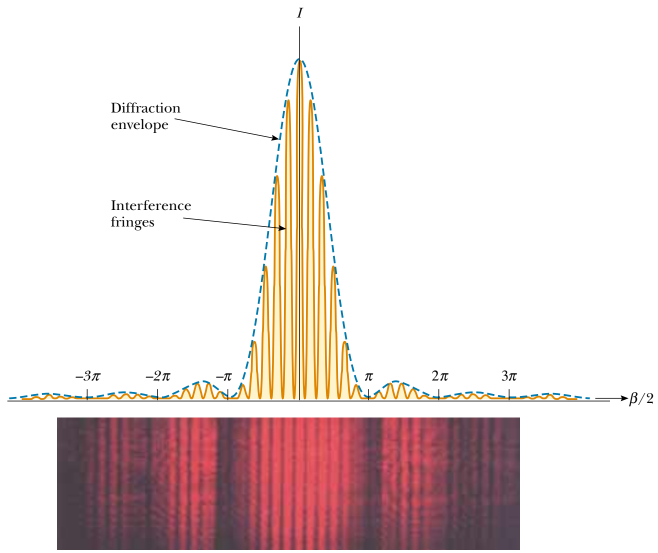


Figure 38.11 The combined effects of diffraction and interference. This is the pattern produced when 650-nm light waves pass through two 3.0-μm slits that are 18 μm apart. Notice how the diffraction pattern acts as an "envelope" and controls the intensity of the regularly spaced interference maxima.

Quick Quiz 38.2

Using Figure 38.11 as a starting point, make a sketch of the combined diffraction and interference pattern for 650-nm light waves striking two 3.0-μm slits located 9.0 μm apart.

38.3 RESOLUTION OF SINGLE-SLIT AND CIRCULAR APERTURES

The ability of optical systems to distinguish between closely spaced objects is limited because of the wave nature of light. To understand this difficulty, let us consider Figure 38.12, which shows two light sources far from a narrow slit of width a. The sources can be considered as two noncoherent point sources S_1 and S_2—for example, they could be two distant stars. If no diffraction occurred, two distinct bright spots (or images) would be observed on the viewing screen. However, because of diffraction, each source is imaged as a bright central region flanked by weaker bright and dark fringes. What is observed on the screen is the sum of two diffraction patterns: one from S_1, and the other from S_2.

If the two sources are far enough apart to keep their central maxima from overlapping, as shown in Figure 38.12a, their images can be distinguished and are said to be *resolved*. If the sources are close together, however, as shown in Figure 38.12b, the two central maxima overlap, and the images are not resolved. In determining whether two images are resolved, the following condition is often used:

When the central maximum of one image falls on the first minimum of the other image, the images are said to be just resolved. This limiting condition of resolution is known as **Rayleigh's criterion.**

Figure 38.13 shows diffraction patterns for three situations. When the objects are far apart, their images are well resolved (Fig. 38.13a). When the angular separation of the objects satisfies Rayleigh's criterion (Fig. 38.13b), the images are just resolved. Finally, when the objects are close together, the images are not resolved (Fig. 38.13c).

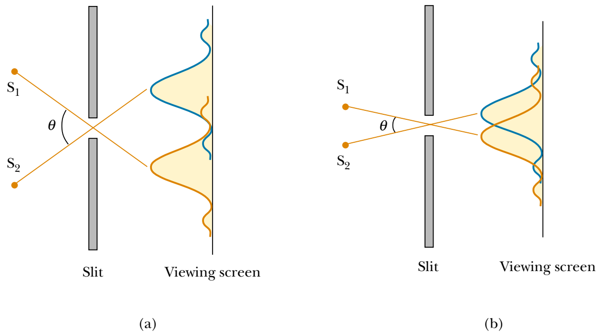


Figure 38.12 Two point sources far from a narrow slit each produce a diffraction pattern. (a) The angle subtended by the sources at the slit is large enough for the diffraction patterns to be distinguishable. (b) The angle subtended by the sources is so small that their diffraction patterns overlap, and the images are not well resolved. (Note that the angles are greatly exaggerated. The drawing is not to scale.)

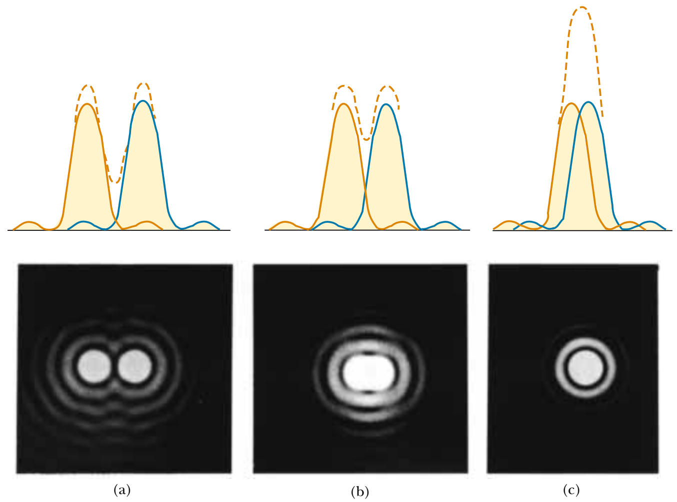


Figure 38.13 Individual diffraction patterns of two point sources (solid curves) and the resultant patterns (dashed curves) for various angular separations of the sources. In each case, the dashed curve is the sum of the two solid curves. (a) The sources are far apart, and the patterns are well resolved. (b) The sources are closer together such that the angular separation just satisfies Rayleigh's criterion, and the patterns are just resolved. (c) The sources are so close together that the patterns are not resolved.

From Rayleigh's criterion, we can determine the minimum angular separation θ_{min} subtended by the sources at the slit for which the images are just resolved. Equation 38.1 indicates that the first minimum in a single-slit diffraction pattern occurs at the angle for which

$$\sin\theta = \frac{\lambda}{a}$$

where a is the width of the slit. According to Rayleigh's criterion, this expression gives the smallest angular separation for which the two images are resolved. Because $\lambda \ll a$ in most situations, $\sin\theta$ is small, and we can use the approximation $\sin\theta \approx \theta$. Therefore, the limiting angle of resolution for a slit of width a is

$$\theta_{min} = \frac{\lambda}{a} \qquad \textbf{(38.8)}$$

where θ_{min} is expressed in radians. Hence, the angle subtended by the two sources at the slit must be greater than λ/a if the images are to be resolved.

Many optical systems use circular apertures rather than slits. The diffraction pattern of a circular aperture, shown in Figure 38.14, consists of a central circular

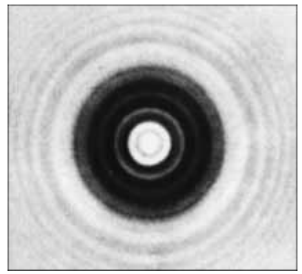


Figure 38.14 The diffraction pattern of a circular aperture consists of a central bright disk surrounded by concentric bright and dark rings.

bright disk surrounded by progressively fainter bright and dark rings. Analysis shows that the limiting angle of resolution of the circular aperture is

Limiting angle of resolution for a circular aperture

$$\theta_{\min} = 1.22 \frac{\lambda}{D} \qquad \textbf{(38.9)}$$

where D is the diameter of the aperture. Note that this expression is similar to Equation 38.8 except for the factor 1.22, which arises from a complex mathematical analysis of diffraction from the circular aperture.

EXAMPLE 38.3 Limiting Resolution of a Microscope

Light of wavelength 589 nm is used to view an object under a microscope. If the aperture of the objective has a diameter of 0.900 cm, (a) what is the limiting angle of resolution?

Solution (a) Using Equation 38.9, we find that the limiting angle of resolution is

$$\theta_{\min} = 1.22\left(\frac{589 \times 10^{-9}\text{ m}}{0.900 \times 10^{-2}\text{ m}}\right) = 7.98 \times 10^{-5}\text{ rad}$$

This means that any two points on the object subtending an angle smaller than this at the objective cannot be distinguished in the image.

(b) If it were possible to use visible light of any wavelength, what would be the maximum limit of resolution for this microscope?

Solution To obtain the smallest limiting angle, we have to use the shortest wavelength available in the visible spectrum. Violet light (400 nm) gives a limiting angle of resolution of

$$\theta_{\min} = 1.22\left(\frac{400 \times 10^{-9}\text{ m}}{0.900 \times 10^{-2}\text{ m}}\right) = 5.42 \times 10^{-5}\text{ rad}$$

(c) Suppose that water ($n = 1.33$) fills the space between the object and the objective. What effect does this have on resolving power when 589-nm light is used?

Solution We find the wavelength of the 589-nm light in the water using Equation 35.7:

$$\lambda_{\text{water}} = \frac{\lambda_{\text{air}}}{n_{\text{water}}} = \frac{589\text{ nm}}{1.33} = 443\text{ nm}$$

The limiting angle of resolution at this wavelength is now smaller than that calculated in part (a):

$$\theta_{\min} = 1.22\left(\frac{443 \times 10^{-9}\text{ m}}{0.900 \times 10^{-2}\text{ m}}\right) = 6.00 \times 10^{-5}\text{ rad}$$

EXAMPLE 38.4 Resolution of a Telescope

The Hale telescope at Mount Palomar has a diameter of 200 in. What is its limiting angle of resolution for 600-nm light?

Solution Because $D = 200$ in. $= 5.08$ m and $\lambda = 6.00 \times 10^{-7}$ m, Equation 38.9 gives

$$\theta_{\min} = 1.22 \frac{\lambda}{D} = 1.22\left(\frac{6.00 \times 10^{-7}\text{ m}}{5.08\text{ m}}\right)$$

$$= 1.44 \times 10^{-7}\text{ rad} \approx 0.03\text{ s of arc}$$

Any two stars that subtend an angle greater than or equal to this value are resolved (if atmospheric conditions are ideal).

The Hale telescope can never reach its diffraction limit because the limiting angle of resolution is always set by atmospheric blurring. This seeing limit is usually about 1 s of arc and is never smaller than about 0.1 s of arc. (This is one of the reasons for the superiority of photographs from the Hubble Space Telescope, which views celestial objects from an orbital position above the atmosphere.)

Exercise The large radio telescope at Arecibo, Puerto Rico, has a diameter of 305 m and is designed to detect 0.75-m radio waves. Calculate the minimum angle of resolution for this telescope and compare your answer with that for the Hale telescope.

Answer 3.0×10^{-3} rad (10 min of arc), more than 10 000 times larger (that is, *worse*) than the Hale minimum.

EXAMPLE 38.5 Resolution of the Eye

Estimate the limiting angle of resolution for the human eye, assuming its resolution is limited only by diffraction.

Solution Let us choose a wavelength of 500 nm, near the center of the visible spectrum. Although pupil diameter

varies from person to person, we estimate a diameter of 2 mm. We use Equation 38.9, taking $\lambda = 500$ nm and $D = 2$ mm:

$$\theta_{\min} = 1.22 \frac{\lambda}{D} = 1.22\left(\frac{5.00 \times 10^{-7}\text{ m}}{2 \times 10^{-3}\text{ m}}\right)$$

$$\approx 3 \times 10^{-4}\text{ rad} \approx 1\text{ min of arc}$$

We can use this result to determine the minimum separation distance d between two point sources that the eye can distinguish if they are a distance L from the observer (Fig. 38.15). Because $\theta_{\min}$ is small, we see that

$$\sin\theta_{\min} \approx \theta_{\min} \approx \frac{d}{L}$$

$$d = L\theta_{\min}$$

For example, if the point sources are 25 cm from the eye (the near point), then

$$d = (25\text{ cm})(3 \times 10^{-4}\text{ rad}) = 8 \times 10^{-3}\text{ cm}$$

This is approximately equal to the thickness of a human hair.

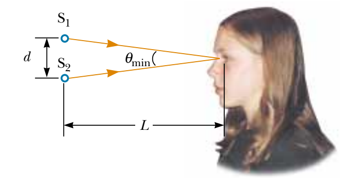


Figure 38.15 Two point sources separated by a distance d as observed by the eye.

Exercise Suppose that the pupil is dilated to a diameter of 5.0 mm and that two point sources 3.0 m away are being viewed. How far apart must the sources be if the eye is to resolve them?

Answer 0.037 cm.

APPLICATION Loudspeaker Design

The three-way speaker system shown in Figure 38.16 contains a woofer, a midrange speaker, and a tweeter. The small-diameter tweeter is for high frequencies, and the large-diameter woofer is for low frequencies. The midrange speaker, of intermediate diameter, is used for the frequency band above the high-frequency cutoff of the woofer and below the low-frequency cutoff of the tweeter. Circuits known as crossover networks include low-pass, midrange, and high-pass filters that direct the electrical signal to the appropriate speaker. The effective aperture size of a speaker is approximately its diameter. Because the wavelengths of sound waves are comparable to the typical sizes of the speakers, diffraction effects determine the angular radiation pattern. To be most useful, a speaker should radiate sound over a broad range of angles so that the listener does not have to stand at a particular spot in the room to hear maximum sound intensity. On the basis of the angular radiation pattern, let us investigate the frequency range for which a 6-in. (0.15-m) midrange speaker is most useful.

The speed of sound in air is 344 m/s, and for a circular aperture, diffraction effects become important when $\lambda = 1.22D$, where D is the speaker diameter. Therefore, we would expect this speaker to radiate non-uniformly for all frequencies above

$$\frac{344\text{ m/s}}{1.22(0.15\text{ m})} = 1\,900\text{ Hz}$$

Suppose our design specifies that the midrange speaker operates between 500 Hz (the high-frequency woofer cutoff) and 2 000 Hz. Measurements of the dispersion of radiated

Figure 38.16 An audio speaker system for high-fidelity sound reproduction. The tweeter is at the top, the midrange speaker is in the middle, and the woofer is at the bottom. *(International Stock Photography)*

sound at a suitably great distance from the speaker yield the angular profiles of sound intensity shown in Figure 38.17. In examining these plots, we see that the dispersion pattern for a 500-Hz sound is fairly uniform. This angular range is sufficiently great for us to say that this midrange speaker satisfies the design criterion. The intensity of a 2 000-Hz sound decreases to about half its maximum value about 30° from the centerline.

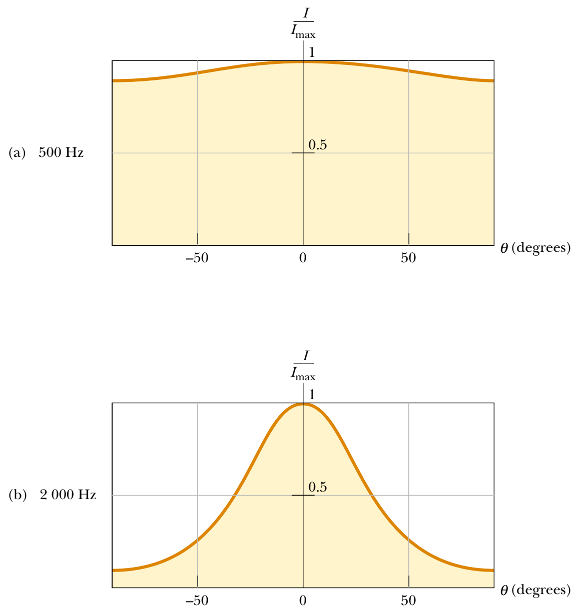


Figure 38.17 Angular dispersion of sound intensity I for a midrange speaker at (a) 500 Hz and (b) 2 000 Hz.

38.4 THE DIFFRACTION GRATING

The **diffraction grating,** a useful device for analyzing light sources, consists of a large number of equally spaced parallel slits. A *transmission grating* can be made by cutting parallel lines on a glass plate with a precision ruling machine. The spaces between the lines are transparent to the light and hence act as separate slits. A *reflection grating* can be made by cutting parallel lines on the surface of a reflective material. The reflection of light from the spaces between the lines is specular, and the reflection from the lines cut into the material is diffuse. Thus, the spaces between the lines act as parallel sources of reflected light, like the slits in a transmission grating. Gratings that have many lines very close to each other can have very small slit spacings. For example, a grating ruled with 5 000 lines/cm has a slit spacing $d = (1/5\,000)$ cm $= 2.00 \times 10^{-4}$ cm.

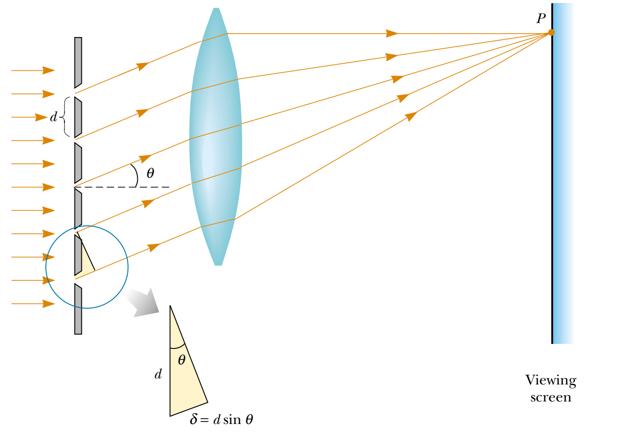


Figure 38.18 Side view of a diffraction grating. The slit separation is d, and the path difference between adjacent slits is $d \sin\theta$.

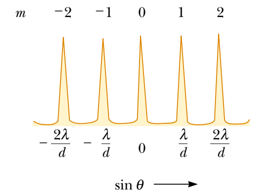


Figure 38.19 Intensity versus $\sin\theta$ for a diffraction grating. The zeroth-, first-, and second-order maxima are shown.

A section of a diffraction grating is illustrated in Figure 38.18. A plane wave is incident from the left, normal to the plane of the grating. A converging lens brings the rays together at point P. The pattern observed on the screen is the result of the combined effects of interference and diffraction. Each slit produces diffraction, and the diffracted beams interfere with one another to produce the final pattern.

The waves from all slits are in phase as they leave the slits. However, for some arbitrary direction θ measured from the horizontal, the waves must travel different path lengths before reaching point P. From Figure 38.18, note that the path difference δ between rays from any two adjacent slits is equal to $d \sin\theta$. If this path difference equals one wavelength or some integral multiple of a wavelength, then waves from all slits are in phase at point P and a bright fringe is observed. Therefore, the condition for maxima in the interference pattern at the angle θ is

$$d \sin\theta = m\lambda \qquad m = 0, 1, 2, 3, \ldots \qquad \textbf{(38.10)}$$

Condition for interference maxima for a grating

We can use this expression to calculate the wavelength if we know the grating spacing and the angle θ. If the incident radiation contains several wavelengths, the mth-order maximum for each wavelength occurs at a specific angle. All wavelengths are seen at $\theta = 0$, corresponding to $m = 0$, the zeroth-order maximum. The first-order maximum ($m = 1$) is observed at an angle that satisfies the relationship $\sin\theta = \lambda/d$; the second-order maximum ($m = 2$) is observed at a larger angle θ, and so on.

The intensity distribution for a diffraction grating obtained with the use of a monochromatic source is shown in Figure 38.19. Note the sharpness of the principal maxima and the broadness of the dark areas. This is in contrast to the broad bright fringes characteristic of the two-slit interference pattern (see Fig. 37.6). Because the principal maxima are so sharp, they are very much brighter than two-slit

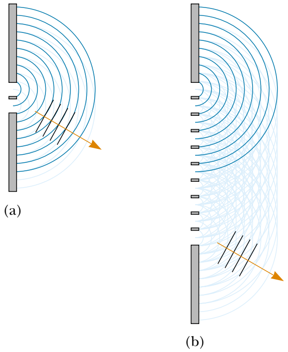


Figure 38.20 (a) Addition of two wave fronts from two slits. (b) Addition of ten wave fronts from ten slits. The resultant wave is much stronger in part (b) than in part (a).

QuickLab

Stand a couple of meters from a lightbulb. Facing away from the light, hold a compact disc about 10 cm from your eye and tilt it until the reflection of the bulb is located in the hole at the disc's center. You should see spectra radiating out from the center, with violet on the inside and red on the outside. Now move the disc away from your eye until the violet band is at the outer edge. Carefully measure the distance from your eye to the center of the disc and also determine the radius of the disc. Use this information to find the angle θ to the first-order maximum for violet light. Now use Equation 38.10 to determine the spacing between the grooves on the disc. The industry standard is 1.6 μm. How close did you come?

interference maxima. The reason for this is illustrated in Figure 38.20, in which the combination of multiple wave fronts for a ten-slit grating is compared with the wave fronts for a two-slit system. Actual gratings have thousands of times more slits, and therefore the maxima are even stronger.

A schematic drawing of a simple apparatus used to measure angles in a diffraction pattern is shown in Figure 38.21. This apparatus is a diffraction grating spectrometer. The light to be analyzed passes through a slit, and a collimated beam of light is incident on the grating. The diffracted light leaves the grating at angles that satisfy Equation 38.10, and a telescope is used to view the image of the slit. The wavelength can be determined by measuring the precise angles at which the images of the slit appear for the various orders.

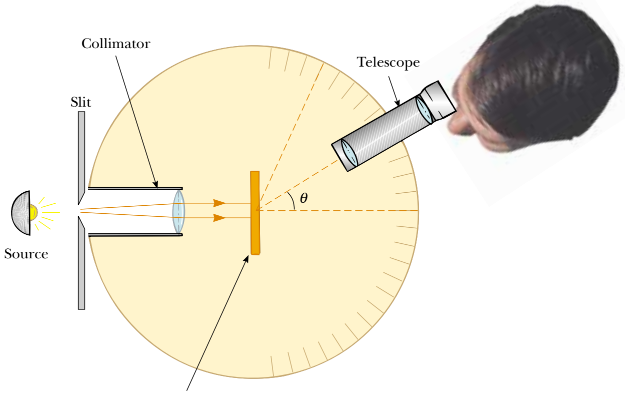


Figure 38.21 Diagram of a diffraction grating spectrometer. The collimated beam incident on the grating is diffracted into the various orders at the angles θ that satisfy the equation $d \sin \theta = m\lambda$, where $m = 0, 1, 2, \ldots$.

CONCEPTUAL EXAMPLE 38.6 A Compact Disc Is a Diffraction Grating

Light reflected from the surface of a compact disc is multicolored, as shown in Figure 38.22. The colors and their intensities depend on the orientation of the disc relative to the eye and relative to the light source. Explain how this works.

Solution The surface of a compact disc has a spiral grooved track (with adjacent grooves having a separation on the order of 1 μm). Thus, the surface acts as a reflection grating. The light reflecting from the regions between these closely spaced grooves interferes constructively only in certain directions that depend on the wavelength and on the direction of the incident light. Any one section of the disc serves as a diffraction grating for white light, sending different colors in different directions. The different colors you see when viewing one section change as the light source, the disc, or you move to change the angles of incidence or diffraction.

Figure 38.22 A compact disc observed under white light. The colors observed in the reflected light and their intensities depend on the orientation of the disc relative to the eye and relative to the light source.

EXAMPLE 38.7 The Orders of a Diffraction Grating

Monochromatic light from a helium-neon laser (λ = 632.8 nm) is incident normally on a diffraction grating containing 6 000 lines per centimeter. Find the angles at which the first-order, second-order, and third-order maxima are observed.

Solution First, we must calculate the slit separation, which is equal to the inverse of the number of lines per centimeter:

$$d = \frac{1}{6\,000}\text{ cm} = 1.667 \times 10^{-4}\text{ cm} = 1\,667\text{ nm}$$

For the first-order maximum ($m = 1$), we obtain

$$\sin \theta_1 = \frac{\lambda}{d} = \frac{632.8\text{ nm}}{1\,667\text{ nm}} = 0.379\,6$$

$$\theta_1 = \boxed{22.31°}$$

For the second-order maximum ($m = 2$), we find

$$\sin \theta_2 = \frac{2\lambda}{d} = \frac{2(632.8\text{ nm})}{1\,667\text{ nm}} = 0.759\,2$$

$$\theta_2 = \boxed{49.39°}$$

For $m = 3$, we find that $\sin \theta_3 = 1.139$. Because $\sin \theta$ cannot exceed unity, this does not represent a realistic solution. Hence, only zeroth-, first-, and second-order maxima are observed for this situation.

Resolving Power of the Diffraction Grating

The diffraction grating is most useful for measuring wavelengths accurately. Like the prism, the diffraction grating can be used to disperse a spectrum into its wavelength components. Of the two devices, the grating is the more precise if one wants to distinguish two closely spaced wavelengths.

For two nearly equal wavelengths λ_1 and λ_2 between which a diffraction grating can just barely distinguish, the **resolving power** R of the grating is defined as

$$R = \frac{\lambda}{\lambda_2 - \lambda_1} = \frac{\lambda}{\Delta\lambda} \qquad \textbf{(38.11)}$$

Resolving power

where $\lambda = (\lambda_1 + \lambda_2)/2$ and $\Delta\lambda = \lambda_2 - \lambda_1$. Thus, a grating that has a high resolving power can distinguish small differences in wavelength. If N lines of the grating are illuminated, it can be shown that the resolving power in the mth-order diffraction is

Resolving power of a grating

$$R = Nm \tag{38.12}$$

Thus, resolving power increases with increasing order number and with increasing number of illuminated slits.

Note that $R = 0$ for $m = 0$; this signifies that all wavelengths are indistinguishable for the zeroth-order maximum. However, consider the second-order diffraction pattern ($m = 2$) of a grating that has 5 000 rulings illuminated by the light source. The resolving power of such a grating in second order is $R = 5\,000 \times 2 = 10\,000$. Therefore, for a mean wavelength of, for example, 600 nm, the minimum wavelength separation between two spectral lines that can be just resolved is $\Delta\lambda = \lambda/R = 6.00 \times 10^{-2}$ nm. For the third-order principal maximum, $R = 15\,000$ and $\Delta\lambda = 4.00 \times 10^{-2}$ nm, and so on.

One of the most interesting applications of diffraction is holography, which is used to create three-dimensional images found practically everywhere, from credit cards to postage stamps. The production of these special diffracting films is discussed in Chapter 42 of the extended version of this text.

EXAMPLE 38.8 Resolving Sodium Spectral Lines

When an element is raised to a very high temperature, the atoms emit radiation having discrete wavelengths. The set of wavelengths for a given element is called its *atomic spectrum.* Two strong components in the atomic spectrum of sodium have wavelengths of 589.00 nm and 589.59 nm. (a) What must be the resolving power of a grating if these wavelengths are to be distinguished?

Solution

$$R = \frac{\lambda}{\Delta\lambda} = \frac{589.30\text{ nm}}{589.59\text{ nm} - 589.00\text{ nm}} = \frac{589.30}{0.59} = \boxed{999}$$

(b) To resolve these lines in the second-order spectrum, how many lines of the grating must be illuminated?

Solution From Equation 38.12 and the results to part (a), we find that

$$N = \frac{R}{m} = \frac{999}{2} = \boxed{500\text{ lines}}$$

Optional Section

38.5 DIFFRACTION OF X-RAYS BY CRYSTALS

In principle, the wavelength of any electromagnetic wave can be determined if a grating of the proper spacing (of the order of λ) is available. X-rays, discovered by Wilhelm Roentgen (1845–1923) in 1895, are electromagnetic waves of very short wavelength (of the order of 0.1 nm). It would be impossible to construct a grating having such a small spacing by the cutting process described at the beginning of Section 38.4. However, the atomic spacing in a solid is known to be about 0.1 nm. In 1913, Max von Laue (1879–1960) suggested that the regular array of atoms in a crystal could act as a three-dimensional diffraction grating for x-rays. Subsequent experiments confirmed this prediction. The diffraction patterns are complex because of the three-dimensional nature of the crystal. Nevertheless, x-ray diffraction

has proved to be an invaluable technique for elucidating crystalline structures and for understanding the structure of matter.[1]

Figure 38.23 is one experimental arrangement for observing x-ray diffraction from a crystal. A collimated beam of x-rays is incident on a crystal. The diffracted beams are very intense in certain directions, corresponding to constructive interference from waves reflected from layers of atoms in the crystal. The diffracted beams can be detected by a photographic film, and they form an array of spots known as a *Laue pattern.* One can deduce the crystalline structure by analyzing the positions and intensities of the various spots in the pattern.

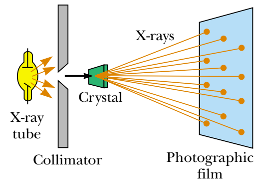


Figure 38.23 Schematic diagram of the technique used to observe the diffraction of x-rays by a crystal. The array of spots formed on the film is called a Laue pattern.

The arrangement of atoms in a crystal of sodium chloride (NaCl) is shown in Figure 38.24. Each unit cell (the geometric solid that repeats throughout the crystal) is a cube having an edge length a. A careful examination of the NaCl structure shows that the ions lie in discrete planes (the shaded areas in Fig. 38.24). Now suppose that an incident x-ray beam makes an angle θ with one of the planes, as shown in Figure 38.25. The beam can be reflected from both the upper plane and the lower one. However, the beam reflected from the lower plane travels farther than the beam reflected from the upper plane. The effective path difference is $2d \sin\theta$. The two beams reinforce each other (constructive interference) when this path difference equals some integer multiple of λ. The same is true for reflection from the entire family of parallel planes. Hence, the condition for constructive interference (maxima in the reflected beam) is

Bragg's law

$$2d\sin\theta = m\lambda \qquad m = 1, 2, 3, \ldots \tag{38.13}$$

This condition is known as **Bragg's law,** after W. L. Bragg (1890–1971), who first derived the relationship. If the wavelength and diffraction angle are measured, Equation 38.13 can be used to calculate the spacing between atomic planes.

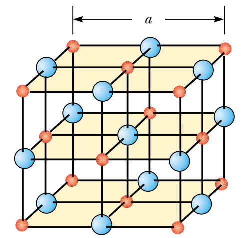


Figure 38.24 Crystalline structure of sodium chloride (NaCl). The blue spheres represent Cl^- ions, and the red spheres represent Na^+ ions. The length of the cube edge is $a = 0.562\,737$ nm.

Quick Quiz 38.3

When you receive a chest x-ray at a hospital, the rays pass through a series of parallel ribs in your chest. Do the ribs act as a diffraction grating for x-rays?

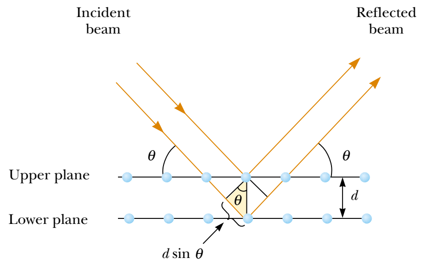


Figure 38.25 A two-dimensional description of the reflection of an x-ray beam from two parallel crystalline planes separated by a distance d. The beam reflected from the lower plane travels farther than the one reflected from the upper plane by a distance $2d\sin\theta$.

[1] For more details on this subject, see Sir Lawrence Bragg, "X-Ray Crystallography," *Sci. Am.* 219:58–70, 1968.

38.6 POLARIZATION OF LIGHT WAVES

In Chapter 34 we described the transverse nature of light and all other electromagnetic waves. Polarization is firm evidence of this transverse nature.

An ordinary beam of light consists of a large number of waves emitted by the atoms of the light source. Each atom produces a wave having some particular orientation of the electric field vector **E**, corresponding to the direction of atomic vibration. The *direction of polarization* of each individual wave is defined to be the direction in which the electric field is vibrating. In Figure 38.26, this direction happens to lie along the *y* axis. However, an individual electromagnetic wave could have its **E** vector in the *yz* plane, making any possible angle with the *y* axis. Because all directions of vibration from a wave source are possible, the resultant electromagnetic wave is a superposition of waves vibrating in many different directions. The result is an **unpolarized** light beam, represented in Figure 38.27a. The direction of wave propagation in this figure is perpendicular to the page. The arrows show a few possible directions of the electric field vectors for the individual waves making up the resultant beam. At any given point and at some instant of time, all these individual electric field vectors add to give one resultant electric field vector.

As noted in Section 34.2, a wave is said to be **linearly polarized** if the resultant electric field **E** vibrates in the same direction *at all times* at a particular point, as shown in Figure 38.27b. (Sometimes, such a wave is described as *plane-polarized,* or simply *polarized.*) The plane formed by **E** and the direction of propagation is called the *plane of polarization* of the wave. If the wave in Figure 38.26 represented the resultant of all individual waves, the plane of polarization is the *xy* plane.

It is possible to obtain a linearly polarized beam from an unpolarized beam by removing all waves from the beam except those whose electric field vectors oscillate in a single plane. We now discuss four processes for producing polarized light from unpolarized light.

Polarization by Selective Absorption

The most common technique for producing polarized light is to use a material that transmits waves whose electric fields vibrate in a plane parallel to a certain direction and that absorbs waves whose electric fields vibrate in all other directions.

In 1938, E. H. Land (1909–1991) discovered a material, which he called *polaroid,* that polarizes light through selective absorption by oriented molecules. This material is fabricated in thin sheets of long-chain hydrocarbons. The sheets are stretched during manufacture so that the long-chain molecules align. After a sheet is dipped into a solution containing iodine, the molecules become good electrical conductors. However, conduction takes place primarily along the hydrocarbon chains because electrons can move easily only along the chains. As a result, the

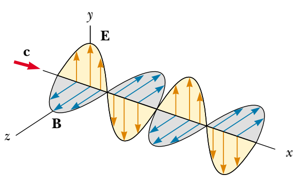


Figure 38.26 Schematic diagram of an electromagnetic wave propagating at velocity **c** in the *x* direction. The electric field vibrates in the *xy* plane, and the magnetic field vibrates in the *xz* plane.

molecules readily absorb light whose electric field vector is parallel to their length and allow light through whose electric field vector is perpendicular to their length.

It is common to refer to the direction perpendicular to the molecular chains as the *transmission axis.* In an ideal polarizer, all light with **E** parallel to the transmission axis is transmitted, and all light with **E** perpendicular to the transmission axis is absorbed.

Figure 38.28 represents an unpolarized light beam incident on a first polarizing sheet, called the *polarizer.* Because the transmission axis is oriented vertically in the figure, the light transmitted through this sheet is polarized vertically. A second polarizing sheet, called the *analyzer,* intercepts the beam. In Figure 38.28, the analyzer transmission axis is set at an angle θ to the polarizer axis. We call the electric field vector of the transmitted beam $\mathbf{E}_0$. The component of $\mathbf{E}_0$ perpendicular to the analyzer axis is completely absorbed. The component of $\mathbf{E}_0$ parallel to the analyzer axis, which is allowed through by the analyzer, is $E_0 \cos\theta$. Because the intensity of the transmitted beam varies as the square of its magnitude, we conclude that the intensity of the (polarized) beam transmitted through the analyzer varies as

$$I = I_{\text{max}} \cos^2\theta \qquad \textbf{(38.14)}$$

where I_{max} is the intensity of the polarized beam incident on the analyzer. This expression, known as **Malus's law,**[2] applies to any two polarizing materials whose transmission axes are at an angle θ to each other. From this expression, note that the intensity of the transmitted beam is maximum when the transmission axes are parallel ($\theta = 0$ or $180°$) and that it is zero (complete absorption by the analyzer) when the transmission axes are perpendicular to each other. This variation in transmitted intensity through a pair of polarizing sheets is illustrated in Figure 38.29. Because the average value of $\cos^2\theta$ is $\frac{1}{2}$, the intensity of the light passed through an ideal polarizer is one-half the intensity of unpolarized light.

Polarization by Reflection

When an unpolarized light beam is reflected from a surface, the reflected light may be completely polarized, partially polarized, or unpolarized, depending on the angle of incidence. If the angle of incidence is 0°, the reflected beam is unpolarized. For other angles of incidence, the reflected light is polarized to some ex-

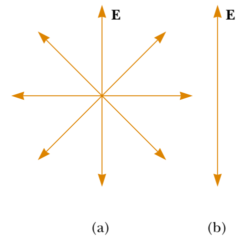


Figure 38.27 (a) An unpolarized light beam viewed along the direction of propagation (perpendicular to the page). The transverse electric field can vibrate in any direction in the plane of the page with equal probability. (b) A linearly polarized light beam with the electric field vibrating in the vertical direction.

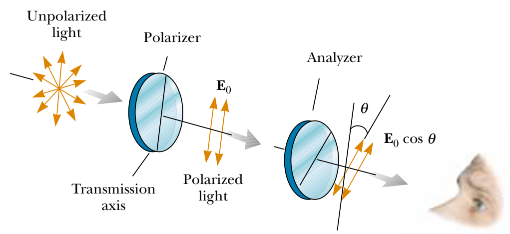
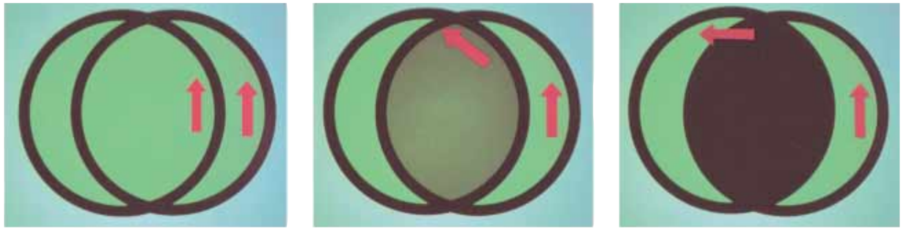


Figure 38.28 Two polarizing sheets whose transmission axes make an angle θ with each other. Only a fraction of the polarized light incident on the analyzer is transmitted through it.

[2] Named after its discoverer, E. L. Malus (1775–1812). Malus discovered that reflected light was polarized by viewing it through a calcite ($CaCO_3$) crystal.



(a) (b) (c)

Figure 38.29 The intensity of light transmitted through two polarizers depends on the relative orientation of their transmission axes. (a) The transmitted light has maximum intensity when the transmission axes are aligned with each other. (b) The transmitted light has lesser intensity when the transmission axes are at an angle of 45° with each other. (c) The transmitted light intensity is a minimum when the transmission axes are at right angles to each other.

tent, and for one particular angle of incidence, the reflected light is completely polarized. Let us now investigate reflection at that special angle.

Suppose that an unpolarized light beam is incident on a surface, as shown in Figure 38.30a. Each individual electric field vector can be resolved into two components: one parallel to the surface (and perpendicular to the page in Fig. 38.30, represented by the dots), and the other (represented by the red arrows) perpendicular both to the first component and to the direction of propagation. Thus, the polarization of the entire beam can be described by two electric field components in these directions. It is found that the parallel component reflects more strongly than the perpendicular component, and this results in a partially polarized reflected beam. Furthermore, the refracted beam is also partially polarized.

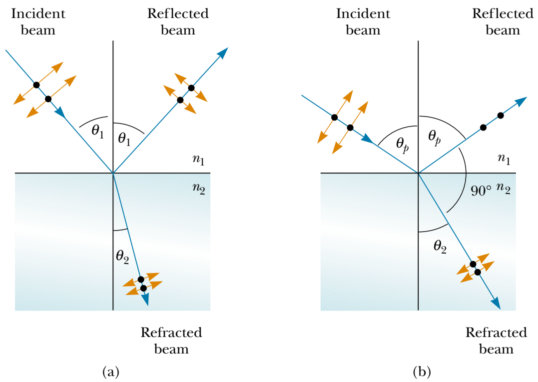


Figure 38.30 (a) When unpolarized light is incident on a reflecting surface, the reflected and refracted beams are partially polarized. (b) The reflected beam is completely polarized when the angle of incidence equals the polarizing angle θ_p, which satisfies the equation $n = \tan\theta_p$.

Now suppose that the angle of incidence θ_1 is varied until the angle between the reflected and refracted beams is 90°, as shown in Figure 38.30b. At this particular angle of incidence, the reflected beam is completely polarized (with its electric field vector parallel to the surface), and the refracted beam is still only partially polarized. The angle of incidence at which this polarization occurs is called the **polarizing angle** θ_p.

Polarizing angle

We can obtain an expression relating the polarizing angle to the index of refraction of the reflecting substance by using Figure 38.30b. From this figure, we see that $\theta_p + 90° + \theta_2 = 180°$; thus, $\theta_2 = 90° - \theta_p$. Using Snell's law of refraction (Eq. 35.8) and taking $n_1 = 1.00$ for air and $n_2 = n$, we have

$$n = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin\theta_p}{\sin\theta_2}$$

Because $\sin\theta_2 = \sin(90° - \theta_p) = \cos\theta_p$, we can write this expression for n as $n = \sin\theta_p/\cos\theta_p$, which means that

$$n = \tan\theta_p \tag{38.15}$$

Brewster's law

This expression is called **Brewster's law,** and the polarizing angle θ_p is sometimes called **Brewster's angle,** after its discoverer, David Brewster (1781–1868). Because n varies with wavelength for a given substance, Brewster's angle is also a function of wavelength.

Polarization by reflection is a common phenomenon. Sunlight reflected from water, glass, and snow is partially polarized. If the surface is horizontal, the electric field vector of the reflected light has a strong horizontal component. Sunglasses made of polarizing material reduce the glare of reflected light. The transmission axes of the lenses are oriented vertically so that they absorb the strong horizontal component of the reflected light. If you rotate sunglasses 90°, they will not be as effective at blocking the glare from shiny horizontal surfaces.

QuickLab

Devise a way to use a protractor, desklamp, and polarizing sunglasses to measure Brewster's angle for the glass in a window. From this, determine the index of refraction of the glass. Compare your results with the values given in Table 35.1.

Polarization by Double Refraction

Solids can be classified on the basis of internal structure. Those in which the atoms are arranged in a specific order are called *crystalline;* the NaCl structure of Figure 38.24 is just one example of a crystalline solid. Those solids in which the atoms are distributed randomly are called *amorphous.* When light travels through an amorphous material, such as glass, it travels with a speed that is the same in all directions. That is, glass has a single index of refraction. In certain crystalline materials, however, such as calcite and quartz, the speed of light is not the same in all directions. Such materials are characterized by two indices of refraction. Hence, they are often referred to as **double-refracting** or **birefringent** materials.

Upon entering a calcite crystal, unpolarized light splits into two plane-polarized rays that travel with different velocities, corresponding to two angles of refraction, as shown in Figure 38.31. The two rays are polarized in two mutually perpendicular directions, as indicated by the dots and arrows. One ray, called the **ordinary (O) ray,** is characterized by an index of refraction n_O that is the same in all directions. This means that if one could place a point source of light inside the crystal, as shown in Figure 38.32, the ordinary waves would spread out from the source as spheres.

The second plane-polarized ray, called the **extraordinary (E) ray,** travels with different speeds in different directions and hence is characterized by an index of refraction n_E that varies with the direction of propagation. The point source in Fig-

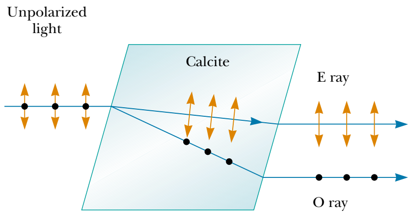


Figure 38.31 Unpolarized light incident on a calcite crystal splits into an ordinary (O) ray and an extraordinary (E) ray. These two rays are polarized in mutually perpendicular directions (drawing not to scale).

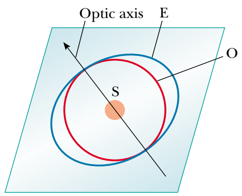


Figure 38.32 A point source S inside a double-refracting crystal produces a spherical wave front corresponding to the ordinary ray and an elliptical wave front corresponding to the extraordinary ray. The two waves propagate with the same velocity along the optic axis.

ure 38.32 sends out an extraordinary wave having wave fronts that are elliptical in cross-section. Note from Figure 38.32 that there is one direction, called the **optic axis,** along which the ordinary and extraordinary rays have the same speed, corresponding to the direction for which $n_O = n_E$. The difference in speed for the two rays is a maximum in the direction perpendicular to the optic axis. For example, in calcite, $n_O = 1.658$ at a wavelength of 589.3 nm, and n_E varies from 1.658 along the optic axis to 1.486 perpendicular to the optic axis. Values for n_O and n_E for various double-refracting crystals are given in Table 38.1.

If we place a piece of calcite on a sheet of paper and then look through the crystal at any writing on the paper, we see two images, as shown in Figure 38.33. As can be seen from Figure 38.31, these two images correspond to one formed by the ordinary ray and one formed by the extraordinary ray. If the two images are viewed through a sheet of rotating polarizing glass, they alternately appear and disappear because the ordinary and extraordinary rays are plane-polarized along mutually perpendicular directions.

Polarization by Scattering

When light is incident on any material, the electrons in the material can absorb and reradiate part of the light. Such absorption and reradiation of light by electrons in the gas molecules that make up air is what causes sunlight reaching an observer on the Earth to be partially polarized. You can observe this effect—called **scattering**—by looking directly up at the sky through a pair of sunglasses whose lenses are made of polarizing material. Less light passes through at certain orientations of the lenses than at others.

Figure 38.34 illustrates how sunlight becomes polarized when it is scattered. An unpolarized beam of sunlight traveling in the horizontal direction (parallel to the ground) strikes a molecule of one of the gases that make up air, setting the electrons of the molecule into vibration. These vibrating charges act like the vibrating charges in an antenna. The horizontal component of the electric field vector in the incident wave results in a horizontal component of the vibration of the charges, and the vertical component of the vector results in a vertical component of vibration. If the observer in Figure 38.34 is looking straight up (perpendicular to the original direction of propagation of the light), the vertical oscillations of the charges send no radiation toward the observer. Thus, the observer sees light that is completely polarized in the horizontal direction, as indicated by the red arrows. If the observer looks in other directions, the light is partially polarized in the horizontal direction.

Figure 38.33 A calcite crystal produces a double image because it is a birefringent (double-refracting) material.

TABLE 38.1 Indices of Refraction for Some Double-Refracting Crystals at a Wavelength of 589.3 nm

Crystal	n_O	n_E	n_O/n_E
Calcite ($CaCO_3$)	1.658	1.486	1.116
Quartz (SiO_2)	1.544	1.553	0.994
Sodium nitrate ($NaNO_3$)	1.587	1.336	1.188
Sodium sulfite ($NaSO_3$)	1.565	1.515	1.033
Zinc chloride ($ZnCl_2$)	1.687	1.713	0.985
Zinc sulfide (ZnS)	2.356	2.378	0.991

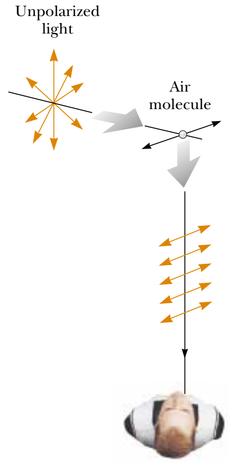


Figure 38.34 The scattering of unpolarized sunlight by air molecules. The scattered light traveling perpendicular to the incident light is plane-polarized because the vertical vibrations of the charges in the air molecule send no light in this direction.

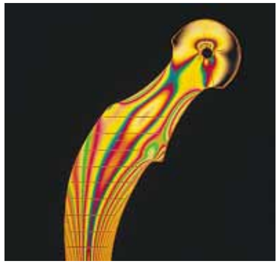
Some phenomena involving the scattering of light in the atmosphere can be understood as follows. When light of various wavelengths λ is incident on gas molecules of diameter d, where $d \ll \lambda$, the relative intensity of the scattered light varies as $1/\lambda^4$. The condition $d \ll \lambda$ is satisfied for scattering from oxygen (O_2) and nitrogen (N_2) molecules in the atmosphere, whose diameters are about 0.2 nm. Hence, short wavelengths (blue light) are scattered more efficiently than long wavelengths (red light). Therefore, when sunlight is scattered by gas molecules in the air, the short-wavelength radiation (blue) is scattered more intensely than the long-wavelength radiation (red).

When you look up into the sky in a direction that is not toward the Sun, you see the scattered light, which is predominantly blue; hence, you see a blue sky. If you look toward the west at sunset (or toward the east at sunrise), you are looking in a direction toward the Sun and are seeing light that has passed through a large distance of air. Most of the blue light has been scattered by the air between you and the Sun. The light that survives this trip through the air to you has had much of its blue component scattered and is thus heavily weighted toward the red end of the spectrum; as a result, you see the red and orange colors of sunset. However, a blue sky is seen by someone to your west for whom it is still a quarter hour before sunset.

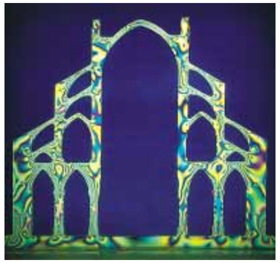
Optical Activity

Many important applications of polarized light involve materials that display **optical activity.** A material is said to be optically active if it rotates the plane of polarization of any light transmitted through the material. The angle through which the light is rotated by a specific material depends on the length of the path through the material and on concentration if the material is in solution. One optically active material is a solution of the common sugar dextrose. A standard method for determining the concentration of sugar solutions is to measure the rotation produced by a fixed length of the solution.

Molecular asymmetry determines whether a material is optically active. For example, some proteins are optically active because of their spiral shape. Other materials, such as glass and plastic, become optically active when stressed. Suppose that an unstressed piece of plastic is placed between a polarizer and an analyzer so that light passes from polarizer to plastic to analyzer. When the plastic is unstressed and the analyzer axis is perpendicular to the polarizer axis, none of the polarized light passes through the analyzer. In other words, the unstressed plastic has no effect on the light passing through it. If the plastic is stressed, however, the regions of greatest stress rotate the polarized light through the largest angles. Hence, a series of bright and dark bands is observed in the transmitted light, with the bright bands corresponding to regions of greatest stress.



(a)



(b)

Figure 38.35 (a) Strain distribution in a plastic model of a hip replacement used in a medical research laboratory. The pattern is produced when the plastic model is viewed between a polarizer and analyzer oriented perpendicular to each other. (b) A plastic model of an arch structure under load conditions observed between perpendicular polarizers. Such patterns are useful in the optimum design of architectural components.

Engineers often use this technique, called *optical stress analysis,* in designing structures ranging from bridges to small tools. They build a plastic model and analyze it under different load conditions to determine regions of potential weakness and failure under stress. Some examples of a plastic model under stress are shown in Figure 38.35.

The liquid crystal displays found in most calculators have their optical activity changed by the application of electric potential across different parts of the display. Try using a pair of polarizing sunglasses to investigate the polarization used in the display of your calculator.

SUMMARY

Diffraction is the deviation of light from a straight-line path when the light passes through an aperture or around an obstacle.

The **Fraunhofer diffraction pattern** produced by a single slit of width a on a distant screen consists of a central bright fringe and alternating bright and dark fringes of much lower intensities. The angles θ at which the diffraction pattern has zero intensity, corresponding to destructive interference, are given by

$$\sin\theta = m\frac{\lambda}{a} \qquad m = \pm 1, \pm 2, \pm 3, \ldots \tag{38.1}$$

How the intensity I of a single-slit diffraction pattern varies with angle θ is given by the expression

$$I = I_{\max}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 \tag{38.4}$$

where $\beta = (2\pi a \sin\theta)/\lambda$ and $I_{\max}$ is the intensity at $\theta = 0$.

Rayleigh's criterion, which is a limiting condition of resolution, states that two images formed by an aperture are just distinguishable if the central maximum of the diffraction pattern for one image falls on the first minimum of the diffraction pattern for the other image. The limiting angle of resolution for a slit of width a is $\theta_{\min} = \lambda/a$, and the limiting angle of resolution for a circular aperture of diameter D is $\theta_{\min} = 1.22\lambda/D$.

A **diffraction grating** consists of a large number of equally spaced, identical slits. The condition for intensity maxima in the interference pattern of a diffraction grating for normal incidence is

$$d\sin\theta = m\lambda \qquad m = 0, 1, 2, 3, \ldots \tag{38.10}$$

where d is the spacing between adjacent slits and m is the order number of the diffraction pattern. The resolving power of a diffraction grating in the mth order of the diffraction pattern is

$$R = Nm \tag{38.12}$$

where N is the number of lines in the grating that are illuminated.

When polarized light of intensity I_0 is emitted by a polarizer and then incident on an analyzer, the light transmitted through the analyzer has an intensity equal to $I_{\max}\cos^2\theta$, where θ is the angle between the polarizer and analyzer transmission axes.

In general, reflected light is partially polarized. However, reflected light is completely polarized when the angle of incidence is such that the angle between the reflected and refracted beams is 90°. This angle of incidence, called the **polarizing angle** θ_p, satisfies **Brewster's law:**

$$n = \tan\theta_p \tag{38.15}$$

where n is the index of refraction of the reflecting medium.

QUESTIONS

1. Why can you hear around corners but not see around them?
2. Observe the shadow of your book when it is held a few inches above a table while illuminated by a lamp several feet above it. Why is the shadow somewhat fuzzy at the edges?
3. Knowing that radio waves travel at the speed of light and that a typical AM radio frequency is 1 000 kHz while an FM radio frequency might be 100 MHz, estimate the wavelengths of typical AM and FM radio signals. Use this information to explain why FM radio stations often fade out when you drive through a short tunnel or underpass but AM radio stations do not.
4. Describe the change in width of the central maximum of the single-slit diffraction pattern as the width of the slit is made narrower.
5. Assuming that the headlights of a car are point sources, estimate the maximum observer-to-car distance at which the headlights are distinguishable from each other.
6. A laser beam is incident at a shallow angle on a machinist's ruler that has a finely calibrated scale. The engraved rulings on the scale give rise to a diffraction pattern on a screen. Discuss how you can use this arrangement to obtain a measure of the wavelength of the laser light.
7. Certain sunglasses use a polarizing material to reduce the intensity of light reflected from shiny surfaces. What orientation of polarization should the material have to be most effective?
8. During the "day" on the Moon (that is, when the Sun is visible), you see a black sky and the stars are clearly visible. During the day on the Earth, you see a blue sky and no stars. Account for this difference.
9. You can make the path of a light beam visible by placing dust in the air (perhaps by shaking a blackboard eraser in the path of the light beam). Explain why you can see the beam under these circumstances.
10. Is light from the sky polarized? Why is it that clouds seen through Polaroid glasses stand out in bold contrast to the sky?
11. If a coin is glued to a glass sheet and the arrangement is held in front of a laser beam, the projected shadow has diffraction rings around its edge and a bright spot in the center. How is this possible?
12. If a fine wire is stretched across the path of a laser beam, is it possible to produce a diffraction pattern?
13. How could the index of refraction of a flat piece of dark obsidian glass be determined?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** ☐ = Computer useful in solving problem ☐ = Interactive Physics
☐ = paired numerical/symbolic problems

Section 38.1 Introduction to Diffraction

Section 38.2 Diffraction from Narrow Slits

1. Helium-neon laser light ($\lambda = 632.8$ nm) is sent through a 0.300-mm-wide single slit. What is the width of the central maximum on a screen 1.00 m from the slit?

2. A beam of green light is diffracted by a slit with a width of 0.550 mm. The diffraction pattern forms on a wall 2.06 m beyond the slit. The distance between the positions of zero intensity on both sides of the central bright fringe is 4.10 mm. Calculate the wavelength of the laser light.

WEB 3. A screen is placed 50.0 cm from a single slit, which is illuminated with 690-nm light. If the distance between the first and third minima in the diffraction pattern is 3.00 mm, what is the width of the slit?

4. Coherent microwaves of wavelength 5.00 cm enter a long, narrow window in a building otherwise essentially opaque to the microwaves. If the window is 36.0 cm wide, what is the distance from the central maximum to the first-order minimum along a wall 6.50 m from the window?

5. Sound with a frequency of 650 Hz from a distant source passes through a doorway 1.10 m wide in a sound-absorbing wall. Find the number and approximate directions of the diffraction-maximum beams radiated into the space beyond.

6. Light with a wavelength of 587.5 nm illuminates a single slit 0.750 mm in width. (a) At what distance from the slit should a screen be located if the first minimum in the diffraction pattern is to be 0.850 mm from the center of the screen? (b) What is the width of the central maximum?

7. A diffraction pattern is formed on a screen 120 cm away from a 0.400-mm-wide slit. Monochromatic 546.1-nm light is used. Calculate the fractional intensity I/I_0 at a point on the screen 4.10 mm from the center of the principal maximum.

8. The second-order bright fringe in a single-slit diffraction pattern is 1.40 mm from the center of the central maximum. The screen is 80.0 cm from a slit of width 0.800 mm. Assuming that the incident light is monochromatic, calculate the light's approximate wavelength.

9. If the light in Figure 38.5 strikes the single slit at an angle β from the perpendicular direction, show that Equation 38.1, the condition for destructive interference, must be modified to read

$$\sin\theta = m\left(\frac{\lambda}{a}\right) - \sin\beta$$

10. Coherent light with a wavelength of 501.5 nm is sent through two parallel slits in a large flat wall. Each slit is 0.700 μm wide, and the slits' centers are 2.80 μm apart. The light falls on a semicylindrical screen, with its axis at the midline between the slits. (a) Predict the direction of each interference maximum on the screen, as an angle away from the bisector of the line joining the slits. (b) Describe the pattern of light on the screen, specifying the number of bright fringes and the location of each. (c) Find the intensity of light on the screen at the center of each bright fringe, expressed as a fraction of the light intensity I_0 at the center of the pattern.

Section 38.3 Resolution of Single-Slit and Circular Apertures

11. The pupil of a cat's eye narrows to a vertical slit of width 0.500 mm in daylight. What is the angular resolution for horizontally separated mice? Assume that the average wavelength of the light is 500 nm.

12. Find the radius of a star image formed on the retina of the eye if the aperture diameter (the pupil) at night is 0.700 cm and the length of the eye is 3.00 cm. Assume that the representative wavelength of starlight in the eye is 500 nm.

WEB 13. A helium-neon laser emits light that has a wavelength of 632.8 nm. The circular aperture through which the beam emerges has a diameter of 0.500 cm. Estimate the diameter of the beam 10.0 km from the laser.

14. On the night of April 18, 1775, a signal was to be sent from the steeple of Old North Church in Boston to Paul Revere, who was 1.80 mi away: "One if by land, two if by sea." At what minimum separation did the sexton have to set the lanterns for Revere to receive the correct message? Assume that Revere's pupils had a diameter of 4.00 mm at night and that the lantern light had a predominant wavelength of 580 nm.

15. The Impressionist painter Georges Seurat created paintings with an enormous number of dots of pure pigment, each of which was approximately 2.00 mm in diameter. The idea was to locate colors such as red and green next to each other to form a scintillating canvas (Fig. P38.15). Outside what distance would one be unable to discern individual dots on the canvas? (Assume that $\lambda = 500$ nm and that the pupil diameter is 4.00 mm.)

16. A binary star system in the constellation Orion has an angular interstellar separation of 1.00×10^{-5} rad. If $\lambda = 500$ nm, what is the smallest diameter a telescope must have to just resolve the two stars?

Figure P38.15 *Sunday Afternoon on the Isle of La Grande Jatte,* by Georges Seurat. *(SuperStock)*

17. A child is standing at the edge of a straight highway watching her grandparents' car driving away at 20.0 m/s. The air is perfectly clear and steady, and after 10.0 min the car's two taillights appear to merge into one. Assuming the diameter of the child's pupils is 5.00 mm, estimate the width of the car.

18. Suppose that you are standing on a straight highway and watching a car moving away from you at a speed v. The air is perfectly clear and steady, and after a time t the taillights appear to merge into one. Assuming the diameter of your pupil is d, estimate the width of the car.

19. A circular radar antenna on a Coast Guard ship has a diameter of 2.10 m and radiates at a frequency of 15.0 GHz. Two small boats are located 9.00 km away from the ship. How close together could the boats be and still be detected as two objects?

20. If we were to send a ruby laser beam ($\lambda = 694.3$ nm) outward from the barrel of a 2.70-m-diameter telescope, what would be the diameter of the big red spot when the beam hit the Moon 384 000 km away? (Neglect atmospheric dispersion.)

21. The angular resolution of a radio telescope is to be 0.100° when the incident waves have a wavelength of 3.00 mm. What minimum diameter is required for the telescope's receiving dish?

22. When Mars is nearest the Earth, the distance separating the two planets is 88.6×10^6 km. Mars is viewed through a telescope whose mirror has a diameter of 30.0 cm. (a) If the wavelength of the light is 590 nm, what is the angular resolution of the telescope? (b) What is the smallest distance that can be resolved between two points on Mars?

Section 38.4 The Diffraction Grating

Note: In the following problems, assume that the light is incident normally on the gratings.

23. White light is spread out into its spectral components by a diffraction grating. If the grating has 2 000 lines per centimeter, at what angle does red light of wavelength 640 nm appear in first order?

24. Light from an argon laser strikes a diffraction grating that has 5 310 lines per centimeter. The central and first-order principal maxima are separated by 0.488 m on a wall 1.72 m from the grating. Determine the wavelength of the laser light.

WEB 25. The hydrogen spectrum has a red line at 656 nm and a violet line at 434 nm. What is the angular separation between two spectral lines obtained with a diffraction grating that has 4 500 lines per centimeter?

26. A helium-neon laser ($\lambda = 632.8$ nm) is used to calibrate a diffraction grating. If the first-order maximum occurs at 20.5°, what is the spacing between adjacent grooves in the grating?

27. Three discrete spectral lines occur at angles of 10.09°, 13.71°, and 14.77° in the first-order spectrum of a grating spectroscope. (a) If the grating has 3 660 slits per centimeter, what are the wavelengths of the light? (b) At what angles are these lines found in the second-order spectrum?

28. A diffraction grating has 800 rulings per millimeter. A beam of light containing wavelengths from 500 to 700 nm hits the grating. Do the spectra of different orders overlap? Explain.

WEB 29. A diffraction grating with a width of 4.00 cm has been ruled with 3 000 grooves per centimeter. (a) What is the resolving power of this grating in the first three orders? (b) If two monochromatic waves incident on this grating have a mean wavelength of 400 nm, what is their wavelength separation if they are just resolved in the third order?

30. Show that, whenever white light is passed through a diffraction grating of any spacing size, the violet end of the continuous visible spectrum in third order always overlaps the red light at the other end of the second-order spectrum.

31. A source emits 531.62-nm and 531.81-nm light. (a) What minimum number of lines is required for a grating that resolves the two wavelengths in the first-order spectrum? (b) Determine the slit spacing for a grating 1.32 cm wide that has the required minimum number of lines.

32. Two wavelengths λ and $\lambda + \Delta\lambda$ (with $\Delta\lambda \ll \lambda$) are incident on a diffraction grating. Show that the angular separation between the spectral lines in the mth order spectrum is

$$\Delta\theta = \frac{\Delta\lambda}{\sqrt{(d/m)^2 - \lambda^2}}$$

where d is the slit spacing and m is the order number.

33. A grating with 250 lines per millimeter is used with an incandescent light source. Assume that the visible spectrum ranges in wavelength from 400 to 700 nm. In how

many orders can one see (a) the entire visible spectrum and (b) the short-wavelength region?

34. A diffraction grating has 4 200 rulings per centimeter. On a screen 2.00 m from the grating, it is found that for a particular order m, the maxima corresponding to two closely spaced wavelengths of sodium (589.0 nm and 589.6 nm) are separated by 1.59 mm. Determine the value of m.

(Optional)
Section 38.5 Diffraction of X-Rays by Crystals

35. Potassium iodide (KI) has the same crystalline structure as NaCl, with $d = 0.353$ nm. A monochromatic x-ray beam shows a diffraction maximum when the grazing angle is 7.60°. Calculate the x-ray wavelength. (Assume first order.)

36. A wavelength of 0.129 nm characterizes K_α x-rays from zinc. When a beam of these x-rays is incident on the surface of a crystal whose structure is similar to that of NaCl, a first-order maximum is observed at 8.15°. Calculate the interplanar spacing on the basis of this information.

WEB **37.** If the interplanar spacing of NaCl is 0.281 nm, what is the predicted angle at which 0.140-nm x-rays are diffracted in a first-order maximum?

38. The first-order diffraction maximum is observed at 12.6° for a crystal in which the interplanar spacing is 0.240 nm. How many other orders can be observed?

39. Monochromatic x-rays of the K_α line from a nickel target ($\lambda = 0.166$ nm) are incident on a potassium chloride (KCl) crystal surface. The interplanar distance in KCl is 0.314 nm. At what angle (relative to the surface) should the beam be directed for a second-order maximum to be observed?

40. In water of uniform depth, a wide pier is supported on pilings in several parallel rows 2.80 m apart. Ocean waves of uniform wavelength roll in, moving in a direction that makes an angle of 80.0° with the rows of posts. Find the three longest wavelengths of waves that will be strongly reflected by the pilings.

Section 38.6 Polarization of Light Waves

41. Unpolarized light passes through two polaroid sheets. The axis of the first is vertical, and that of the second is at 30.0° to the vertical. What fraction of the initial light is transmitted?

42. Three polarizing disks whose planes are parallel are centered on a common axis. The direction of the transmission axis in each case is shown in Figure P38.42 relative to the common vertical direction. A plane-polarized beam of light with E_0 parallel to the vertical reference direction is incident from the left on the first disk with an intensity of $I_i = 10.0$ units (arbitrary). Calculate the transmitted intensity I_f when (a) $\theta_1 = 20.0°$, $\theta_2 = 40.0°$, and $\theta_3 = 60.0°$; (b) $\theta_1 = 0°$, $\theta_2 = 30.0°$, and $\theta_3 = 60.0°$.

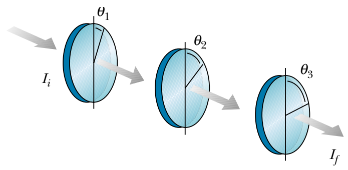


Figure P38.42 Problems 42 and 48.

43. Plane-polarized light is incident on a single polarizing disk with the direction of E_0 parallel to the direction of the transmission axis. Through what angle should the disk be rotated so that the intensity in the transmitted beam is reduced by a factor of (a) 3.00, (b) 5.00, (c) 10.0?

44. The angle of incidence of a light beam onto a reflecting surface is continuously variable. The reflected ray is found to be completely polarized when the angle of incidence is 48.0°. What is the index of refraction of the reflecting material?

45. The critical angle for total internal reflection for sapphire surrounded by air is 34.4°. Calculate the polarizing angle for sapphire.

46. For a particular transparent medium surrounded by air, show that the critical angle for total internal reflection and the polarizing angle are related by the expression $\cot \theta_p = \sin \theta_c$.

47. How far above the horizon is the Moon when its image reflected in calm water is completely polarized? ($n_{\text{water}} = 1.33$.)

ADDITIONAL PROBLEMS

48. In Figure P38.42, suppose that the transmission axes of the left and right polarizing disks are perpendicular to each other. Also, let the center disk be rotated on the common axis with an angular speed ω. Show that if unpolarized light is incident on the left disk with an intensity $I_{\max}$, the intensity of the beam emerging from the right disk is

$$I = \frac{1}{16} I_{\max}(1 - \cos 4\omega t)$$

This means that the intensity of the emerging beam is modulated at a rate that is four times the rate of rotation of the center disk. [*Hint:* Use the trigonometric identities $\cos^2\theta = (1 + \cos 2\theta)/2$ and $\sin^2\theta = (1 - \cos 2\theta)/2$, and recall that $\theta = \omega t$.]

49. You want to rotate the plane of polarization of a polarized light beam by 45.0° with a maximum intensity reduction of 10.0%. (a) How many sheets of perfect polarizers do you need to achieve your goal? (b) What is the angle between adjacent polarizers?

50. Figure P38.50 shows a megaphone in use. Construct a theoretical description of how a megaphone works. You may assume that the sound of your voice radiates just through the opening of your mouth. Most of the information in speech is carried not in a signal at the fundamental frequency, but rather in noises and in harmonics, with frequencies of a few thousand hertz. Does your theory allow any prediction that is simple to test?

Figure P38.50 *(Susan Allen Sigmon/Allsport USA)*

51. Light from a helium-neon laser ($\lambda = 632.8$ nm) is incident on a single slit. What is the maximum width for which no diffraction minima are observed?

52. What are the approximate dimensions of the smallest object on Earth that astronauts can resolve by eye when they are orbiting 250 km above the Earth? Assume that $\lambda = 500$ nm and that a pupil's diameter is 5.00 mm.

53. **Review Problem.** A beam of 541-nm light is incident on a diffraction grating that has 400 lines per millimeter. (a) Determine the angle of the second-order ray. (b) If the entire apparatus is immersed in water, what is the new second-order angle of diffraction? (c) Show that the two diffracted rays of parts (a) and (b) are related through the law of refraction.

54. The Very Large Array is a set of 27 radio telescope dishes in Caton and Socorro Counties, New Mexico (Fig. P38.54). The antennas can be moved apart on railroad tracks, and their combined signals give the resolving power of a synthetic aperture 36.0 km in diameter. (a) If the detectors are tuned to a frequency of 1.40 GHz, what is the angular resolution of the VLA? (b) Clouds of hydrogen radiate at this frequency. What must be the separation distance for two clouds at the center of the galaxy, 26 000 lightyears away, if they are to be resolved? (c) As the telescope looks up, a circling hawk looks down. For comparison, find the angular resolution of the hawk's eye. Assume that it is most sensitive to green light having a wavelength of 500 nm and that it has a pupil with a diameter of 12.0 mm. (d) A mouse is on the ground 30.0 m below. By what distance must the mouse's whiskers be separated for the hawk to resolve them?

Figure P38.54 A rancher in New Mexico rides past one of the 27 radio telescopes that make up the Very Large Array (VLA). *© Danny Lehman)*

55. Grote Reber was a pioneer in radio astronomy. He constructed a radio telescope with a 10.0-m diameter receiving dish. What was the telescope's angular resolution for 2.00-m radio waves?

56. A 750-nm light beam hits the flat surface of a certain liquid, and the beam is split into a reflected ray and a refracted ray. If the reflected ray is completely polarized at 36.0°, what is the wavelength of the refracted ray?

57. Light of wavelength 500 nm is incident normally on a diffraction grating. If the third-order maximum of the diffraction pattern is observed at 32.0°, (a) what is the number of rulings per centimeter for the grating? (b) Determine the total number of primary maxima that can be observed in this situation.

58. Light strikes a water surface at the polarizing angle. The part of the beam refracted into the water strikes a submerged glass slab (index of refraction, 1.50), as shown in Figure P38.58. If the light reflected from the upper surface of the slab is completely polarized, what is the angle between the water surface and the glass slab?

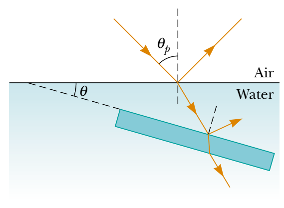


Figure P38.58

59. An American standard television picture is composed of about 485 horizontal lines of varying light intensity. Assume that your ability to resolve the lines is limited only by the Rayleigh criterion and that the pupils of your eyes are 5.00 mm in diameter. Calculate the ratio of minimum viewing distance to the vertical dimension of the picture such that you will not be able to resolve the lines. Assume that the average wavelength of the light coming from the screen is 550 nm.

60. (a) If light traveling in a medium for which the index of refraction is n_1 is incident at an angle θ on the surface of a medium of index n_2 so that the angle between the reflected and refracted rays is β, show that

$$\tan\theta = \frac{n_2 \sin\beta}{n_1 - n_2\cos\beta}$$

[*Hint:* Use the identity $\sin(A + B) = \sin A\cos B + \cos A\sin B$.] (b) Show that this expression for $\tan\theta$ reduces to Brewster's law when $\beta = 90°$, $n_1 = 1$, and $n_2 = n$.

61. Suppose that the single slit in Figure 38.6 is 6.00 cm wide and in front of a microwave source operating at 7.50 GHz. (a) Calculate the angle subtended by the first minimum in the diffraction pattern. (b) What is the relative intensity I/I_{max} at $\theta = 15.0°$? (c) Consider the case when there are two such sources, separated laterally by 20.0 cm, behind the slit. What must the maximum distance between the plane of the sources and the slit be if the diffraction patterns are to be resolved? (In this case, the approximation $\sin\theta \approx \tan\theta$ is not valid because of the relatively small value of a/λ.)

62. Two polarizing sheets are placed together with their transmission axes crossed so that no light is transmitted. A third sheet is inserted between them with its transmission axis at an angle of 45.0° with respect to each of the other axes. Find the fraction of incident unpolarized light intensity transmitted by the three-sheet combination. (Assume that each polarizing sheet is ideal.)

63. Figure P38.63a is a three-dimensional sketch of a birefringent crystal. The dotted lines illustrate how a thin parallel-faced slab of material could be cut from the larger specimen with the optic axis of the crystal parallel to the faces of the plate. A section cut from the crystal in this manner is known as a *retardation plate.* When a beam of light is incident on the plate perpendicular to the direction of the optic axis, as shown in Figure P38.63b, the O ray and the E ray travel along a single straight line but with different speeds. (a) Letting the thickness of the plate be d, show that the phase difference between the O ray and the E ray is

$$\theta = \frac{2\pi d}{\lambda}(n_O - n_E)$$

where λ is the wavelength in air. (b) If in a particular case the incident light has a wavelength of 550 nm, what is the minimum value of d for a quartz plate for which $\theta = \pi/2$? Such a plate is called a *quarter-wave plate.* (Use values of n_O and n_E from Table 38.1.)

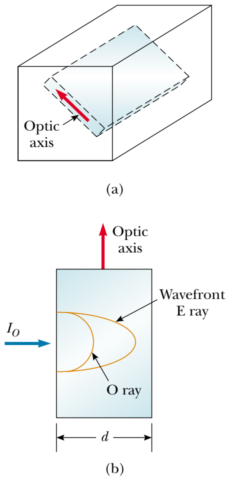


Figure P38.63

64. Derive Equation 38.12 for the resolving power of a grating, $R = Nm$, where N is the number of lines illuminated and m is the order in the diffraction pattern. Remember that Rayleigh's criterion (see Section 38.3) states that two wavelengths will be resolved when the principal maximum for one falls on the first minimum for the other.

65. Light of wavelength 632.8 nm illuminates a single slit, and a diffraction pattern is formed on a screen 1.00 m from the slit. Using the data in the table on the following page, plot relative intensity versus distance. Choose an appropriate value for the slit width a, and on the same graph used for the experimental data, plot the theoretical expression for the relative intensity

$$\frac{I}{I_{max}} = \frac{\sin^2(\beta/2)}{(\beta/2)^2}$$

What value of a gives the best fit of theory and experiment?

66. How much diffraction spreading does a light beam undergo? One quantitative answer is the *full width at half maximum* of the central maximum of the Fraunhofer diffraction pattern of a single slit. You can evaluate this angle of spreading in this problem and in the next. (a) In Equation 38.4, define $\beta/2 = \phi$ and show that, at the point where $I = 0.5I_{max}$, we must have $\sin\phi = \phi/\sqrt{2}$. (b) Let $y_1 = \sin\phi$ and $y_2 = \phi/\sqrt{2}$. Plot y_1 and y_2 on the same set of axes over a range from $\phi = 1$ rad to $\phi = \pi/2$ rad. Determine ϕ from the point of inter-

Relative Intensity	Distance from Center of Central Maximum (mm)
1.00	0
0.95	0.8
0.80	1.6
0.60	2.4
0.39	3.2
0.21	4.0
0.079	4.8
0.014	5.6
0.003	6.5
0.015	7.3
0.036	8.1
0.047	8.9
0.043	9.7
0.029	10.5
0.013	11.3
0.002	12.1
0.000 3	12.9
0.005	13.7
0.012	14.5
0.016	15.3
0.015	16.1
0.010	16.9
0.004 4	17.7
0.000 6	18.5
0.000 3	19.3
0.003	20.2

section of the two curves. (c) Then show that, if the fraction λ/a is not large, the angular full width at half maximum of the central diffraction maximum is $\Delta\theta = 0.886\lambda/a$.

67. Another method to solve the equation $\phi = \sqrt{2}\sin\phi$ in Problem 66 is to use a calculator, guess a first value of ϕ, see if it fits, and continue to update your estimate until the equation balances. How many steps (iterations) does this take?

68. In the diffraction pattern of a single slit, described by the equation

$$I_\theta = I_{max}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

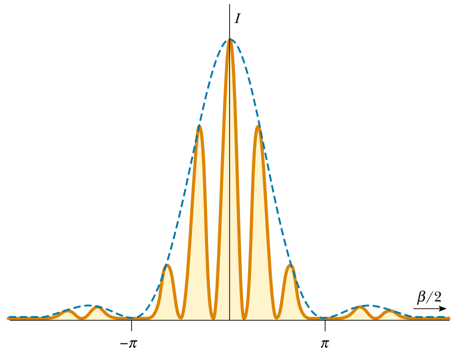
with $\beta = (2\pi a\sin\theta)/\lambda$, the central maximum is at $\beta = 0$ and the side maxima are *approximately* at $\beta/2 = (m + \frac{1}{2})\pi$ for $m = 1, 2, 3, \ldots$. Determine more precisely (a) the location of the first side maximum, where $m = 1$, and (b) the location of the second side maximum. Observe in Figure 38.10a that the graph of intensity versus $\beta/2$ has a horizontal tangent at maxima and also at minima. You will need to solve a transcendental equation.

69. A *pinhole camera* has a small circular aperture of diameter D. Light from distant objects passes through the aperture into an otherwise dark box, falling upon a screen located a distance L away. If D is too large, the display on the screen will be fuzzy because a bright point in the field of view will send light onto a circle of diameter slightly larger than D. On the other hand, if D is too small, diffraction will blur the display on the screen. The screen shows a reasonably sharp image if the diameter of the central disk of the diffraction pattern, specified by Equation 38.9, is equal to D at the screen. (a) Show that for monochromatic light with plane wave fronts and $L \gg D$, the condition for a sharp view is fulfilled if $D^2 = 2.44\,\lambda L$. (b) Find the optimum pinhole diameter if 500-nm light is projected onto a screen 15.0 cm away.

ANSWERS TO QUICK QUIZZES

38.1 The space between the slightly open door and the doorframe acts as a single slit. Sound waves have wavelengths that are approximately the same size as the opening and so are diffracted and spread throughout the room you are in. Because light wavelengths are much smaller than the slit width, they are virtually undiffracted. As a result, you must have a direct line of sight to detect the light waves.

38.2 The situation is like that depicted in Figure 38.11 except that now the slits are only half as far apart. The diffraction pattern is the same, but the interference pattern is stretched out because d is smaller. Because $d/a = 3$, the third interference maximum coincides with the first diffraction minimum. Your sketch should look like the figure to the right.



38.3 Yes, but no diffraction effects are observed because the separation distance between adjacent ribs is so much greater than the wavelength of the x-rays.

PUZZLER

The wristwatches worn by the people in this commercial jetliner properly record the passage of time as experienced by the travelers. Amazingly, however, the duration of the trip as measured by an Earth-bound observer is very slightly longer. How can high-speed travel affect something as regular as the ticking of a clock? *(© Larry Mulvehill/Photo Researchers, Inc.)*

chapter

39 Relativity

Chapter Outline

Most of our everyday experiences and observations have to do with objects that move at speeds much less than the speed of light. Newtonian mechanics was formulated to describe the motion of such objects, and this formalism is still very successful in describing a wide range of phenomena that occur at low speeds. It fails, however, when applied to particles whose speeds approach that of light.

Experimentally, the predictions of Newtonian theory can be tested at high speeds by accelerating electrons or other charged particles through a large electric potential difference. For example, it is possible to accelerate an electron to a speed of $0.99c$ (where c is the speed of light) by using a potential difference of several million volts. According to Newtonian mechanics, if the potential difference is increased by a factor of 4, the electron's kinetic energy is four times greater and its speed should double to $1.98c$. However, experiments show that the speed of the electron—as well as the speed of any other particle in the Universe—always remains less than the speed of light, regardless of the size of the accelerating voltage. Because it places no upper limit on speed, Newtonian mechanics is contrary to modern experimental results and is clearly a limited theory.

In 1905, at the age of only 26, Einstein published his special theory of relativity. Regarding the theory, Einstein wrote:

> The relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties . . . . [1]

Although Einstein made many other important contributions to science, the special theory of relativity alone represents one of the greatest intellectual achievements of all time. With this theory, experimental observations can be correctly predicted over the range of speeds from $v = 0$ to speeds approaching the speed of light. At low speeds, Einstein's theory reduces to Newtonian mechanics as a limiting situation. It is important to recognize that Einstein was working on electromagnetism when he developed the special theory of relativity. He was convinced that Maxwell's equations were correct, and in order to reconcile them with one of his postulates, he was forced into the bizarre notion of assuming that space and time are not absolute.

This chapter gives an introduction to the special theory of relativity, with emphasis on some of its consequences. The special theory covers phenomena such as the slowing down of clocks and the contraction of lengths in moving reference frames as measured by a stationary observer. We also discuss the relativistic forms of momentum and energy, as well as some consequences of the famous mass–energy formula, $E = mc^2$.

In addition to its well-known and essential role in theoretical physics, the special theory of relativity has practical applications, including the design of nuclear power plants and modern global positioning system (GPS) units. These devices do not work if designed in accordance with nonrelativistic principles.

We shall have occasion to use relativity in some subsequent chapters of the extended version of this text, most often presenting only the outcome of relativistic effects.

[1] A. Einstein and L. Infeld, *The Evolution of Physics,* New York, Simon and Schuster, 1961.

39.1 THE PRINCIPLE OF GALILEAN RELATIVITY

To describe a physical event, it is necessary to establish a frame of reference. You should recall from Chapter 5 that Newton's laws are valid in all inertial frames of reference. Because an inertial frame is defined as one in which Newton's first law is valid, we can say that **an inertial frame of reference is one in which an object is observed to have no acceleration when no forces act on it.** Furthermore, any system moving with constant velocity with respect to an inertial system must also be an inertial system.

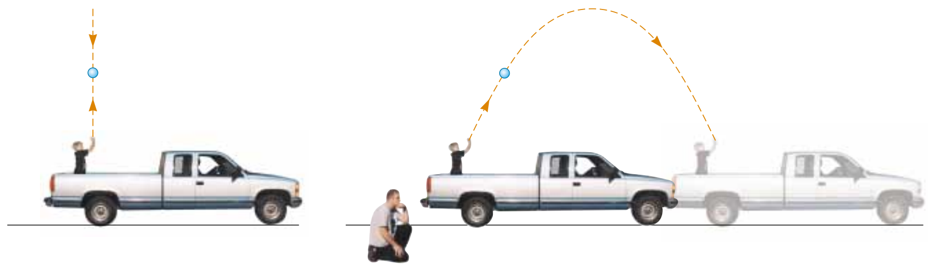
Inertial frame of reference

There is no preferred inertial reference frame. This means that the results of an experiment performed in a vehicle moving with uniform velocity will be identical to the results of the same experiment performed in a stationary vehicle. The formal statement of this result is called the **principle of Galilean relativity:**

> The laws of mechanics must be the same in all inertial frames of reference.

Let us consider an observation that illustrates the equivalence of the laws of mechanics in different inertial frames. A pickup truck moves with a constant velocity, as shown in Figure 39.1a. If a passenger in the truck throws a ball straight up, and if air effects are neglected, the passenger observes that the ball moves in a vertical path. The motion of the ball appears to be precisely the same as if the ball were thrown by a person at rest on the Earth. The law of gravity and the equations of motion under constant acceleration are obeyed whether the truck is at rest or in uniform motion.

Now consider the same situation viewed by an observer at rest on the Earth. This stationary observer sees the path of the ball as a parabola, as illustrated in Figure 39.1b. Furthermore, according to this observer, the ball has a horizontal component of velocity equal to the velocity of the truck. Although the two observers disagree on certain aspects of the situation, they agree on the validity of Newton's laws and on such classical principles as conservation of energy and conservation of linear momentum. This agreement implies that no mechanical experiment can detect any difference between the two inertial frames. The only thing that can be detected is the relative motion of one frame with respect to the other. That is, the notion of absolute motion through space is meaningless, as is the notion of a preferred reference frame.



(a) (b)

Figure 39.1 (a) The observer in the truck sees the ball move in a vertical path when thrown upward. (b) The Earth observer sees the path of the ball as a parabola.

Quick Quiz 39.1

Which observer in Figure 39.1 is right about the ball's path?

Suppose that some physical phenomenon, which we call an *event,* occurs in an inertial system. The event's location and time of occurrence can be specified by the four coordinates (x, y, z, t). We would like to be able to transform these coordinates from one inertial system to another one moving with uniform relative velocity.

Consider two inertial systems S and S′ (Fig. 39.2). The system S′ moves with a constant velocity **v** along the xx' axes, where **v** is measured relative to S. We assume that an event occurs at the point P and that the origins of S and S′ coincide at $t = 0$. An observer in S describes the event with space–time coordinates (x, y, z, t), whereas an observer in S′ uses the coordinates (x', y', z', t') to describe the same event. As we see from Figure 39.2, the relationships between these various coordinates can be written

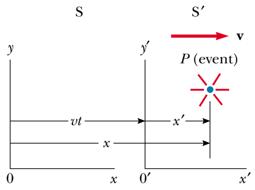


Figure 39.2 An event occurs at a point P. The event is seen by two observers in inertial frames S and S′, where S′ moves with a velocity **v** relative to S.

$$x' = x - vt$$
$$y' = y \qquad \textbf{(39.1)}$$
$$z' = z$$
$$t' = t$$

Galilean space–time transformation equations

These equations are the **Galilean space–time transformation equations.** Note that time is assumed to be the same in both inertial systems. That is, within the framework of classical mechanics, all clocks run at the same rate, regardless of their velocity, so that the time at which an event occurs for an observer in S is the same as the time for the same event in S′. Consequently, the time interval between two successive events should be the same for both observers. Although this assumption may seem obvious, it turns out to be incorrect in situations where v is comparable to the speed of light.

Now suppose that a particle moves a distance dx in a time interval dt as measured by an observer in S. It follows from Equations 39.1 that the corresponding distance dx' measured by an observer in S′ is $dx' = dx - v\,dt$, where frame S′ is moving with speed v relative to frame S. Because $dt = dt'$, we find that

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

or

$$u_x' = u_x - v \qquad \textbf{(39.2)}$$

Galilean velocity transformation equation

where u_x and u_x' are the x components of the velocity relative to S and S′, respectively. (We use the symbol **u** for particle velocity rather than **v**, which is used for the relative velocity of two reference frames.) This is the **Galilean velocity transformation equation.** It is used in everyday observations and is consistent with our intuitive notion of time and space. As we shall soon see, however, it leads to serious contradictions when applied to electromagnetic waves.

Quick Quiz 39.2

Applying the Galilean velocity transformation equation, determine how fast (relative to the Earth) a baseball pitcher with a 90-mi/h fastball can throw a ball while standing in a boxcar moving at 110 mi/h.

The Speed of Light

It is quite natural to ask whether the principle of Galilean relativity also applies to electricity, magnetism, and optics. Experiments indicate that the answer is no. Recall from Chapter 34 that Maxwell showed that the speed of light in free space is $c = 3.00 \times 10^8$ m/s. Physicists of the late 1800s thought that light waves moved through a medium called the *ether* and that the speed of light was c only in a special, absolute frame at rest with respect to the ether. The Galilean velocity transformation equation was expected to hold in any frame moving at speed v relative to the absolute ether frame.

Because the existence of a preferred, absolute ether frame would show that light was similar to other classical waves and that Newtonian ideas of an absolute frame were true, considerable importance was attached to establishing the existence of the ether frame. Prior to the late 1800s, experiments involving light traveling in media moving at the highest laboratory speeds attainable at that time were not capable of detecting changes as small as $c \pm v$. Starting in about 1880, scientists decided to use the Earth as the moving frame in an attempt to improve their chances of detecting these small changes in the speed of light.

As observers fixed on the Earth, we can say that we are stationary and that the absolute ether frame containing the medium for light propagation moves past us with speed v. Determining the speed of light under these circumstances is just like determining the speed of an aircraft traveling in a moving air current, or wind; consequently, we speak of an "ether wind" blowing through our apparatus fixed to the Earth.

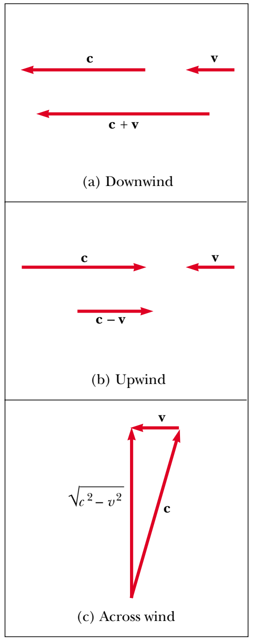


Figure 39.3 If the velocity of the ether wind relative to the Earth is **v** and the velocity of light relative to the ether is **c**, then the speed of light relative to the Earth is (a) $c + v$ in the downwind direction, (b) $c - v$ in the upwind direction, and (c) $(c^2 - v^2)^{1/2}$ in the direction perpendicular to the wind.

A direct method for detecting an ether wind would use an apparatus fixed to the Earth to measure the wind's influence on the speed of light. If v is the speed of the ether relative to the Earth, then the speed of light should have its maximum value, $c + v$, when propagating downwind, as shown in Figure 39.3a. Likewise, the speed of light should have its minimum value, $c - v$, when propagating upwind, as shown in Figure 39.3b, and an intermediate value, $(c^2 - v^2)^{1/2}$, in the direction perpendicular to the ether wind, as shown in Figure 39.3c. If the Sun is assumed to be at rest in the ether, then the velocity of the ether wind would be equal to the orbital velocity of the Earth around the Sun, which has a magnitude of approximately 3×10^4 m/s. Because $c = 3 \times 10^8$ m/s, it should be possible to detect a change in speed of about 1 part in 10^4 for measurements in the upwind or downwind directions. However, as we shall see in the next section, all attempts to detect such changes and establish the existence of the ether wind (and hence the absolute frame) proved futile! (You may want to return to Problem 40 in Chapter 4 to see a situation in which the Galilean velocity transformation equation does hold.)

If it is assumed that the laws of electricity and magnetism are the same in all inertial frames, a paradox concerning the speed of light immediately arises. We can understand this by recognizing that Maxwell's equations seem to imply that the speed of light always has the fixed value 3.00×10^8 m/s in all inertial frames, a result in direct contradiction to what is expected based on the Galilean velocity transformation equation. According to Galilean relativity, the speed of light should not be the same in all inertial frames.

For example, suppose a light pulse is sent out by an observer S′ standing in a boxcar moving with a velocity **v** relative to a stationary observer standing alongside the track (Fig. 39.4). The light pulse has a speed c relative to S′. According to Galilean relativity, the pulse speed relative to S should be $c + v$. This is in contradiction to Einstein's special theory of relativity, which, as we shall see, postulates that the speed of the pulse is the same for all observers.

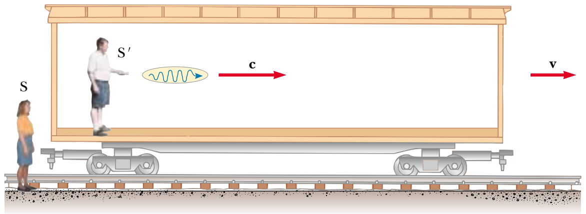


Figure 39.4 A pulse of light is sent out by a person in a moving boxcar. According to Galilean relativity, the speed of the pulse should be $c + v$ relative to a stationary observer.

To resolve this contradiction in theories, we must conclude that either (1) the laws of electricity and magnetism are not the same in all inertial frames or (2) the Galilean velocity transformation equation is incorrect. If we assume the first alternative, then a preferred reference frame in which the speed of light has the value c must exist and the measured speed must be greater or less than this value in any other reference frame, in accordance with the Galilean velocity transformation equation. If we assume the second alternative, then we are forced to abandon the notions of absolute time and absolute length that form the basis of the Galilean space–time transformation equations.

39.2 THE MICHELSON–MORLEY EXPERIMENT

The most famous experiment designed to detect small changes in the speed of light was first performed in 1881 by Albert A. Michelson (see Section 37.7) and later repeated under various conditions by Michelson and Edward W. Morley (1838–1923). We state at the outset that the outcome of the experiment contradicted the ether hypothesis.

The experiment was designed to determine the velocity of the Earth relative to that of the hypothetical ether. The experimental tool used was the Michelson interferometer, which was discussed in Section 37.7 and is shown again in Figure 39.5. Arm 2 is aligned along the direction of the Earth's motion through space. The Earth moving through the ether at speed v is equivalent to the ether flowing past the Earth in the opposite direction with speed v. This ether wind blowing in the direction opposite the direction of Earth's motion should cause the speed of light measured in the Earth frame to be $c - v$ as the light approaches mirror M_2 and $c + v$ after reflection, where c is the speed of light in the ether frame.

The two beams reflected from M_1 and M_2 recombine, and an interference pattern consisting of alternating dark and bright fringes is formed. The interference pattern was observed while the interferometer was rotated through an angle of 90°. This rotation supposedly would change the speed of the ether wind along the arms of the interferometer. The rotation should have caused the fringe pattern to shift slightly but measurably, but measurements failed to show any change in the interference pattern! The Michelson–Morley experiment was repeated at different times of the year when the ether wind was expected to change direction

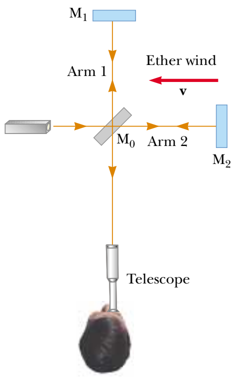
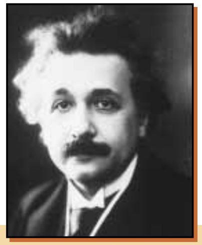


Figure 39.5 According to the ether wind theory, the speed of light should be $c - v$ as the beam approaches mirror M_2 and $c + v$ after reflection.

and magnitude, but the results were always the same: **no fringe shift of the magnitude required was ever observed.**[2]

The negative results of the Michelson–Morley experiment not only contradicted the ether hypothesis but also showed that it was impossible to measure the absolute velocity of the Earth with respect to the ether frame. However, as we shall see in the next section, Einstein offered a postulate for his special theory of relativity that places quite a different interpretation on these null results. In later years, when more was known about the nature of light, the idea of an ether that permeates all of space was relegated to the ash heap of worn-out concepts. **Light is now understood to be an electromagnetic wave, which requires no medium for its propagation.** As a result, the idea of an ether in which these waves could travel became unnecessary.



Albert Einstein (1879–1955)
Einstein, one of the greatest physicists of all times, was born in Ulm, Germany. In 1905, at the age of 26, he published four scientific papers that revolutionized physics. Two of these papers were concerned with what is now considered his most important contribution: the special theory of relativity.

In 1916, Einstein published his work on the general theory of relativity. The most dramatic prediction of this theory is the degree to which light is deflected by a gravitational field. Measurements made by astronomers on bright stars in the vicinity of the eclipsed Sun in 1919 confirmed Einstein's prediction, and as a result Einstein became a world celebrity.

Einstein was deeply disturbed by the development of quantum mechanics in the 1920s despite his own role as a scientific revolutionary. In particular, he could never accept the probabilistic view of events in nature that is a central feature of quantum theory. The last few decades of his life were devoted to an unsuccessful search for a unified theory that would combine gravitation and electromagnetism. *(AIP Niels Bohr Library)*

Optional Section

Details of the Michelson–Morley Experiment

To understand the outcome of the Michelson–Morley experiment, let us assume that the two arms of the interferometer in Figure 39.5 are of equal length L. We shall analyze the situation as if there were an ether wind, because that is what Michelson and Morley expected to find. As noted above, the speed of the light beam along arm 2 should be $c - v$ as the beam approaches M_2 and $c + v$ after the beam is reflected. Thus, the time of travel to the right is $L/(c - v)$, and the time of travel to the left is $L/(c + v)$. The total time needed for the round trip along arm 2 is

$$t_1 = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2Lc}{c^2 - v^2} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

Now consider the light beam traveling along arm 1, perpendicular to the ether wind. Because the speed of the beam relative to the Earth is $(c^2 - v^2)^{1/2}$ in this case (see Fig. 39.3), the time of travel for each half of the trip is $L/(c^2 - v^2)^{1/2}$, and the total time of travel for the round trip is

$$t_2 = \frac{2L}{(c^2 - v^2)^{1/2}} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Thus, the time difference between the horizontal round trip (arm 2) and the vertical round trip (arm 1) is

$$\Delta t = t_1 - t_2 = \frac{2L}{c}\left[\left(1 - \frac{v^2}{c^2}\right)^{-1} - \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right]$$

Because $v^2/c^2 \ll 1$, we can simplify this expression by using the following binomial expansion after dropping all terms higher than second order:

$$(1 - x)^n \approx 1 - nx \qquad \text{for } x \ll 1$$

In our case, $x = v^2/c^2$, and we find that

$$\Delta t = t_1 - t_2 \approx \frac{Lv^2}{c^3} \qquad \textbf{(39.3)}$$

This time difference between the two instants at which the reflected beams arrive at the viewing telescope gives rise to a phase difference between the beams,

[2] From an Earth observer's point of view, changes in the Earth's speed and direction of motion in the course of a year are viewed as ether wind shifts. Even if the speed of the Earth with respect to the ether were zero at some time, six months later the speed of the Earth would be 60 km/s with respect to the ether, and as a result a fringe shift should be noticed. No shift has ever been observed, however.

producing an interference pattern when they combine at the position of the telescope. A shift in the interference pattern should be detected when the interferometer is rotated through 90° in a horizontal plane, so that the two beams exchange roles. This results in a time difference twice that given by Equation 39.3. Thus, the path difference that corresponds to this time difference is

$$\Delta d = c(2\,\Delta t) = \frac{2Lv^2}{c^2}$$

Because a change in path length of one wavelength corresponds to a shift of one fringe, the corresponding fringe shift is equal to this path difference divided by the wavelength of the light:

$$\text{Shift} = \frac{2Lv^2}{\lambda c^2} \tag{39.4}$$

In the experiments by Michelson and Morley, each light beam was reflected by mirrors many times to give an effective path length L of approximately 11 m. Using this value and taking v to be equal to 3.0×10^4 m/s, the speed of the Earth around the Sun, we obtain a path difference of

$$\Delta d = \frac{2(11\text{ m})(3.0 \times 10^4\text{ m/s})^2}{(3.0 \times 10^8\text{ m/s})^2} = 2.2 \times 10^{-7}\text{ m}$$

This extra travel distance should produce a noticeable shift in the fringe pattern. Specifically, using 500-nm light, we expect a fringe shift for rotation through 90° of

$$\text{Shift} = \frac{\Delta d}{\lambda} = \frac{2.2 \times 10^{-7}\text{ m}}{5.0 \times 10^{-7}\text{ m}} \approx 0.44$$

The instrument used by Michelson and Morley could detect shifts as small as 0.01 fringe. However, **it detected no shift whatsoever in the fringe pattern.** Since then, the experiment has been repeated many times by different scientists under a wide variety of conditions, and no fringe shift has ever been detected. Thus, it was concluded that the motion of the Earth with respect to the postulated ether cannot be detected.

Many efforts were made to explain the null results of the Michelson–Morley experiment and to save the ether frame concept and the Galilean velocity transformation equation for light. All proposals resulting from these efforts have been shown to be wrong. No experiment in the history of physics received such valiant efforts to explain the absence of an expected result as did the Michelson–Morley experiment. The stage was set for Einstein, who solved the problem in 1905 with his special theory of relativity.

39.3 EINSTEIN'S PRINCIPLE OF RELATIVITY

In the previous section we noted the impossibility of measuring the speed of the ether with respect to the Earth and the failure of the Galilean velocity transformation equation in the case of light. Einstein proposed a theory that boldly removed these difficulties and at the same time completely altered our notion of space and time.[3] He based his special theory of relativity on two postulates:

[3] A. Einstein, "On the Electrodynamics of Moving Bodies," *Ann. Physik* 17:891, 1905. For an English translation of this article and other publications by Einstein, see the book by H. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity,* Dover, 1958.

The postulates of the special theory of relativity

1. **The principle of relativity:** The laws of physics must be the same in all inertial reference frames.
2. **The constancy of the speed of light:** The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

The first postulate asserts that *all* the laws of physics—those dealing with mechanics, electricity and magnetism, optics, thermodynamics, and so on—are the same in all reference frames moving with constant velocity relative to one another. This postulate is a sweeping generalization of the principle of Galilean relativity, which refers only to the laws of mechanics. From an experimental point of view, Einstein's principle of relativity means that any kind of experiment (measuring the speed of light, for example) performed in a laboratory at rest must give the same result when performed in a laboratory moving at a constant velocity past the first one. Hence, no preferred inertial reference frame exists, and it is impossible to detect absolute motion.

Note that postulate 2 is required by postulate 1: If the speed of light were not the same in all inertial frames, measurements of different speeds would make it possible to distinguish between inertial frames; as a result, a preferred, absolute frame could be identified, in contradiction to postulate 1.

Although the Michelson–Morley experiment was performed before Einstein published his work on relativity, it is not clear whether or not Einstein was aware of the details of the experiment. Nonetheless, the null result of the experiment can be readily understood within the framework of Einstein's theory. According to his principle of relativity, the premises of the Michelson–Morley experiment were incorrect. In the process of trying to explain the expected results, we stated that when light traveled against the ether wind its speed was $c - v$, in accordance with the Galilean velocity transformation equation. However, if the state of motion of the observer or of the source has no influence on the value found for the speed of light, one always measures the value to be c. Likewise, the light makes the return trip after reflection from the mirror at speed c, not at speed $c + v$. Thus, the motion of the Earth does not influence the fringe pattern observed in the Michelson–Morley experiment, and a null result should be expected.

If we accept Einstein's theory of relativity, we must conclude that relative motion is unimportant when measuring the speed of light. At the same time, we shall see that we must alter our common-sense notion of space and time and be prepared for some bizarre consequences. It may help as you read the pages ahead to keep in mind that our common-sense ideas are based on a lifetime of everyday experiences and not on observations of objects moving at hundreds of thousands of kilometers per second.

39.4 CONSEQUENCES OF THE SPECIAL THEORY OF RELATIVITY

Before we discuss the consequences of Einstein's special theory of relativity, we must first understand how an observer located in an inertial reference frame describes an event. As mentioned earlier, an event is an occurrence describable by three space coordinates and one time coordinate. Different observers in different inertial frames usually describe the same event with different coordinates.

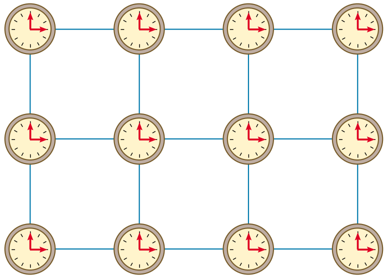


Figure 39.6 In studying relativity, we use a reference frame consisting of a coordinate grid and a set of synchronized clocks.

The reference frame used to describe an event consists of a coordinate grid and a set of synchronized clocks located at the grid intersections, as shown in Figure 39.6 in two dimensions. The clocks can be synchronized in many ways with the help of light signals. For example, suppose an observer is located at the origin with a master clock and sends out a pulse of light at $t = 0$. The pulse takes a time r/c to reach a clock located a distance r from the origin. Hence, this clock is synchronized with the master clock if this clock reads r/c at the instant the pulse reaches it. This procedure of synchronization assumes that the speed of light has the same value in all directions and in all inertial frames. Furthermore, the procedure concerns an event recorded by an observer in a specific inertial reference frame. An observer in some other inertial frame would assign different space–time coordinates to events being observed by using another coordinate grid and another array of clocks.

As we examine some of the consequences of relativity in the remainder of this section, we restrict our discussion to the concepts of simultaneity, time, and length, all three of which are quite different in relativistic mechanics from what they are in Newtonian mechanics. For example, in relativistic mechanics the distance between two points and the time interval between two events depend on the frame of reference in which they are measured. That is, **in relativistic mechanics there is no such thing as absolute length or absolute time.** Furthermore, **events at different locations that are observed to occur simultaneously in one frame are not observed to be simultaneous in another frame moving uniformly past the first.**

Simultaneity and the Relativity of Time

A basic premise of Newtonian mechanics is that a universal time scale exists that is the same for all observers. In fact, Newton wrote that "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external." Thus, Newton and his followers simply took simultaneity for granted. In his special theory of relativity, Einstein abandoned this assumption.

Einstein devised the following thought experiment to illustrate this point. A boxcar moves with uniform velocity, and two lightning bolts strike its ends, as illustrated in Figure 39.7a, leaving marks on the boxcar and on the ground. The marks on the boxcar are labeled A' and B', and those on the ground are labeled A and B. An observer O' moving with the boxcar is midway between A' and B', and a ground observer O is midway between A and B. The events recorded by the observers are the striking of the boxcar by the two lightning bolts.

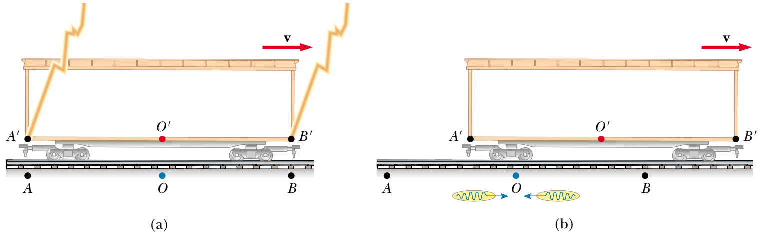


Figure 39.7 (a) Two lightning bolts strike the ends of a moving boxcar. (b) The events appear to be simultaneous to the stationary observer O, standing midway between A and B. The events do not appear to be simultaneous to observer O', who claims that the front of the car is struck before the rear. Note that in (b) the leftward-traveling light signal has already passed O' but the rightward-traveling signal has not yet reached O'.

The light signals recording the instant at which the two bolts strike reach observer O at the same time, as indicated in Figure 39.7b. This observer realizes that the signals have traveled at the same speed over equal distances, and so rightly concludes that the events at A and B occurred simultaneously. Now consider the same events as viewed by observer O'. By the time the signals have reached observer O, observer O' has moved as indicated in Figure 39.7b. Thus, the signal from B' has already swept past O', but the signal from A' has not yet reached O'. In other words, O' sees the signal from B' before seeing the signal from A'. According to Einstein, *the two observers must find that light travels at the same speed.* Therefore, observer O' concludes that the lightning strikes the front of the boxcar before it strikes the back.

This thought experiment clearly demonstrates that the two events that appear to be simultaneous to observer O do not appear to be simultaneous to observer O'. In other words,

> two events that are simultaneous in one reference frame are in general not simultaneous in a second frame moving relative to the first. That is, simultaneity is not an absolute concept but rather one that depends on the state of motion of the observer.

Quick Quiz 39.3

Which observer in Figure 39.7 is correct?

The central point of relativity is this: Any inertial frame of reference can be used to describe events and do physics. **There is no preferred inertial frame of reference.** However, observers in different inertial frames always measure different time intervals with their clocks and different distances with their meter sticks. Nevertheless, all observers agree on the forms of the laws of physics in their respective frames because these laws must be the same for all observers in uniform motion. For example, the relationship $F = ma$ in a frame S has the same form $F' = ma'$ in a frame S′ that is moving at constant velocity relative to frame S. It is the alteration of time and space that allows the laws of physics (including Maxwell's equations) to be the same for all observers in uniform motion.

Time Dilation

We can illustrate the fact that observers in different inertial frames always measure different time intervals between a pair of events by considering a vehicle moving to the right with a speed v, as shown in Figure 39.8a. A mirror is fixed to the ceiling of the vehicle, and observer O' at rest in this system holds a laser a distance d below the mirror. At some instant, the laser emits a pulse of light directed toward the mirror (event 1), and at some later time after reflecting from the mirror, the pulse arrives back at the laser (event 2). Observer O' carries a clock C' and uses it to measure the time interval Δt_p between these two events. (The subscript p stands for *proper,* as we shall see in a moment.) Because the light pulse has a speed c, the time it takes the pulse to travel from O' to the mirror and back to O' is

$$\Delta t_p = \frac{\text{Distance traveled}}{\text{Speed}} = \frac{2d}{c} \qquad \textbf{(39.5)}$$

This time interval Δt_p measured by O' requires only a single clock C' located at the same place as the laser in this frame.

Now consider the same pair of events as viewed by observer O in a second frame, as shown in Figure 39.8b. According to this observer, the mirror and laser are moving to the right with a speed v, and as a result the sequence of events appears entirely different. By the time the light from the laser reaches the mirror, the mirror has moved to the right a distance $v\,\Delta t/2$, where Δt is the time it takes the light to travel from O' to the mirror and back to O' as measured by O. In other words, O concludes that, because of the motion of the vehicle, if the light is to hit the mirror, it must leave the laser at an angle with respect to the vertical direction. Comparing Figure 39.8a and b, we see that the light must travel farther in (b) than in (a). (Note that neither observer "knows" that he or she is moving. Each is at rest in his or her own inertial frame.)

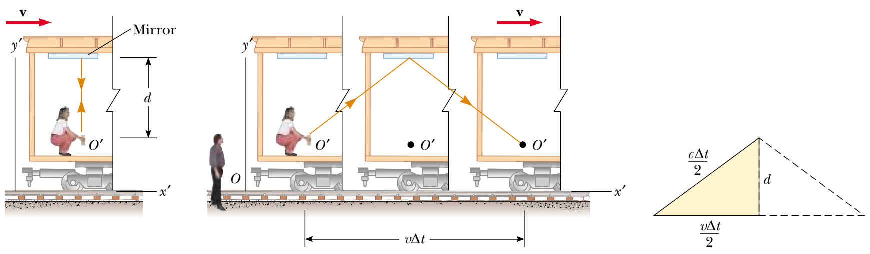


Figure 39.8 (a) A mirror is fixed to a moving vehicle, and a light pulse is sent out by observer O' at rest in the vehicle. (b) Relative to a stationary observer O standing alongside the vehicle, the mirror and O' move with a speed v. Note that what observer O measures for the distance the pulse travels is greater than $2d$. (c) The right triangle for calculating the relationship between Δt and Δt_p.

According to the second postulate of the special theory of relativity, both observers must measure c for the speed of light. Because the light travels farther in the frame of O, it follows that the time interval Δt measured by O is longer than the time interval Δt_p measured by O'. To obtain a relationship between these two time intervals, it is convenient to use the right triangle shown in Figure 39.8c. The Pythagorean theorem gives

$$\left(\frac{c\,\Delta t}{2}\right)^2 = \left(\frac{v\,\Delta t}{2}\right)^2 + d^2$$

Solving for Δt gives

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{1 - \dfrac{v^2}{c^2}}} \qquad \textbf{(39.6)}$$

Because $\Delta t_p = 2d/c$, we can express this result as

Time dilation

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \gamma\,\Delta t_p \qquad \textbf{(39.7)}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2} \qquad \textbf{(39.8)}$$

Because γ is always greater than unity, this result says that **the time interval Δt measured by an observer moving with respect to a clock is longer than the time interval Δt_p measured by an observer at rest with respect to the clock.** (That is, $\Delta t > \Delta t_p$.) This effect is known as **time dilation.** Figure 39.9 shows that as the velocity approaches the speed of light, γ increases dramatically. Note that for speeds less than one tenth the speed of light, γ is very nearly equal to unity.

The time interval Δt_p in Equations 39.5 and 39.7 is called the **proper time.** (In German, Einstein used the term *Eigenzeit,* which means "own-time.") In general, **proper time is the time interval between two events measured by an observer who sees the events occur at the same point in space.** Proper time is always the time measured with a single clock (clock C' in our case) at rest in the frame in which the events take place.

If a clock is moving with respect to you, it appears to fall behind (tick more slowly than) the clocks it is passing in the grid of synchronized clocks in your reference frame. Because the time interval $\gamma(2d/c)$, the interval between ticks of a moving

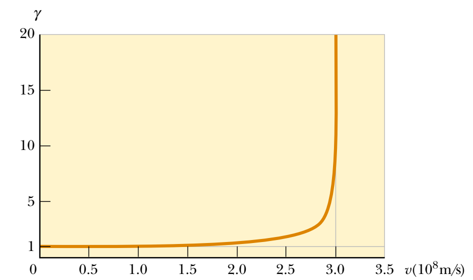


Figure 39.9 Graph of γ versus v. As the velocity approaches the speed of light, γ increases rapidly.

clock, is observed to be longer than $2d/c$, the time interval between ticks of an identical clock in your reference frame, it is often said that a moving clock runs more slowly than a clock in your reference frame by a factor γ. This is true for mechanical clocks as well as for the light clock just described. We can generalize this result by stating that all physical processes, including chemical and biological ones, slow down relative to a stationary clock when those processes occur in a moving frame. For example, the heartbeat of an astronaut moving through space would keep time with a clock inside the spaceship. Both the astronaut's clock and heartbeat would be slowed down relative to a stationary clock back on the Earth (although the astronaut would have no sensation of life slowing down in the spaceship).

Quick Quiz 39.4

A rocket has a clock built into its control panel. Use Figure 39.9 to determine approximately how fast the rocket must be moving before its clock appears to an Earth-bound observer to be ticking at one fifth the rate of a clock on the wall at Mission Control. What does an astronaut in the rocket observe?

Bizarre as it may seem, time dilation is a verifiable phenomenon. An experiment reported by Hafele and Keating provided direct evidence of time dilation.[4] Time intervals measured with four cesium atomic clocks in jet flight were compared with time intervals measured by Earth-based reference atomic clocks. In order to compare these results with theory, many factors had to be considered, including periods of acceleration and deceleration relative to the Earth, variations in direction of travel, and the fact that the gravitational field experienced by the flying clocks was weaker than that experienced by the Earth-based clock. The results were in good agreement with the predictions of the special theory of relativity and can be explained in terms of the relative motion between the Earth and the jet aircraft. In their paper, Hafele and Keating stated that "Relative to the atomic time scale of the U.S. Naval Observatory, the flying clocks lost 59 ± 10 ns during the eastward trip and gained 273 ± 7 ns during the westward trip . . . . These results provide an unambiguous empirical resolution of the famous clock paradox with macroscopic clocks."

Another interesting example of time dilation involves the observation of *muons,* unstable elementary particles that have a charge equal to that of the electron and a mass 207 times that of the electron. Muons can be produced by the collision of cosmic radiation with atoms high in the atmosphere. These particles have a lifetime of 2.2 μs when measured in a reference frame in which they are at rest or moving slowly. If we take 2.2 μs as the average lifetime of a muon and assume that its speed is close to the speed of light, we find that these particles travel only approximately 600 m before they decay (Fig. 39.10a). Hence, they cannot reach the Earth from the upper atmosphere where they are produced. However, experiments show that a large number of muons do reach the Earth. The phenomenon of time dilation explains this effect. Relative to an observer on the Earth, the muons have a lifetime equal to $\gamma\tau_p$, where $\tau_p = 2.2\ \mu$s is the lifetime in the frame traveling with the muons or the proper lifetime. For example, for a muon speed of $v = 0.99c$, $\gamma \approx 7.1$ and $\gamma\tau_p \approx 16\ \mu$s. Hence, the average distance traveled as measured by an observer on the Earth is $\gamma v \tau_p \approx 4\,800$ m, as indicated in Figure 39.10b.

In 1976, at the laboratory of the European Council for Nuclear Research

[4] J. C. Hafele and R. E. Keating, "Around the World Atomic Clocks: Relativistic Time Gains Observed," *Science,* 177:168, 1972.

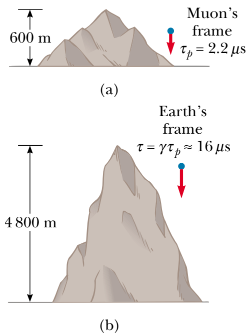


Figure 39.10 (a) Muons moving with a speed of 0.99c travel approximately 600 m as measured in the reference frame of the muons, where their lifetime is about 2.2 μs. (b) The muons travel approximately 4 800 m as measured by an observer on the Earth.

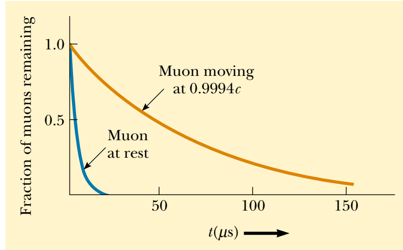


Figure 39.11 Decay curves for muons at rest and for muons traveling at a speed of 0.9994c.

(CERN) in Geneva, muons injected into a large storage ring reached speeds of approximately 0.9994c. Electrons produced by the decaying muons were detected by counters around the ring, enabling scientists to measure the decay rate and hence the muon lifetime. The lifetime of the moving muons was measured to be approximately 30 times as long as that of the stationary muon (Fig. 39.11), in agreement

EXAMPLE 39.1 What Is the Period of the Pendulum?

The period of a pendulum is measured to be 3.0 s in the reference frame of the pendulum. What is the period when measured by an observer moving at a speed of 0.95c relative to the pendulum?

Solution Instead of the observer moving at 0.95c, we can take the equivalent point of view that the observer is at rest and the pendulum is moving at 0.95c past the stationary observer. Hence, the pendulum is an example of a moving clock.

The proper time is $\Delta t_p = 3.0$ s. Because a moving clock runs more slowly than a stationary clock by a factor γ, Equation 39.7 gives

$$\Delta t = \gamma \Delta t_p = \frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}}\Delta t_p = \frac{1}{\sqrt{1-0.902}}\Delta t_p$$

$$= (3.2)(3.0\text{ s}) = \boxed{9.6\text{ s}}$$

That is, a moving pendulum takes longer to complete a period than a pendulum at rest does.

EXAMPLE 39.2 How Long Was Your Trip?

Suppose you are driving your car on a business trip and are traveling at 30 m/s. Your boss, who is waiting at your destination, expects the trip to take 5.0 h. When you arrive late, your excuse is that your car clock registered the passage of 5.0 h but that you were driving fast and so your clock ran more slowly than your boss's clock. If your car clock actually did indicate a 5.0-h trip, how much time passed on your boss's clock, which was at rest on the Earth?

Solution We begin by calculating γ from Equation 39.8:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(3\times 10^1\text{ m/s})^2}{(3\times 10^8\text{ m/s})^2}}} = \frac{1}{\sqrt{1-10^{-14}}}$$

If you try to determine this value on your calculator, you will probably get $\gamma = 1$. However, if we perform a binomial expansion, we can more precisely determine the value as

$$\gamma = (1-10^{-14})^{-1/2} \approx 1 + \tfrac{1}{2}(10^{-14}) = 1 + 5.0\times 10^{-15}$$

This result indicates that at typical automobile speeds, γ is not much different from 1.

Applying Equation 39.7, we find Δt, the time interval measured by your boss, to be

$$\Delta t = \gamma \Delta t_p = (1 + 5.0\times 10^{-15})(5.0\text{ h})$$

$$= 5.0\text{ h} + 2.5\times 10^{-14}\text{ h} = \boxed{5.0\text{ h} + 0.09\text{ ns}}$$

Your boss's clock would be only 0.09 ns ahead of your car clock. You might want to try another excuse!

with the prediction of relativity to within two parts in a thousand.

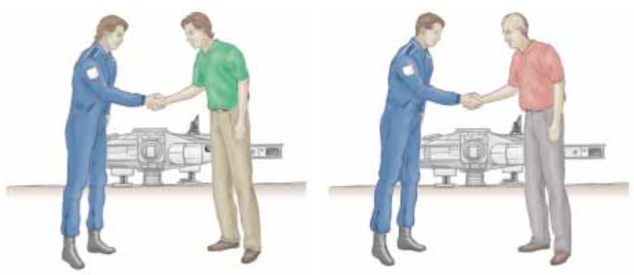
The Twins Paradox

An intriguing consequence of time dilation is the so-called *twins paradox* (Fig. 39.12). Consider an experiment involving a set of twins named Speedo and Goslo. When they are 20 yr old, Speedo, the more adventuresome of the two, sets out on an epic journey to Planet X, located 20 ly from the Earth. Furthermore, his spaceship is capable of reaching a speed of $0.95c$ relative to the inertial frame of his twin brother back home. After reaching Planet X, Speedo becomes homesick and immediately returns to the Earth at the same speed $0.95c$. Upon his return, Speedo is shocked to discover that Goslo has aged 42 yr and is now 62 yr old. Speedo, on the other hand, has aged only 13 yr.

At this point, it is fair to raise the following question—which twin is the traveler and which is really younger as a result of this experiment? From Goslo's frame of reference, he was at rest while his brother traveled at a high speed. But from Speedo's perspective, it is he who was at rest while Goslo was on the high-speed space journey. According to Speedo, he himself remained stationary while Goslo and the Earth raced away from him on a 6.5-yr journey and then headed back for another 6.5 yr. This leads to an apparent contradiction. Which twin has developed signs of excess aging?

To resolve this apparent paradox, recall that the special theory of relativity deals with inertial frames of reference moving relative to each other at uniform speed. However, the trip in our current problem is not symmetrical. Speedo, the space traveler, must experience a series of accelerations during his journey. As a result, his speed is not always uniform, and consequently he is not in an inertial frame. He cannot be regarded as always being at rest while Goslo is in uniform motion because to do so would be an incorrect application of the special theory of relativity. Therefore, there is no paradox. During each passing year noted by Goslo, slightly less than 4 months elapsed for Speedo.

The conclusion that Speedo is in a noninertial frame is inescapable. Each twin observes the other as accelerating, but it is Speedo that actually undergoes dynamical acceleration due to the real forces acting on him. The time required to accelerate and decelerate Speedo's spaceship may be made very small by using large rockets, so that Speedo can claim that he spends most of his time traveling to Planet X



(a) (b)

Figure 39.12 (a) As one twin leaves his brother on the Earth, both are the same age. (b) When Speedo returns from his journey to Planet X, he is younger than his twin Goslo.

at $0.95c$ in an inertial frame. However, Speedo must slow down, reverse his motion, and return to the Earth in an altogether different inertial frame. At the very best, Speedo is in two different inertial frames during his journey. Only Goslo, who is in a single inertial frame, can apply the simple time-dilation formula to Speedo's trip. Thus, Goslo finds that instead of aging 42 yr, Speedo ages only $(1 - v^2/c^2)^{1/2}(42\text{ yr}) = 13\text{ yr}$. Conversely, Speedo spends 6.5 yr traveling to Planet X and 6.5 yr returning, for a total travel time of 13 yr, in agreement with our earlier statement.

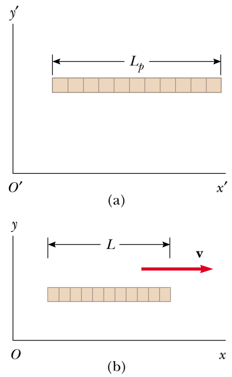


Figure 39.13 (a) A stick measured by an observer in a frame attached to the stick (that is, both have the same velocity) has its proper length L_p. (b) The stick measured by an observer in a frame in which the stick has a velocity **v** relative to the frame is shorter than its proper length L_p by a factor $(1 - v^2/c^2)^{1/2}$.

Quick Quiz 39.5

Suppose astronauts are paid according to the amount of time they spend traveling in space. After a long voyage traveling at a speed approaching c, would a crew rather be paid according to an Earth-based clock or their spaceship's clock?

Length Contraction

Length contraction

The measured distance between two points also depends on the frame of reference. **The proper length L_p of an object is the length measured by someone at rest relative to the object.** The length of an object measured by someone in a reference frame that is moving with respect to the object is always less than the proper length. This effect is known as **length contraction.**

Consider a spaceship traveling with a speed v from one star to another. There are two observers: one on the Earth and the other in the spaceship. The observer at rest on the Earth (and also assumed to be at rest with respect to the two stars) measures the distance between the stars to be the proper length L_p. According to this observer, the time it takes the spaceship to complete the voyage is $\Delta t = L_p/v$. Because of time dilation, the space traveler measures a smaller time of travel by the spaceship clock: $\Delta t_p = \Delta t/\gamma$. The space traveler claims to be at rest and sees the destination star moving toward the spaceship with speed v. Because the space traveler reaches the star in the time Δt_p, he or she concludes that the distance L between the stars is shorter than L_p. This distance measured by the space traveler is

$$L = v\,\Delta t_p = v\frac{\Delta t}{\gamma}$$

Because $L_p = v\,\Delta t$, we see that

$$L = \frac{L_p}{\gamma} = L_p\left(1 - \frac{v^2}{c^2}\right)^{1/2} \qquad \textbf{(39.9)}$$

If an object has a proper length L_p when it is at rest, then when it moves with speed v in a direction parallel to its length, it contracts to the length $L = L_p(1 - v^2/c^2)^{1/2} = L_p/\gamma$.

where $(1 - v^2/c^2)^{1/2}$ is a factor less than unity. This result may be interpreted as follows:

For example, suppose that a stick moves past a stationary Earth observer with speed v, as shown in Figure 39.13. The length of the stick as measured by an observer in a frame attached to the stick is the proper length L_p shown in Figure 39.13a. The length of the stick L measured by the Earth observer is shorter than L_p by the factor $(1 - v^2/c^2)^{1/2}$. Furthermore, length contraction is a symmetrical effect: If the stick is at rest on the Earth, an observer in a moving frame would measure its length to be shorter by the same factor $(1 - v^2/c^2)^{1/2}$. Note that **length contraction takes place only along the direction of motion.**

It is important to emphasize that proper length and proper time are measured in different reference frames. As an example of this point, let us return to the decaying muons moving at speeds close to the speed of light. An observer in the muon reference frame measures the proper lifetime (that is, the time interval τ_p), whereas an Earth-based observer measures a dilated lifetime. However, the Earth-based observer measures the proper height (the length L_p) of the mountain in Figure 39.10b. In the muon reference frame, this height is less than L_p, as the figure shows. Thus, in the muon frame, length contraction occurs but time dilation does not. In the Earth-based reference frame, time dilation occurs but length contraction does not. Thus, when calculations on the muon are performed in both frames, the effect of "offsetting penalties" is seen, and the outcome of the experiment in one frame is the same as the outcome in the other frame!

EXAMPLE 39.3 The Contraction of a Spaceship

A spaceship is measured to be 120.0 m long and 20.0 m in diameter while at rest relative to an observer. If this spaceship now flies by the observer with a speed of 0.99c, what length and diameter does the observer measure?

Solution From Equation 39.9, the length measured by the observer is

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}} = (120.0 \text{ m})\sqrt{1 - \frac{(0.99c)^2}{c^2}} = \boxed{17 \text{ m}}$$

The diameter measured by the observer is still 20.0 m because the diameter is a dimension perpendicular to the motion and length contraction occurs only along the direction of motion.

Exercise If the ship moves past the observer with a speed of 0.100 0c, what length does the observer measure?

Answer 119.4 m.

EXAMPLE 39.4 How Long Was Your Car?

In Example 39.2, you were driving at 30 m/s and claimed that your clock was running more slowly than your boss's stationary clock. Although your statement was true, the time dilation was negligible. If your car is 4.3 m long when it is parked, how much shorter does it appear to a stationary roadside observer as you drive by at 30 m/s?

Solution The observer sees the horizontal length of the car to be contracted to a length

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}} \approx L_p\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

where we have again used the binomial expansion for the factor $\sqrt{1 - \frac{v^2}{c^2}}$. The roadside observer sees the car's length as having changed by an amount $L_p - L$:

$$L_p - L \approx \frac{L_p}{2}\left(\frac{v^2}{c^2}\right) = \left(\frac{4.3 \text{ m}}{2}\right)\left(\frac{3.0 \times 10^1 \text{ m/s}}{3.0 \times 10^8 \text{ m/s}}\right)^2$$

$$= \boxed{2.2 \times 10^{-14} \text{ m}}$$

This is much smaller than the diameter of an atom!

EXAMPLE 39.5 A Voyage to Sirius

An astronaut takes a trip to Sirius, which is located a distance of 8 lightyears from the Earth. (Note that 1 lightyear (ly) is the distance light travels through free space in 1 yr.) The astronaut measures the time of the one-way journey to be 6 yr. If the spaceship moves at a constant speed of 0.8c, how can the 8-ly distance be reconciled with the 6-yr trip time measured by the astronaut?

Solution The 8 ly represents the proper length from the Earth to Sirius measured by an observer seeing both bodies nearly at rest. The astronaut sees Sirius approaching her at 0.8c but also sees the distance contracted to

$$\frac{8 \text{ ly}}{\gamma} = (8 \text{ ly})\sqrt{1 - \frac{v^2}{c^2}} = (8 \text{ ly})\sqrt{1 - \frac{(0.8c)^2}{c^2}} = 5 \text{ ly}$$

Thus, the travel time measured on her clock is

$$t = \frac{d}{v} = \frac{5 \text{ ly}}{0.8c} = 6 \text{ yr}$$

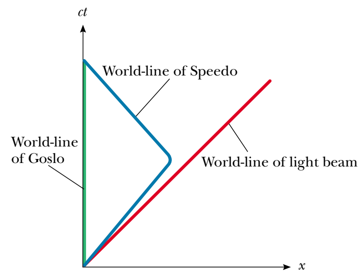


Figure 39.14 The twins paradox on a space–time graph. The twin who stays on the Earth has a world-line along the t axis. The path of the traveling twin through space–time is represented by a world-line that changes direction.

Space–Time Graphs

It is sometimes helpful to make a *space–time graph,* in which time is the ordinate and displacement is the abscissa. The twins paradox is displayed in such a graph in Figure 39.14. A path through space–time is called a **world-line.** At the origin, the world-lines of Speedo and Goslo coincide because the twins are in the same location at the same time. After Speedo leaves on his trip, his world-line diverges from that of his brother. At their reunion, the two world-lines again come together. Note that Goslo's world-line is vertical, indicating no displacement from his original location. Also note that it would be impossible for Speedo to have a world-line that crossed the path of a light beam that left the Earth when he did. To do so would require him to have a speed greater than c.

World-lines for light beams are diagonal lines on space–time graphs, typically drawn at 45° to the right or left of vertical, depending on whether the light beam is traveling in the direction of increasing or decreasing x. These two world-lines means that all possible future events for Goslo and Speedo lie within two 45° lines extending from the origin. Either twin's presence at an event outside this "light cone" would require that twin to move at a speed greater than c, which, as we shall see in Section 39.5, is not possible. Also, the only past events that Goslo and Speedo could have experienced occurred within two similar 45° world-lines that approach the origin from below the x axis.

Quick Quiz 39.6

How is acceleration indicated on a space–time graph?

The Relativistic Doppler Effect

Another important consequence of time dilation is the shift in frequency found for light emitted by atoms in motion as opposed to light emitted by atoms at rest. This phenomenon, known as the Doppler effect, was introduced in Chapter 17 as it pertains to sound waves. In the case of sound, the motion of the source with respect to the medium of propagation can be distinguished from the motion of the observer with respect to the medium. Light waves must be analyzed differently, however, because they require no medium of propagation, and no method exists for distinguishing the motion of a light source from the motion of the observer.

If a light source and an observer approach each other with a relative speed v, the frequency f_{obs} measured by the observer is

$$f_{\text{obs}} = \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}} f_{\text{source}} \quad \textbf{(39.10)}$$

where f_{source} is the frequency of the source measured in its rest frame. Note that this relativistic Doppler shift formula, unlike the Doppler shift formula for sound, depends only on the relative speed v of the source and observer and holds for relative speeds as great as c. As you might expect, the formula predicts that $f_{\text{obs}} > f_{\text{source}}$ when the source and observer approach each other. We obtain the expression for the case in which the source and observer recede from each other by replacing v with $-v$ in Equation 39.10.

The most spectacular and dramatic use of the relativistic Doppler effect is the measurement of shifts in the frequency of light emitted by a moving astronomical object such as a galaxy. Spectral lines normally found in the extreme violet region for galaxies at rest with respect to the Earth are shifted by about 100 nm toward the red end of the spectrum for distant galaxies—indicating that these galaxies are *receding* from us. The American astronomer Edwin Hubble (1889–1953) performed extensive measurements of this *red shift* to confirm that most galaxies are moving away from us, indicating that the Universe is expanding.

39.5 THE LORENTZ TRANSFORMATION EQUATIONS

We have seen that the Galilean transformation equations are not valid when v approaches the speed of light. In this section, we state the correct transformation equations that apply for all speeds in the range $0 \le v < c$.

Suppose that an event that occurs at some point P is reported by two observers, one at rest in a frame S and the other in a frame S′ that is moving to the right with speed v, as in Figure 39.15. The observer in S reports the event with space–time coordinates (x, y, z, t), and the observer in S′ reports the same event using the coordinates (x', y', z', t'). We would like to find a relationship between these coordinates that is valid for all speeds.

The equations that are valid from $v = 0$ to $v = c$ and enable us to transform coordinates from S to S′ are the **Lorentz transformation equations:**

Lorentz transformation equations for S → S′

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z \quad \textbf{(39.11)}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

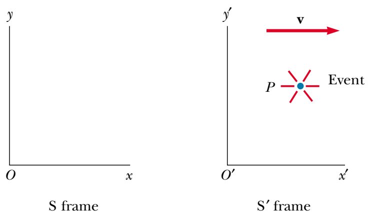


Figure 39.15 An event that occurs at some point P is observed by two persons, one at rest in the S frame and the other in the S′ frame, which is moving to the right with a speed v.

These transformation equations were developed by Hendrik A. Lorentz (1853–1928) in 1890 in connection with electromagnetism. However, it was Einstein who recognized their physical significance and took the bold step of interpreting them within the framework of the special theory of relativity.

Note the difference between the Galilean and Lorentz time equations. In the Galilean case, $t = t'$, but in the Lorentz case the value for t' assigned to an event by an observer O' standing at the origin of the S′ frame in Figure 39.15 depends both on the time t and on the coordinate x as measured by an observer O standing in the S frame. This is consistent with the notion that an event is characterized by four space–time coordinates (x, y, z, t). In other words, in relativity, space and time are not separate concepts but rather are closely interwoven with each other.

If we wish to transform coordinates in the S′ frame to coordinates in the S frame, we simply replace v by $-v$ and interchange the primed and unprimed coordinates in Equations 39.11:

Inverse Lorentz transformation equations for S′ → S

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z' \quad \textbf{(39.12)}$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

When $v \ll c$, the Lorentz transformation equations should reduce to the Galilean equations. To verify this, note that as v approaches zero, $v/c \ll 1$ and $v^2/c^2 \ll 1$; thus, $\gamma = 1$, and Equations 39.11 reduce to the Galilean space–time transformation equations:

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

In many situations, we would like to know the difference in coordinates between two events or the time interval between two events as seen by observers O and O'. We can accomplish this by writing the Lorentz equations in a form suitable for describing pairs of events. From Equations 39.11 and 39.12, we can express the differences between the four variables x, x', t, and t' in the form

$$\left.\begin{aligned} \Delta x' &= \gamma(\Delta x - v\,\Delta t) \\ \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \end{aligned}\right\} \text{S} \rightarrow \text{S}' \quad \textbf{(39.13)}$$

$$\left.\begin{aligned} \Delta x &= \gamma(\Delta x' + v\,\Delta t') \\ \Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \end{aligned}\right\} \text{S}' \rightarrow \text{S} \quad \textbf{(39.14)}$$

[5] Although relative motion of the two frames along the x axis does not change the y and z coordinates of an object, it does change the y and z velocity components of an object moving in either frame, as we shall soon see.

EXAMPLE 39.6 Simultaneity and Time Dilation Revisited

Use the Lorentz transformation equations in difference form to show that (a) simultaneity is not an absolute concept and that (b) moving clocks run more slowly than stationary clocks.

Solution (a) Suppose that two events are simultaneous according to a moving observer O', such that $\Delta t' = 0$. From the expression for Δt given in Equation 39.14, we see that in this case the time interval Δt measured by a stationary observer O is $\Delta t = \gamma v\,\Delta x'/c^2$. That is, the time interval for the same two events as measured by O is nonzero, and so the events do not appear to be simultaneous to O.

(b) Suppose that observer O' finds that two events occur at the same place ($\Delta x' = 0$) but at different times ($\Delta t' \neq 0$). In this situation, the expression for Δt given in Equation 39.14 becomes $\Delta t = \gamma\,\Delta t'$. This is the equation for time dilation found earlier (Eq. 39.7), where $\Delta t' = \Delta t_p$ is the proper time measured by a clock located in the moving frame of observer O'.

Exercise Use the Lorentz transformation equations in difference form to confirm that $L = L_p/\gamma$ (Eq. 39.9).

where $\Delta x' = x_2' - x_1'$ and $\Delta t' = t_2' - t_1'$ are the differences measured by observer O' and $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$ are the differences measured by observer O. (We have not included the expressions for relating the y and z coordinates because they are unaffected by motion along the x direction.[5])

Derivation of the Lorentz Velocity Transformation Equation

Once again S is our stationary frame of reference, and S′ is our frame moving at a speed v relative to S. Suppose that an object has a speed u_x' measured in the S′ frame, where

$$u_x' = \frac{dx'}{dt'} \qquad \textbf{(39.15)}$$

Using Equation 39.11, we have

$$dx' = \gamma(dx - v\,dt)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}\,dx\right)$$

Substituting these values into Equation 39.15 gives

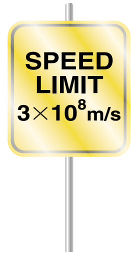
$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \dfrac{v}{c^2}\,dx} = \frac{\dfrac{dx}{dt} - v}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}$$

But dx/dt is just the velocity component u_x of the object measured by an observer in S, and so this expression becomes

Lorentz velocity transformation equation for S → S′

$$u_x' = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \qquad \textbf{(39.16)}$$

If the object has velocity components along the y and z axes, the components as measured by an observer in S′ are



The speed of light is the speed limit of the Universe. It is the maximum possible speed for energy transfer and for information transfer. Any object with mass must move at a lower speed.

Lorentz velocity transformation equations for S′ → S

$$u_y' = \frac{u_y}{\gamma\left(1 - \dfrac{u_x v}{c^2}\right)} \quad \text{and} \quad u_z' = \frac{u_z}{\gamma\left(1 - \dfrac{u_x v}{c^2}\right)} \qquad \textbf{(39.17)}$$

Note that u_y' and u_z' do not contain the parameter v in the numerator because the relative velocity is along the x axis.

When u_x and v are both much smaller than c (the nonrelativistic case), the denominator of Equation 39.16 approaches unity, and so $u_x' \approx u_x - v$, which is the Galilean velocity transformation equation. In the other extreme, when $u_x = c$, Equation 39.16 becomes

$$u_x' = \frac{c - v}{1 - \dfrac{cv}{c^2}} = \frac{c\left(1 - \dfrac{v}{c}\right)}{1 - \dfrac{v}{c}} = c$$

From this result, we see that an object moving with a speed c relative to an observer in S also has a speed c relative to an observer in S′—independent of the relative motion of S and S′. Note that this conclusion is consistent with Einstein's second postulate—that the speed of light must be c relative to all inertial reference frames. Furthermore, the speed of an object can never exceed c. That is, the speed of light is the ultimate speed. We return to this point later when we consider the energy of a particle.

EXAMPLE 39.7 Relative Velocity of Spaceships

Two spaceships A and B are moving in opposite directions, as shown in Figure 39.16. An observer on the Earth measures the speed of ship A to be $0.750c$ and the speed of ship B to be $0.850c$. Find the velocity of ship B as observed by the crew on ship A.

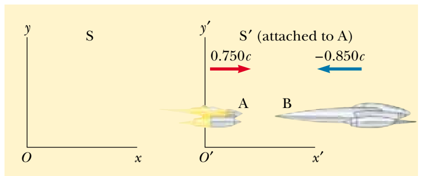


Figure 39.16 Two spaceships A and B move in opposite directions. The speed of B relative to A is *less* than c and is obtained from the relativistic velocity transformation equation.

Solution We can solve this problem by taking the S′ frame as being attached to ship A, so that $v = 0.750c$ relative to the Earth (the S frame). We can consider ship B as moving with a velocity $u_x = -0.850c$ relative to the Earth. Hence, we can obtain the velocity of ship B relative to ship A by using Equation 39.16:

$$u_x' = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} = \frac{-0.850c - 0.750c}{1 - \dfrac{(-0.850c)(0.750c)}{c^2}} = \boxed{-0.977c}$$

The negative sign indicates that ship B is moving in the negative x direction as observed by the crew on ship A. Note that the speed is less than c. That is, a body whose speed is less than c in one frame of reference must have a speed less than c in any other frame. (If the Galilean velocity transformation equation were used in this example, we would find that $u_x' = u_x - v = -0.850c - 0.750c = -1.60c$, which is impossible. The Galilean transformation equation does not work in relativistic situations.)

EXAMPLE 39.8 The Speeding Motorcycle

Imagine a motorcycle moving with a speed $0.80c$ past a stationary observer, as shown in Figure 39.17. If the rider tosses a ball in the forward direction with a speed of $0.70c$ relative to himself, what is the speed of the ball relative to the stationary observer?

Solution The speed of the motorcycle relative to the stationary observer is $v = 0.80c$. The speed of the ball in the frame of reference of the motorcyclist is $u_x' = 0.70c$. Therefore, the speed u_x of the ball relative to the stationary observer is

$$u_x = \frac{u_x' + v}{1 + \dfrac{u_x' v}{c^2}} = \frac{0.70c + 0.80c}{1 + \dfrac{(0.70c)(0.80c)}{c^2}} = \boxed{0.96c}$$

Exercise Suppose that the motorcyclist turns on the headlight so that a beam of light moves away from him with a speed c in the forward direction. What does the stationary observer measure for the speed of the light?

Answer c.

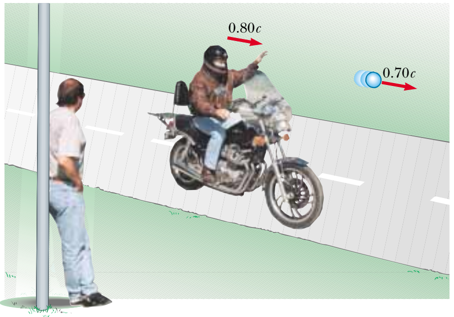


Figure 39.17 A motorcyclist moves past a stationary observer with a speed of $0.80c$ and throws a ball in the direction of motion with a speed of $0.70c$ relative to himself.

EXAMPLE 39.9 Relativistic Leaders of the Pack

Two motorcycle pack leaders named David and Emily are racing at relativistic speeds along perpendicular paths, as shown in Figure 39.18. How fast does Emily recede as seen by David over his right shoulder?

Solution Figure 39.18 represents the situation as seen by a police officer at rest in frame S, who observes the following:

David: $u_x = 0.75c \quad u_y = 0$

Emily: $u_x = 0 \quad u_y = -0.90c$

To calculate Emily's speed of recession as seen by David, we take S′ to move along with David and then calculate u_x' and u_y' for Emily using Equations 39.16 and 39.17:

$$u_x' = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} = \frac{0 - 0.75c}{1 - \dfrac{(0)(0.75c)}{c^2}} = -0.75c$$

$$u_y' = \frac{u_y}{\gamma\left(1 - \dfrac{u_x v}{c^2}\right)} = \frac{\sqrt{1 - \dfrac{(0.75c)^2}{c^2}}\,(-0.90c)}{\left(1 - \dfrac{(0)(0.75c)}{c^2}\right)} = -0.60c$$

Thus, the speed of Emily as observed by David is

$$u' = \sqrt{(u_x')^2 + (u_y')^2} = \sqrt{(-0.75c)^2 + (-0.60c)^2} = \boxed{0.96c}$$

Note that this speed is less than c, as required by the special theory of relativity.

Exercise Use the Galilean velocity transformation equation to calculate the classical speed of recession for Emily as observed by David.

Answer $1.2c$.

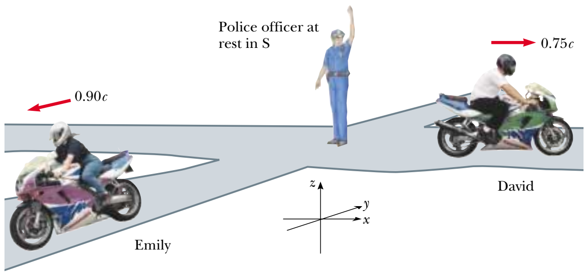


Figure 39.18 David moves to the east with a speed $0.75c$ relative to the police officer, and Emily travels south at a speed $0.90c$ relative to the officer.

To obtain u_x in terms of u_x', we replace v by $-v$ in Equation 39.16 and interchange the roles of u_x and u_x':

$$u_x = \frac{u_x' + v}{1 + \dfrac{u_x' v}{c^2}} \qquad \textbf{(39.18)}$$

39.6 RELATIVISTIC LINEAR MOMENTUM AND THE RELATIVISTIC FORM OF NEWTON'S LAWS

We have seen that in order to describe properly the motion of particles within the framework of the special theory of relativity, we must replace the Galilean transformation equations by the Lorentz transformation equations. Because the laws of physics must remain unchanged under the Lorentz transformation, we must generalize Newton's laws and the definitions of linear momentum and energy to conform to the Lorentz transformation equations and the principle of relativity. These generalized definitions should reduce to the classical (nonrelativistic) definitions for $v \ll c$.

First, recall that the law of conservation of linear momentum states that when two isolated objects collide, their combined total momentum remains constant. Suppose that the collision is described in a reference frame S in which linear momentum is conserved. If we calculate the velocities in a second reference frame S′ using the Lorentz velocity transformation equation and the classical definition of linear momentum, $\mathbf{p} = m\mathbf{u}$ (where $\mathbf{u}$ is the velocity of either object), we find that linear momentum is *not* conserved in S′. However, because the laws of physics are the same in all inertial frames, linear momentum must be conserved in all frames. In view of this condition and assuming that the Lorentz velocity transformation equation is correct, we must modify the definition of linear momentum to satisfy the following conditions:

Definition of relativistic linear momentum

- Linear momentum $\mathbf{p}$ must be conserved in all collisions.
- The relativistic value calculated for $\mathbf{p}$ must approach the classical value $m\mathbf{u}$ as $\mathbf{u}$ approaches zero.

For any particle, the correct relativistic equation for linear momentum that satisfies these conditions is

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \gamma m\mathbf{u} \qquad \textbf{(39.19)}$$

where $\mathbf{u}$ is the velocity of the particle and m is the mass of the particle. When u is much less than c, $\gamma = (1 - u^2/c^2)^{-1/2}$ approaches unity and $\mathbf{p}$ approaches $m\mathbf{u}$. Therefore, the relativistic equation for $\mathbf{p}$ does indeed reduce to the classical expression when u is much smaller than c.

The relativistic force $\mathbf{F}$ acting on a particle whose linear momentum is $\mathbf{p}$ is defined as

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{dt} \qquad \textbf{(39.20)}$$

where $\mathbf{p}$ is given by Equation 39.19. This expression, which is the relativistic form of Newton's second law, is reasonable because it preserves classical mechanics in

EXAMPLE 39.10 Linear Momentum of an Electron

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of $0.750c$. Find its relativistic momentum and compare this value with the momentum calculated from the classical expression.

Solution Using Equation 39.19 with $u = 0.750c$, we have

$$p = \frac{m_e u}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$= \frac{(9.11 \times 10^{-31}\ \text{kg})(0.750 \times 3.00 \times 10^8\ \text{m/s})}{\sqrt{1 - \frac{(0.750c)^2}{c^2}}}$$

$$= 3.10 \times 10^{-22}\ \text{kg}\cdot\text{m/s}$$

The (incorrect) classical expression gives

$$p_{\text{classical}} = m_e u = 2.05 \times 10^{-22}\ \text{kg}\cdot\text{m/s}$$

Hence, the correct relativistic result is 50% greater than the classical result!

the limit of low velocities and requires conservation of linear momentum for an isolated system ($\mathbf{F} = 0$) both relativistically and classically.

It is left as an end-of-chapter problem (Problem 63) to show that under relativistic conditions, the acceleration **a** of a particle decreases under the action of a constant force, in which case $a \propto (1 - u^2/c^2)^{3/2}$. From this formula, note that as the particle's speed approaches c, the acceleration caused by any finite force approaches zero. Hence, it is impossible to accelerate a particle from rest to a speed $u \geq c$.

39.7 RELATIVISTIC ENERGY

We have seen that the definition of linear momentum and the laws of motion require generalization to make them compatible with the principle of relativity. This implies that the definition of kinetic energy must also be modified.

To derive the relativistic form of the work–kinetic energy theorem, let us first use the definition of relativistic force, Equation 39.20, to determine the work done on a particle by a force F:

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx \qquad \textbf{(39.21)}$$

for force and motion both directed along the x axis. In order to perform this integration and find the work done on the particle and the relativistic kinetic energy as a function of u, we first evaluate dp/dt:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{m(du/dt)}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}$$

Substituting this expression for dp/dt and $dx = u\,dt$ into Equation 39.21 gives

$$W = \int_0^t \frac{m(du/dt)u\,dt}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}} = m\int_0^u \frac{u}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}\,du$$

where we use the limits 0 and u in the rightmost integral because we have assumed

that the particle is accelerated from rest to some final speed u. Evaluating the integral, we find that

Relativistic kinetic energy

$$W = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 \qquad \textbf{(39.22)}$$

Recall from Chapter 7 that the work done by a force acting on a particle equals the change in kinetic energy of the particle. Because of our assumption that the initial speed of the particle is zero, we know that the initial kinetic energy is zero. We therefore conclude that the work W is equivalent to the relativistic kinetic energy K:

$$K = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 = \gamma mc^2 - mc^2 \qquad \textbf{(39.23)}$$

This equation is routinely confirmed by experiments using high-energy particle accelerators.

At low speeds, where $u/c \ll 1$, Equation 39.23 should reduce to the classical expression $K = \frac{1}{2}mu^2$. We can check this by using the binomial expansion $(1 - x^2)^{-1/2} \approx 1 + \frac{1}{2}x^2 + \ldots$ for $x \ll 1$, where the higher-order powers of x are neglected in the expansion. In our case, $x = u/c$, so that

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$$

Substituting this into Equation 39.23 gives

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) - mc^2 = \frac{1}{2}mu^2$$

Definition of total energy

which is the classical expression for kinetic energy. A graph comparing the relativistic and nonrelativistic expressions is given in Figure 39.19. In the relativistic case, the particle speed never exceeds c, regardless of the kinetic energy. The two curves are in good agreement when $u \ll c$.

The constant term mc^2 in Equation 39.23, which is independent of the speed of the particle, is called the **rest energy** E_R of the particle (see Section 8.9). The term γmc^2, which does depend on the particle speed, is therefore the sum of the kinetic and rest energies. We define γmc^2 to be the **total energy** E:

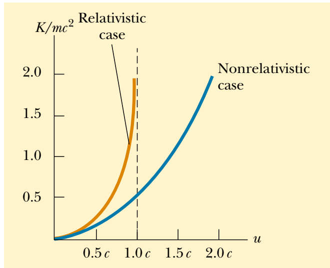


Figure 39.19 A graph comparing relativistic and nonrelativistic kinetic energy. The energies are plotted as a function of speed. In the relativistic case, u is always less than c.

Total energy = kinetic energy + rest energy

$$E = \gamma mc^2 = K + mc^2 \qquad \textbf{(39.24)}$$

or

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad \textbf{(39.25)}$$

This is Einstein's famous equation about mass–energy equivalence.

The relationship $E = K + mc^2$ shows that **mass is a form of energy,** where c^2 in the rest energy term is just a constant conversion factor. This expression also shows that a small mass corresponds to an enormous amount of energy, a concept fundamental to nuclear and elementary-particle physics.

Energy–momentum relationship

In many situations, the linear momentum or energy of a particle is measured rather than its speed. It is therefore useful to have an expression relating the total energy E to the relativistic linear momentum p. This is accomplished by using the expressions $E = \gamma mc^2$ and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate u (Problem 39). The result, after some algebra, is[6]

$$E^2 = p^2c^2 + (mc^2)^2 \qquad \textbf{(39.26)}$$

When the particle is at rest, $p = 0$ and so $E = E_R = mc^2$. For particles that have zero mass, such as photons, we set $m = 0$ in Equation 39.26 and see that

$$E = pc \qquad \textbf{(39.27)}$$

This equation is an exact expression relating total energy and linear momentum for photons, which always travel at the speed of light.

Finally, note that because the mass m of a particle is independent of its motion, m must have the same value in all reference frames. For this reason, m is often called the **invariant mass.** On the other hand, because the total energy and linear momentum of a particle both depend on velocity, these quantities depend on the reference frame in which they are measured.

Because m is a constant, we conclude from Equation 39.26 that the quantity $E^2 - p^2c^2$ must have the same value in all reference frames. That is, $E^2 - p^2c^2$ is invariant under a Lorentz transformation. (Equations 39.26 and 39.27 do not make provision for potential energy.)

When we are dealing with subatomic particles, it is convenient to express their

EXAMPLE 39.11 The Energy of a Speedy Electron

An electron in a television picture tube typically moves with a speed $u = 0.250c$. Find its total energy and kinetic energy in electron volts.

Solution Using the fact that the rest energy of the electron is 0.511 MeV together with Equation 39.25, we have

$$E = \frac{m_ec^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{0.511\text{ MeV}}{\sqrt{1 - \frac{(0.250c)^2}{c^2}}}$$

$$= 1.03(0.511\text{ MeV}) = \boxed{0.528\text{ MeV}}$$

This is 3% greater than the rest energy.

We obtain the kinetic energy by subtracting the rest energy from the total energy:

$$K = E - m_ec^2 = 0.528\text{ MeV} - 0.511\text{ MeV} = \boxed{0.017\text{ MeV}}$$

[6] One way to remember this relationship is to draw a right triangle having a hypotenuse of length E and legs of lengths pc and mc^2.

EXAMPLE 39.12 The Energy of a Speedy Proton

(a) Find the rest energy of a proton in electron volts.

Solution

$$E_R = m_pc^2 = (1.67 \times 10^{-27}\text{ kg})(3.00 \times 10^8\text{ m/s})^2$$
$$= (1.50 \times 10^{-10}\text{ J})(1.00\text{ eV}/1.60 \times 10^{-19}\text{ J})$$
$$= \boxed{938\text{ MeV}}$$

(b) If the total energy of a proton is three times its rest energy, with what speed is the proton moving?

Solution Equation 39.25 gives

$$E = 3m_pc^2 = \frac{m_pc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$3 = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Solving for u gives

$$1 - \frac{u^2}{c^2} = \frac{1}{9}$$

$$\frac{u^2}{c^2} = \frac{8}{9}$$

$$u = \frac{\sqrt{8}}{3}c = \boxed{2.83 \times 10^8\text{ m/s}}$$

(c) Determine the kinetic energy of the proton in electron volts.

Solution From Equation 39.24,

$$K = E - m_pc^2 = 3m_pc^2 - m_pc^2 = 2m_pc^2$$

Because $m_pc^2 = 938$ MeV, $K = \boxed{1\,880\text{ MeV}}$

(d) What is the proton's momentum?

Solution We can use Equation 39.26 to calculate the momentum with $E = 3m_pc^2$:

$$E^2 = p^2c^2 + (m_pc^2)^2 = (3m_pc^2)^2$$

$$p^2c^2 = 9(m_pc^2)^2 - (m_pc^2)^2 = 8(m_pc^2)^2$$

$$p = \sqrt{8}\,\frac{m_pc^2}{c} = \sqrt{8}\,\frac{(938\text{ MeV})}{c} = \boxed{2\,650\text{ MeV}/c}$$

The unit of momentum is written MeV/c for convenience.

energy in electron volts because the particles are usually given this energy by acceleration through a potential difference. The conversion factor, as you recall from Equation 25.5, is

$$1\text{ eV} = 1.602 \times 10^{-19}\text{ J}$$

For example, the mass of an electron is 9.109×10^{-31} kg. Hence, the rest energy of the electron is

$$m_ec^2 = (9.109 \times 10^{-31}\text{ kg})(2.9979 \times 10^8\text{ m/s})^2 = 8.187 \times 10^{-14}\text{ J}$$
$$= (8.187 \times 10^{-14}\text{ J})(1\text{ eV}/1.602 \times 10^{-19}\text{ J}) = 0.5110\text{ MeV}$$

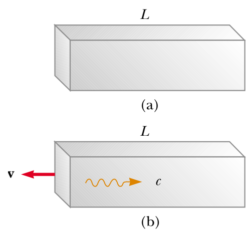


Figure 39.20 (a) A box of length L at rest. (b) When a light pulse directed to the right is emitted at the left end of the box, the box recoils to the left until the pulse strikes the right end.

39.8 EQUIVALENCE OF MASS AND ENERGY

To understand the equivalence of mass and energy, consider the following thought experiment proposed by Einstein in developing his famous equation $E = mc^2$. Imagine an isolated box of mass M_{box} and length L initially at rest, as shown in Figure 39.20a. Suppose that a pulse of light is emitted from the left side of the box, as depicted in Figure 39.20b. From Equation 39.27, we know that light of energy E carries linear momentum $p = E/c$. Hence, if momentum is to be conserved, the box must recoil to the left with a speed v. If it is assumed that the box is very massive, the recoil speed is much less than the speed of light, and conservation of momentum gives $M_{\text{box}}v = E/c$, or

$$v = \frac{E}{M_{\text{box}}c}$$

The time it takes the light pulse to move the length of the box is approximately $\Delta t = L/c$. In this time interval, the box moves a small distance Δx to the left, where

$$\Delta x = v\,\Delta t = \frac{EL}{M_{\text{box}}c^2}$$

The light then strikes the right end of the box and transfers its momentum to the box, causing the box to stop. With the box in its new position, its center of mass appears to have moved to the left. However, its center of mass cannot have moved because the box is an isolated system. Einstein resolved this perplexing situation by assuming that in addition to energy and momentum, light also carries mass. If M_{pulse} is the effective mass carried by the pulse of light and if the center of mass of the system (box plus pulse of light) is to remain fixed, then

$$M_{\text{pulse}}L = M_{\text{box}}\Delta x$$

Solving for M_{pulse}, and using the previous expression for Δx, we obtain

$$M_{\text{pulse}} = \frac{M_{\text{box}}\Delta x}{L} = \frac{M_{\text{box}}}{L}\frac{EL}{M_{\text{box}}c^2} = \frac{E}{c^2}$$

or

$$E = M_{\text{pulse}}c^2$$

the energy of a system of particles before interaction must equal the energy of the system after interaction, where energy of the ith particle is given by the expression

$$E_i = \frac{m_ic^2}{\sqrt{1 - \frac{u_i^2}{c^2}}} = \gamma m_ic^2$$

Conversion of mass–energy

Thus, Einstein reached the profound conclusion that "if a body gives off the energy E in the form of radiation, its mass diminishes by E/c^2, . . ."

Although we derived the relationship $E = mc^2$ for light energy, the equivalence of mass and energy is universal. Equation 39.24, $E = \gamma mc^2$, which represents the total energy of any particle, suggests that even when a particle is at rest ($\gamma = 1$) it still possesses enormous energy because it has mass. Probably the clearest experimental proof of the equivalence of mass and energy occurs in nuclear and elementary particle interactions, where large amounts of energy are released and the energy release is accompanied by a decrease in mass. Because energy and mass are related, we see that the laws of conservation of energy and conservation of mass are one and the same. Simply put, this law states that

The release of enormous quantities of energy from subatomic particles, accompanied by changes in their masses, is the basis of all nuclear reactions. In a conventional nuclear reactor, a uranium nucleus undergoes *fission,* a reaction that creates several lighter fragments having considerable kinetic energy. The combined mass of all the fragments is less than the mass of the parent uranium nucleus by an amount Δm. The corresponding energy Δmc^2 associated with this mass difference is exactly equal to the total kinetic energy of the fragments. This kinetic energy raises the temperature of water in the reactor, converting it to steam for the

CONCEPTUAL EXAMPLE 39.13

Because mass is a measure of energy, can we conclude that the mass of a compressed spring is greater than the mass of the same spring when it is not compressed?

Solution Recall that when a spring of force constant k is compressed (or stretched) from its equilibrium position a distance x, it stores elastic potential energy $U = kx^2/2$. According to the special theory of relativity, any change in the total energy of a system is equivalent to a change in the mass of the system. Therefore, the mass of a compressed (or stretched) spring is greater than the mass of the spring in its equilibrium position by an amount U/c^2.

EXAMPLE 39.14 Binding Energy of the Deuteron

A deuteron, which is the nucleus of a deuterium atom, contains one proton and one neutron and has a mass of 2.013 553 u. This total deuteron mass is not equal to the sum of the masses of the proton and neutron. Calculate the mass difference and determine its energy equivalence, which is called the *binding energy* of the nucleus.

Solution Using atomic mass units (u), we have

$$m_p = \text{mass of proton} = 1.007\,276\text{ u}$$

$$m_n = \text{mass of neutron} = 1.008\,665\text{ u}$$

$$m_p + m_n = 2.015\,941\text{ u}$$

The mass difference Δm is therefore 0.002 388 u. By definition, 1 u = 1.66×10^{-27} kg, and therefore

$$\Delta m = 0.002\,388\text{ u} = \boxed{3.96 \times 10^{-30}\text{ kg}}$$

Using $E = \Delta mc^2$, we find that the binding energy is

$$E = \Delta mc^2 = (3.96 \times 10^{-30}\text{ kg})(3.00 \times 10^8\text{ m/s})^2$$

$$= 3.56 \times 10^{-13}\text{ J} = \boxed{2.23\text{ MeV}}$$

Therefore, the minimum energy required to separate the proton from the neutron of the deuterium nucleus (the binding energy) is 2.23 MeV.

generation of electric power.

In the nuclear reaction called *fusion,* two atomic nuclei combine to form a single nucleus. The fusion reaction in which two deuterium nuclei fuse to form a helium nucleus is of major importance in current research and the development of controlled-fusion reactors. The decrease in mass that results from the creation of one helium nucleus from two deuterium nuclei is $\Delta m = 4.25 \times 10^{-29}$ kg. Hence, the corresponding excess energy that results from one fusion reaction is $\Delta mc^2 = 3.83 \times 10^{-12}\text{ J} = 23.9$ MeV. To appreciate the magnitude of this result, note that if 1 g of deuterium is converted to helium, the energy released is about 10^{12} J! At the current cost of electrical energy, this quantity of energy would be worth about $70 000.

39.9 RELATIVITY AND ELECTROMAGNETISM

Consider two frames of reference S and S′ that are in relative motion, and assume that a single charge q is at rest in the S′ frame of reference. According to an ob-

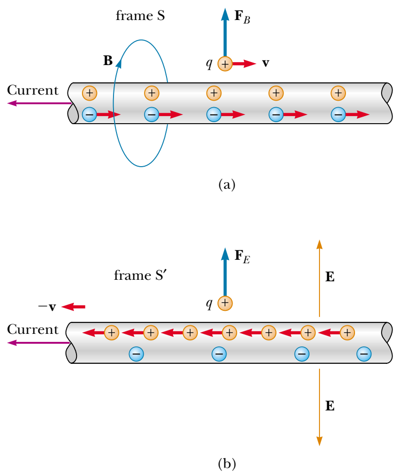


Figure 39.21 (a) In frame S, the positive charge q moves to the right with a velocity **v**, and the current-carrying wire is stationary. A magnetic field **B** surrounds the wire, and charge experiences a *magnetic* force directed away from the wire. (b) In frame S′, the wire moves to the left with a velocity −**v**, and the charge q is stationary. The wire creates an electric field **E**, and the charge experiences an *electric* force directed away from the wire.

server in this frame, an electric field surrounds the charge. However, an observer in frame S says that the charge is in motion and therefore measures both an electric field and a magnetic field. The magnetic field measured by the observer in frame S is created by the moving charge, which constitutes an electric current. In other words, electric and magnetic fields are viewed differently in frames of reference that are moving relative to each other. We now describe one situation that shows how an electric field in one frame of reference is viewed as a magnetic field in another frame of reference.

A positive test charge q is moving parallel to a current-carrying wire with velocity **v** relative to the wire in frame S, as shown in Figure 39.21a. We assume that the net charge on the wire is zero and that the electrons in the wire also move with velocity **v** in a straight line. The leftward current in the wire produces a magnetic field that forms circles around the wire and is directed into the page at the location of the moving test charge. Therefore, a magnetic force $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$ directed away from the wire is exerted on the test charge. However, no electric force acts on the test charge because the net charge on the wire is zero when viewed in this frame.

Now consider the same situation as viewed from frame S′, where the test charge is at rest (Figure 39.21b). In this frame, the positive charges in the wire move to the left, the electrons in the wire are at rest, and the wire still carries a current. Because the test charge is not moving in this frame, $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B} = 0$; there is no magnetic force exerted on the test charge when viewed in this frame. However, if a force is exerted on the test charge in frame S′, the frame of the wire, as described earlier, a force must be exerted on it in any other frame. What is the origin of this force in frame S, the frame of the test charge?

The answer to this question is provided by the special theory of relativity. When the situation is viewed in frame S, as in Figure 39.21a, the positive charges are at rest and the electrons in the wire move to the right with a velocity **v**. Because of length contraction, the electrons appear to be closer together than their proper separation. Because there is no net charge on the wire this contracted separation must equal the separation between the stationary positive charges. The situation is quite different when viewed in frame S′, shown in Figure 39.21b. In this frame, the positive charges appear closer together because of length contraction, and the electrons in the wire are at rest with a separation that is greater than that viewed in frame S. Therefore, there is a net positive charge on the wire when viewed in frame S′. This net positive charge produces an electric field pointing away from the wire toward the test charge, and so the test charge experiences an electric force directed away from the wire. Thus, what was viewed as a magnetic field (and a corresponding magnetic force) in the frame of the wire transforms into an electric field (and a corresponding electric force) in the frame of the test charge.

Optional Section

39.10 THE GENERAL THEORY OF RELATIVITY

Up to this point, we have sidestepped a curious puzzle. Mass has two seemingly different properties: a *gravitational attraction* for other masses and an *inertial* property that resists acceleration. To designate these two attributes, we use the subscripts g and i and write

Gravitational property	$F_g = m_g g$
Inertial property	$\Sigma F = m_i a$

The value for the gravitational constant G was chosen to make the magnitudes of m_g and m_i numerically equal. Regardless of how G is chosen, however, the strict proportionality of m_g and m_i has been established experimentally to an extremely high degree: a few parts in 10^{12}. Thus, it appears that gravitational mass and inertial mass may indeed be exactly proportional.

But why? They seem to involve two entirely different concepts: a force of mutual gravitational attraction between two masses, and the resistance of a single mass to being accelerated. This question, which puzzled Newton and many other physicists over the years, was answered when Einstein published his theory of gravitation, known as his *general theory of relativity,* in 1916. Because it is a mathematically complex theory, we offer merely a hint of its elegance and insight.

In Einstein's view, the remarkable coincidence that m_g and m_i seemed to be proportional to each other was evidence of an intimate and basic connection between the two concepts. He pointed out that no mechanical experiment (such as dropping a mass) could distinguish between the two situations illustrated in Figure 39.22a and b. In each case, the dropped briefcase undergoes a downward acceleration g relative to the floor.

Einstein carried this idea further and proposed that *no* experiment, mechanical or otherwise, could distinguish between the two cases. This extension to in-

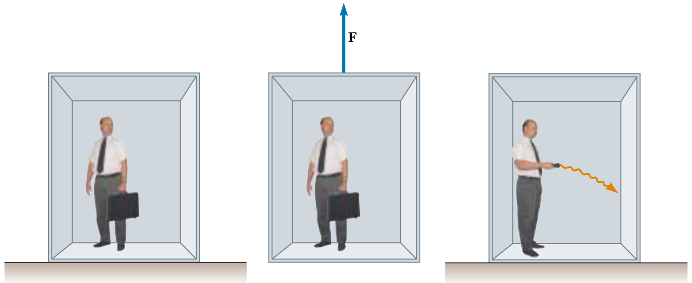


Figure 39.22 (a) The observer is at rest in a uniform gravitational field **g**. (b) The observer is in a region where gravity is negligible, but the frame of reference is accelerated by an external force **F** that produces an acceleration **g**. According to Einstein, the frames of reference in parts (a) and (b) are equivalent in every way. No local experiment can distinguish any difference between the two frames. (c) If parts (a) and (b) are truly equivalent, as Einstein proposed, then a ray of light should bend in a gravitational field.

clude all phenomena (not just mechanical ones) has interesting consequences. For example, suppose that a light pulse is sent horizontally across the elevator. During the time it takes the light to make the trip, the right wall of the elevator has accelerated upward. This causes the light to arrive at a location lower on the wall than the spot it would have hit if the elevator were not accelerating. Thus, in the frame of the elevator, the trajectory of the light pulse bends downward as the elevator accelerates upward to meet it. Because the accelerating elevator cannot be distinguished from a nonaccelerating one located in a gravitational field, Einstein proposed that a beam of light *should also be bent downward by a gravitational field,* as shown in Figure 39.22c. Experiments have verified the effect, although the bending is small. A laser aimed at the horizon falls less than 1 cm after traveling 6 000 km. (No such bending is predicted in Newton's theory of gravitation.)

The two postulates of Einstein's **general theory of relativity** are

- All the laws of nature have the same form for observers in any frame of reference, whether accelerated or not.

This Global Positioning System (GPS) unit incorporates relativistically corrected time calculations in its analysis of signals it receives from orbiting satellites. These corrections allow the unit to determine its position on the Earth's surface to within a few meters. If the corrections were not made, the location error would be about 1 km. *(Courtesy of Trimble Navigation Limited)*

- In the vicinity of any point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects. (This is the *principle of equivalence.*)

The second postulate implies that gravitational mass and inertial mass are completely equivalent, not just proportional. What were thought to be two different types of mass are actually identical.

One interesting effect predicted by the general theory is that time scales are altered by gravity. A clock in the presence of gravity runs more slowly than one located where gravity is negligible. Consequently, the frequencies of radiation emitted by atoms in the presence of a strong gravitational field are *red-shifted* to lower frequencies when compared with the same emissions in the presence of a weak field. This gravitational red shift has been detected in spectral lines emitted by atoms in massive stars. It has also been verified on the Earth by comparison of the frequencies of gamma rays (a high-energy form of electromagnetic radiation) emitted from nuclei separated vertically by about 20 m.

Quick Quiz 39.7

Two identical clocks are in the same house, one upstairs in a bedroom and the other downstairs in the kitchen. Which clock runs more slowly?

The second postulate suggests that a gravitational field may be "transformed away" at any point if we choose an appropriate accelerated frame of reference—a freely falling one. Einstein developed an ingenious method of describing the acceleration necessary to make the gravitational field "disappear." He specified a concept, the *curvature of space–time,* that describes the gravitational effect at every point. In fact, the curvature of space–time completely replaces Newton's gravitational theory. According to Einstein, there is no such thing as a gravitational force. Rather, the presence of a mass causes a curvature of space–time in the vicinity of the mass, and this curvature dictates the space–time path that all freely moving objects must follow. In 1979, John Wheeler summarized Einstein's general theory of relativity in a single sentence: "Space tells matter how to move and matter tells space how to curve."

Consider two travelers on the surface of the Earth walking directly toward the

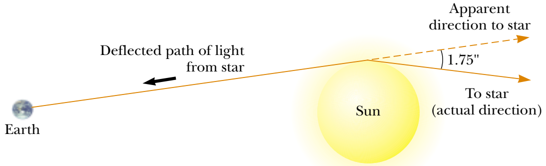
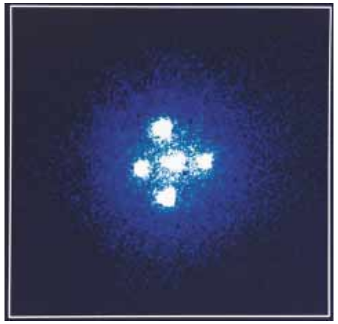


Figure 39.23 Deflection of starlight passing near the Sun. Because of this effect, the Sun or some other remote object can act as a *gravitational lens.* In his general theory of relativity, Einstein calculated that starlight just grazing the Sun's surface should be deflected by an angle of 1.75″.



Einstein's cross. The four bright spots are images of the same galaxy that have been bent around a massive object located between the galaxy and the Earth. The massive object acts like a lens, causing the rays of light that were diverging from the distant galaxy to converge on the Earth. (If the intervening massive object had a uniform mass distribution, we would see a bright ring instead of four spots.)

North Pole but from different starting locations. Even though both say they are walking due north, and thus should be on parallel paths, they see themselves getting closer and closer together, as if they were somehow attracted to each other. The curvature of the Earth causes this effect. In a similar way, what we are used to thinking of as the gravitational attraction between two masses is, in Einstein's view, two masses curving space–time and as a result moving toward each other, much like two bowling balls on a mattress rolling together.

One prediction of the general theory of relativity is that a light ray passing near the Sun should be deflected into the curved space–time created by the Sun's mass. This prediction was confirmed when astronomers detected the bending of starlight near the Sun during a total solar eclipse that occurred shortly after World War I (Fig. 39.23). When this discovery was announced, Einstein became an international celebrity.

If the concentration of mass becomes very great, as is believed to occur when a large star exhausts its nuclear fuel and collapses to a very small volume, a **black hole** may form. Here, the curvature of space–time is so extreme that, within a certain distance from the center of the black hole, all matter and light become trapped.

SUMMARY

The two basic postulates of the special theory of relativity are

- The laws of physics must be the same in all inertial reference frames.
- The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

Three consequences of the special theory of relativity are

- Events that are simultaneous for one observer are not simultaneous for another observer who is in motion relative to the first.
- Clocks in motion relative to an observer appear to be slowed down by a factor $\gamma = (1 - v^2/c^2)^{-1/2}$. This phenomenon is known as **time dilation.**
- The length of objects in motion appears to be contracted in the direction of

motion by a factor $1/\gamma = (1 - v^2/c^2)^{1/2}$. This phenomenon is known as **length contraction.**

To satisfy the postulates of special relativity, the Galilean transformation equations must be replaced by the **Lorentz transformation equations:**

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z \qquad \textbf{(39.11)}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

The relativistic form of the **velocity transformation equation** is

$$u_x' = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \qquad \textbf{(39.16)}$$

where u_x is the speed of an object as measured in the S frame and u_x' is its speed measured in the S′ frame.

The relativistic expression for the **linear momentum** of a particle moving with a velocity **u** is

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \gamma m\mathbf{u} \qquad \textbf{(39.19)}$$

The relativistic expression for the **kinetic energy** of a particle is

QUESTIONS

1. What two speed measurements do two observers in relative motion always agree on?
2. A spaceship in the shape of a sphere moves past an observer on the Earth with a speed 0.5*c*. What shape does the observer see as the spaceship moves past?
3. An astronaut moves away from the Earth at a speed close to the speed of light. If an observer on Earth measures the astronaut's dimensions and pulse rate, what changes (if any) would the observer measure? Would the astronaut measure any changes about himself?
4. Two identical clocks are synchronized. One is then put in orbit directed eastward around the Earth while the other remains on Earth. Which clock runs slower? When the moving clock returns to Earth, are the two still synchronized?
5. Two lasers situated on a moving spacecraft are triggered simultaneously. An observer on the spacecraft claims to see the pulses of light simultaneously. What condition is necessary so that a second observer agrees?
6. When we say that a moving clock runs more slowly than a stationary one, does this imply that there is something physically unusual about the moving clock?
7. List some ways our day-to-day lives would change if the speed of light were only 50 m/s.
8. Give a physical argument that shows that it is impossible to accelerate an object of mass *m* to the speed of light, even if it has a continuous force acting on it.
9. It is said that Einstein, in his teenage years, asked the question, "What would I see in a mirror if I carried it in my hands and ran at the speed of light?" How would you answer this question?
10. Some distant star-like objects, called *quasars,* are receding from us at half the speed of light (or greater). What is the speed of the light we receive from these quasars?
11. How is it possible that photons of light, which have zero mass, have momentum?
12. With regard to reference frames, how does general relativity differ from special relativity?
13. Describe how the results of Example 39.7 would change if, instead of fast spaceships, two ordinary cars were approaching each other at highway speeds.
14. Two objects are identical except that one is hotter than the other. Compare how they respond to identical forces.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 39.1 The Principle of Galilean Relativity

1. A 2 000-kg car moving at 20.0 m/s collides and locks together with a 1 500-kg car at rest at a stop sign. Show that momentum is conserved in a reference frame moving at 10.0 m/s in the direction of the moving car.

2. A ball is thrown at 20.0 m/s inside a boxcar moving along the tracks at 40.0 m/s. What is the speed of the ball relative to the ground if the ball is thrown (a) forward? (b) backward? (c) out the side door?

3. In a laboratory frame of reference, an observer notes that Newton's second law is valid. Show that it is also valid for an observer moving at a constant speed, small compared with the speed of light, relative to the laboratory frame.

4. Show that Newton's second law is *not* valid in a reference frame moving past the laboratory frame of Problem 3 with a constant acceleration.

Section 39.2 The Michelson–Morley Experiment
Section 39.3 Einstein's Principle of Relativity
Section 39.4 Consequences of the Special Theory of Relativity

5. How fast must a meter stick be moving if its length is observed to shrink to 0.500 m?

6. At what speed does a clock have to move if it is to be seen to run at a rate that is one-half the rate of a clock at rest?

7. An astronaut is traveling in a space vehicle that has a speed of $0.500c$ relative to the Earth. The astronaut measures his pulse rate at 75.0 beats per minute. Signals generated by the astronaut's pulse are radioed to Earth when the vehicle is moving in a direction perpendicular to a line that connects the vehicle with an observer on the Earth. What pulse rate does the Earth observer measure? What would be the pulse rate if the speed of the space vehicle were increased to $0.990c$?

8. The proper length of one spaceship is three times that of another. The two spaceships are traveling in the same direction and, while both are passing overhead, an Earth observer measures the two spaceships to have the same length. If the slower spaceship is moving with a speed of $0.350c$, determine the speed of the faster spaceship.

9. An atomic clock moves at 1 000 km/h for 1 h as measured by an identical clock on Earth. How many nanoseconds slow will the moving clock be at the end of the 1-h interval?

10. If astronauts could travel at $v = 0.950c$, we on Earth would say it takes $(4.20/0.950) = 4.42$ yr to reach Alpha Centauri, 4.20 ly away. The astronauts disagree. (a) How much time passes on the astronauts' clocks? (b) What distance to Alpha Centauri do the astronauts measure?

WEB **11.** A spaceship with a proper length of 300 m takes 0.750 μs to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.

12. A spaceship of proper length L_p takes time t to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.

13. A muon formed high in the Earth's atmosphere travels at speed $v = 0.990c$ for a distance of 4.60 km before it decays into an electron, a neutrino, and an antineutrino $(\mu^- \rightarrow e^- + \nu + \bar{\nu})$. (a) How long does the muon live, as measured in its reference frame? (b) How far does the muon travel, as measured in its frame?

14. Review Problem. In 1962, when Mercury astronaut Scott Carpenter orbited the Earth 22 times, the press stated that for each orbit he aged 2 millionths of a second less than he would have had he remained on Earth. (a) Assuming that he was 160 km above the Earth in a circular orbit, determine the time difference between someone on Earth and the orbiting astronaut for the 22 orbits. You will need to use the approximation $\sqrt{1 - x} \approx 1 - x/2$ for small x. (b) Did the press report accurate information? Explain.

15. The pion has an average lifetime of 26.0 ns when at rest. In order for it to travel 10.0 m, how fast must it move?

16. For what value of v does $\gamma = 1.01$? Observe that for speeds less than this value, time dilation and length contraction are less-than-one-percent effects.

17. A friend passes by you in a spaceship traveling at a high speed. He tells you that his ship is 20.0 m long and that the identically constructed ship you are sitting in is 19.0 m long. According to your observations, (a) how long is your ship, (b) how long is your friend's ship, and (c) what is the speed of your friend's ship?

18. An interstellar space probe is launched from Earth. After a brief period of acceleration it moves with a constant velocity, 70.0% of the speed of light. Its nuclear-powered batteries supply the energy to keep its data transmitter active continuously. The batteries have a lifetime of 15.0 yr as measured in a rest frame. (a) How long do the batteries on the space probe last as measured by Mission Control on Earth? (b) How far is the probe from Earth when its batteries fail, as measured by Mission Control? (c) How far is the probe from Earth when its batteries fail, as measured by its built-in trip odometer? (d) For what total time after launch are data received from the probe by Mission Control? Note that radio waves travel at the speed of light and fill the space between the probe and Earth at the time of battery failure.

19. Review Problem. An alien civilization occupies a brown dwarf, nearly stationary relative to the Sun, several lightyears away. The extraterrestrials have come to love original broadcasts of *The Ed Sullivan Show*, on our television channel 2, at carrier frequency 57.0 MHz. Their line of sight to us is in the plane of the Earth's orbit. Find the difference between the highest and lowest frequencies they receive due to the Earth's orbital motion around the Sun.

20. Police radar detects the speed of a car (Fig. P39.20) as follows: Microwaves of a precisely known frequency are broadcast toward the car. The moving car reflects the microwaves with a Doppler shift. The reflected waves are received and combined with an attenuated version of the transmitted wave. Beats occur between the two microwave signals. The beat frequency is measured. (a) For an electromagnetic wave reflected back to its source from a mirror approaching at speed v, show that the reflected wave has frequency

$$f = f_{\text{source}} \frac{c + v}{c - v}$$

where f_{source} is the source frequency. (b) When v is much less than c, the beat frequency is much less than the transmitted frequency. In this case, use the approximation $f + f_{\text{source}} \cong 2f_{\text{source}}$ and show that the beat frequency can be written as $f_b = 2v/\lambda$. (c) What beat frequency is measured for a car speed of 30.0 m/s if the microwaves have frequency 10.0 GHz? (d) If the beat frequency measurement is accurate to ±5 Hz, how accurate is the velocity measurement?

Figure P39.20 (Trent Steffler/David R. Frazier Photolibrary)

21. *The red shift.* A light source recedes from an observer with a speed v_{source}, which is small compared with c. (a) Show that the fractional shift in the measured wavelength is given by the approximate expression

$$\frac{\Delta\lambda}{\lambda} \approx \frac{v_{\text{source}}}{c}$$

This phenomenon is known as the red shift because the visible light is shifted toward the red. (b) Spectroscopic measurements of light at $\lambda = 397$ nm coming from a galaxy in Ursa Major reveal a red shift of 20.0 nm. What is the recessional speed of the galaxy?

Section 39.5 The Lorentz Transformation Equations

22. A spaceship travels at $0.750c$ relative to Earth. If the spaceship fires a small rocket in the forward direction, how fast (relative to the ship) must it be fired for it to travel at $0.950c$ relative to Earth?

WEB **23.** Two jets of material from the center of a radio galaxy fly away in opposite directions. Both jets move at $0.750c$ relative to the galaxy. Determine the speed of one jet relative to that of the other.

24. A moving rod is observed to have a length of 2.00 m, and to be oriented at an angle of 30.0° with respect to the direction of motion (Fig. P39.24). The rod has a speed of $0.995c$. (a) What is the proper length of the rod? (b) What is the orientation angle in the proper frame?

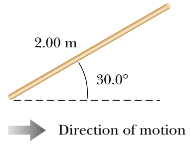


Figure P39.24

25. A Klingon space ship moves away from the Earth at a speed of $0.800c$ (Fig. P39.25). The starship *Enterprise* pursues at a speed of $0.900c$ relative to the Earth. Observers on Earth see the *Enterprise* overtaking the Klingon ship at a relative speed of $0.100c$. With what speed is the *Enterprise* overtaking the Klingon ship as seen by the crew of the *Enterprise*?

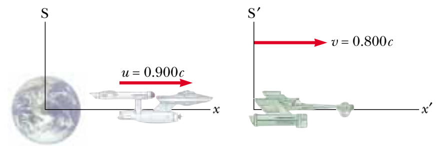


Figure P39.25

26. A red light flashes at position $x_R = 3.00$ m and time $t_R = 1.00 \times 10^{-9}$ s, and a blue light flashes at $x_B = 5.00$ m and $t_B = 9.00 \times 10^{-9}$ s (all values are measured in the S reference frame). Reference frame S′ has its origin at the same point as S at $t = t' = 0$; frame S′ moves constantly to the right. Both flashes are observed to occur at the same place in S′. (a) Find the relative velocity between S and S′. (b) Find the location of the two flashes in frame S′. (c) At what time does the red flash occur in the S′ frame?

Section 39.6 Relativistic Linear Momentum and the Relativistic Form of Newton's Laws

27. Calculate the momentum of an electron moving with a speed of (a) 0.010 0c, (b) 0.500c, (c) 0.900c.

28. The nonrelativistic expression for the momentum of a particle, $p = mu$, can be used if $u \ll c$. For what speed does the use of this formula yield an error in the momentum of (a) 1.00 percent and (b) 10.0 percent?

29. A golf ball travels with a speed of 90.0 m/s. By what fraction does its relativistic momentum p differ from its classical value mu? That is, find the ratio $(p - mu)/mu$.

30. Show that the speed of an object having momentum p and mass m is

$$u = \frac{c}{\sqrt{1 + (mc/p)^2}}$$

WEB **31.** An unstable particle at rest breaks into two fragments of *unequal* mass. The mass of the lighter fragment is 2.50×10^{-28} kg, and that of the heavier fragment is 1.67×10^{-27} kg. If the lighter fragment has a speed of 0.893c after the breakup, what is the speed of the heavier fragment?

Section 39.7 Relativistic Energy

32. Determine the energy required to accelerate an electron (a) from 0.500c to 0.900c and (b) from 0.900c to 0.990c.

33. Find the momentum of a proton in MeV/c units if its total energy is twice its rest energy.

34. Show that, for any object moving at less than one-tenth the speed of light, the relativistic kinetic energy agrees with the result of the classical equation $K = mu^2/2$ to within less than 1%. Thus, for most purposes, the classical equation is good enough to describe these objects, whose motion we call *nonrelativistic*.

WEB **35.** A proton moves at 0.950c. Calculate its (a) rest energy, (b) total energy, and (c) kinetic energy.

36. An electron has a kinetic energy five times greater than its rest energy. Find (a) its total energy and (b) its speed.

37. A cube of steel has a volume of 1.00 cm^3 and a mass of 8.00 g when at rest on the Earth. If this cube is now given a speed $u = 0.900c$, what is its density as measured by a stationary observer? Note that relativistic density is E_R/c^2V.

38. An unstable particle with a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocities of 0.987c and $-0.868c$. Find the masses of the fragments. (*Hint:* Conserve both mass–energy and momentum.)

39. Show that the energy–momentum relationship $E^2 = p^2c^2 + (mc^2)^2$ follows from the expressions $E = \gamma mc^2$ and $p = \gamma mu$.

40. A proton in a high-energy accelerator is given a kinetic energy of 50.0 GeV. Determine (a) its momentum and (b) its speed.

41. In a typical color television picture tube, the electrons are accelerated through a potential difference of 25 000 V. (a) What speed do the electrons have when they strike the screen? (b) What is their kinetic energy in joules?

42. Electrons are accelerated to an energy of 20.0 GeV in the 3.00-km-long Stanford Linear Accelerator. (a) What is the γ factor for the electrons? (b) What is their speed? (c) How long does the accelerator appear to them?

43. A pion at rest ($m_\pi = 270m_e$) decays to a muon ($m_\mu = 206m_e$) and an antineutrino ($m_{\bar{\nu}} \approx 0$). The reaction is written $\pi^- \rightarrow \mu^- + \bar{\nu}$. Find the kinetic energy of the muon and the antineutrino in electron volts. (*Hint:* Relativistic momentum is conserved.)

Section 39.8 Equivalence of Mass and Energy

44. Make an order-of-magnitude estimate of the ratio of mass increase to the original mass of a flag as you run it up a flagpole. In your solution explain what quantities you take as data and the values you estimate or measure for them.

45. When 1.00 g of hydrogen combines with 8.00 g of oxygen, 9.00 g of water is formed. During this chemical reaction, 2.86×10^5 J of energy is released. How much mass do the constituents of this reaction lose? Is the loss of mass likely to be detectable?

46. A spaceship of mass 1.00×10^6 kg is to be accelerated to 0.600c. (a) How much energy does this require? (b) How many kilograms of matter would it take to provide this much energy?

47. In a nuclear power plant the fuel rods last 3 yr before they are replaced. If a plant with rated thermal power 1.00 GW operates at 80.0% capacity for the 3 yr, what is the loss of mass of the fuel?

48. A ^{57}Fe nucleus at rest emits a 14.0-keV photon. Use the conservation of energy and momentum to deduce the kinetic energy of the recoiling nucleus in electron volts. (Use $Mc^2 = 8.60 \times 10^{-9}$ J for the final state of the ^{57}Fe nucleus.)

49. The power output of the Sun is 3.77×10^{26} W. How much mass is converted to energy in the Sun each second?

50. A gamma ray (a high-energy photon of light) can produce an electron (e^-) and a positron (e^+) when it enters the electric field of a heavy nucleus: $\gamma \rightarrow e^+ + e^-$. What minimum γ-ray energy is required to accomplish this task? (*Hint:* The masses of the electron and the positron are equal.)

Section 39.9 Relativity and Electromagnetism

51. As measured by observers in a reference frame S, a particle having charge q moves with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$. The resulting force on the particle is then measured to be $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Another observer moves along with the charged particle and also measures its charge to be q but measures the electric field to be $\mathbf{E}'$. If both observers are to measure the same force $\mathbf{F}$, show that $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$.

ADDITIONAL PROBLEMS

52. An electron has a speed of 0.750c. Find the speed of a proton that has (a) the same kinetic energy as the electron; (b) the same momentum as the electron.

WEB **53.** The cosmic rays of highest energy are protons, which have kinetic energy on the order of 10^{13} MeV. (a) How long would it take a proton of this energy to travel across the Milky Way galaxy, having a diameter of $\sim 10^5$ ly, as measured in the proton's frame? (b) From the point of view of the proton, how many kilometers across is the galaxy?

54. A spaceship moves away from the Earth at 0.500c and fires a shuttle craft in the forward direction at 0.500c relative to the ship. The pilot of the shuttle craft launches a probe at forward speed 0.500c relative to the shuttle craft. Determine (a) the speed of the shuttle craft relative to the Earth and (b) the speed of the probe relative to the Earth.

55. The net nuclear fusion reaction inside the Sun can be written as $4\,^1H \rightarrow \,^4He + \Delta E$. If the rest energy of each hydrogen atom is 938.78 MeV and the rest energy of the helium-4 atom is 3 728.4 MeV, what is the percentage of the starting mass that is released as energy?

56. An astronaut wishes to visit the Andromeda galaxy (2.00 million lightyears away), making a one-way trip that will take 30.0 yr in the spaceship's frame of reference. If his speed is constant, how fast must he travel relative to the Earth?

57. An alien spaceship traveling at 0.600c toward the Earth launches a landing craft with an advance guard of purchasing agents. The lander travels in the same direction with a velocity 0.800c relative to the spaceship. As observed on the Earth, the spaceship is 0.200 ly from the Earth when the lander is launched. (a) With what velocity is the lander observed to be approaching by observers on the Earth? (b) What is the distance to the Earth at the time of lander launch, as observed by the aliens? (c) How long does it take the lander to reach the Earth as observed by the aliens on the mother ship? (d) If the lander has a mass of 4.00×10^5 kg, what is its kinetic energy as observed in the Earth reference frame?

58. A physics professor on the Earth gives an exam to her students, who are on a rocket ship traveling at speed v relative to the Earth. The moment the ship passes the professor, she signals the start of the exam. She wishes her students to have time T_0 (rocket time) to complete the exam. Show that she should wait a time (Earth time) of

$$T = T_0\sqrt{\frac{1 - v/c}{1 + v/c}}$$

before sending a light signal telling them to stop. (*Hint:* Remember that it takes some time for the second light signal to travel from the professor to the students.)

59. Spaceship I, which contains students taking a physics exam, approaches the Earth with a speed of 0.600c (relative to the Earth), while spaceship II, which contains professors proctoring the exam, moves at 0.280c (relative to the Earth) directly toward the students. If the professors stop the exam after 50.0 min have passed on their clock, how long does the exam last as measured by (a) the students? (b) an observer on the Earth?

60. Energy reaches the upper atmosphere of the Earth from the Sun at the rate of 1.79×10^{17} W. If all of this energy were absorbed by the Earth and not re-emitted, how much would the mass of the Earth increase in 1 yr?

61. A supertrain (proper length, 100 m) travels at a speed of 0.950c as it passes through a tunnel (proper length, 50.0 m). As seen by a trackside observer, is the train ever completely within the tunnel? If so, with how much space to spare?

62. Imagine that the entire Sun collapses to a sphere of radius R_g such that the work required to remove a small mass m from the surface would be equal to its rest energy mc^2. This radius is called the *gravitational radius* for the Sun. Find R_g. (It is believed that the ultimate fate of very massive stars is to collapse beyond their gravitational radii into black holes.)

63. A charged particle moves along a straight line in a uniform electric field $\mathbf{E}$ with a speed of u. If the motion and the electric field are both in the x direction, (a) show that the acceleration of the charge q in the x direction is given by

$$a = \frac{du}{dt} = \frac{qE}{m}\left(1 - \frac{u^2}{c^2}\right)^{3/2}$$

(b) Discuss the significance of the dependence of the acceleration on the speed. (c) If the particle starts from rest at $x = 0$ at $t = 0$, how would you proceed to find the speed of the particle and its position after a time t has elapsed?

64. (a) Show that the Doppler shift $\Delta\lambda$ in the wavelength of light is described by the expression

$$\frac{\Delta\lambda}{\lambda} + 1 = \sqrt{\frac{c - v}{c + v}}$$

where λ is the source wavelength and v is the speed of relative approach between source and observer. (b) How fast would a motorist have to be going for a red light to appear green? Take 650 nm as a typical wavelength for red light, and one of 550 nm as typical for green.

65. A rocket moves toward a mirror at $0.800c$ relative to the reference frame S in Figure P39.65. The mirror is stationary relative to S. A light pulse emitted by the rocket travels toward the mirror and is reflected back to the rocket. The front of the rocket is 1.80×10^{12} m from the mirror (as measured by observers in S) at the moment the light pulse leaves the rocket. What is the total travel time of the pulse as measured by observers in (a) the S frame and (b) the front of the rocket?

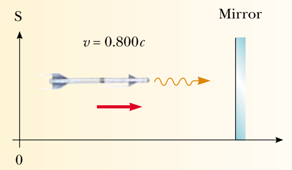


Figure P39.65 Problems 65 and 66.

66. An observer in a rocket moves toward a mirror at speed v relative to the reference frame labeled by S in Figure P39.65. The mirror is stationary with respect to S. A light pulse emitted by the rocket travels toward the mirror and is reflected back to the rocket. The front of the rocket is a distance d from the mirror (as measured by observers in S) at the moment the light pulse leaves the rocket. What is the total travel time of the pulse as measured by observers in (a) the S frame and (b) the front of the rocket?

67. Ted and Mary are playing a game of catch in frame S′, which is moving at $0.600c$, while Jim in frame S watches the action (Fig. P39.67). Ted throws the ball to Mary at $0.800c$ (according to Ted) and their separation (measured in S′) is 1.80×10^{12} m. (a) According to Mary, how fast is the ball moving? (b) According to Mary, how long does it take the ball to reach her? (c) According to Jim, how far apart are Ted and Mary, and how fast is the ball moving? (d) According to Jim, how long does it take the ball to reach Mary?

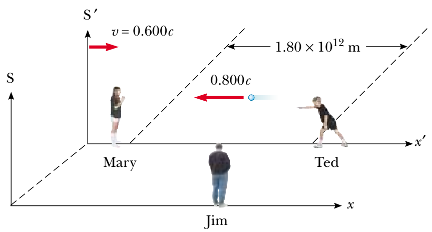


Figure P39.67

68. A rod of length L_0 moving with a speed v along the horizontal direction makes an angle θ_0 with respect to the x' axis. (a) Show that the length of the rod as measured by a stationary observer is $L = L_0[1 - (v^2/c^2)\cos^2\theta_0]^{1/2}$. (b) Show that the angle that the rod makes with the x axis is given by $\tan\theta = \gamma\tan\theta_0$. These results show that the rod is both contracted and rotated. (Take the lower end of the rod to be at the origin of the primed coordinate system.)

69. Consider two inertial reference frames S and S′, where S′ is moving to the right with a constant speed of $0.600c$ as measured by an observer in S. A stick of proper length 1.00 m moves to the left toward the origins of both S and S′, and the length of the stick is 50.0 cm as measured by an observer in S′. (a) Determine the speed of the stick as measured by observers in S and S′. (b) What is the length of the stick as measured by an observer in S?

70. Suppose our Sun is about to explode. In an effort to escape, we depart in a spaceship at $v = 0.800c$ and head toward the star Tau Ceti, 12.0 ly away. When we reach the midpoint of our journey from the Earth, we see our Sun explode and, unfortunately, at the same instant we see Tau Ceti explode as well. (a) In the spaceship's frame of reference, should we conclude that the two explosions occurred simultaneously? If not, which occurred first? (b) In a frame of reference in which the Sun and Tau Ceti are at rest, did they explode simultaneously? If not, which exploded first?

71. The light emitted by a galaxy shows a continuous distribution of wavelengths because the galaxy is composed of billions of different stars and other thermal emitters. Nevertheless, some narrow gaps occur in the continuous spectrum where light has been absorbed by cooler gases in the outer photospheres of normal stars. In particular, ionized calcium atoms at rest produce strong absorption at a wavelength of 394 nm. For a galaxy in the constellation Hydra, 2 billion lightyears away, this absorption line is shifted to 475 nm. How fast is the galaxy moving away from the Earth? (*Note:* The assumption that the recession speed is small compared with c, as made in Problem 21, is not a good approximation here.)

72. Prepare a graph of the relativistic kinetic energy and the classical kinetic energy, both as a function of speed, for an object with a mass of your choice. At what speed does the classical kinetic energy underestimate the relativistic value by 1 percent? By 5 percent? By 50 percent?

73. The total volume of water in the oceans is approximately 1.40×10^9 km^3. The density of sea water is 1 030 kg/m^3, and the specific heat of the water is 4 186 J/(kg·°C). Find the increase in mass of the oceans produced by an increase in temperature of 10.0°C.

ANSWERS TO QUICK QUIZZES

39.1 They both are because they can report only what they see. They agree that the person in the truck throws the ball up and then catches it a bit later.

39.2 It depends on the direction of the throw. Taking the direction in which the train is traveling as the positive x direction, use the values $u_x' = +90$ mi/h and $v = +110$ mi/h, with u_x in Equation 39.2 being the value you are looking for. If the pitcher throws the ball in the same direction as the train, a person at rest on the Earth sees the ball moving at 110 mi/h + 90 mi/h = 200 mi/h. If the pitcher throws in the opposite direction, the person on the Earth sees the ball moving in the same direction as the train but at only 110 mi/h − 90 mi/h = 20 mi/h.

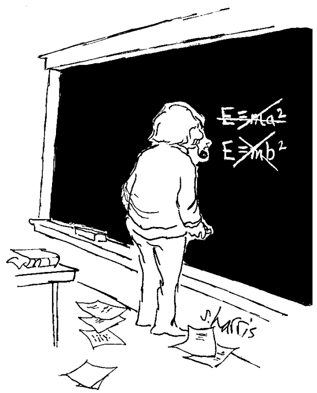
39.3 Both are correct. Although the two observers reach different conclusions, each is correct in her or his own reference frame because the concept of simultaneity is not absolute.

39.4 About 2.9×10^8 m/s, because this is the speed at which $\gamma = 5$. For every 5 s ticking by on the Mission Control clock, the Earth-bound observer (with a powerful telescope!) sees the rocket clock ticking off 1 s. The astronaut sees her own clock operating at a normal rate. To her, Mission Control is moving away from her at a speed of 2.9×10^8 m/s, and she sees the Mission Control clock as running slow. Strange stuff, this relativity!

39.5 If their on-duty time is based on clocks that remain on the Earth, they will have larger paychecks. Less time will have passed for the astronauts in their frame of reference than for their employer back on the Earth.

39.6 By a curved line. This can be seen in the middle of Speedo's world-line in Figure 39.14, where he turns around and begins his trip home.

39.7 The downstairs clock runs more slowly because it is closer to the Earth and hence experiences a stronger gravitational field than the upstairs clock does.



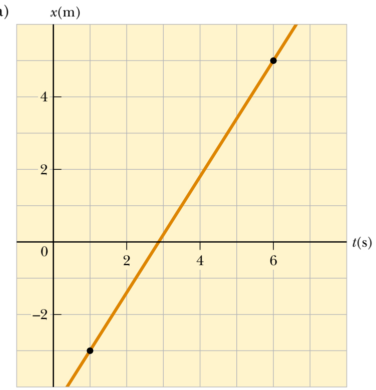
Answers to Odd-Numbered Problems

Chapter 1

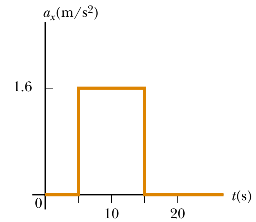
1. 2.15×10^4 kg/m^3
3. 184 g
5. (a) 7.10 cm^3 (b) 1.18×10^{-29} m^3 (c) 0.228 nm (d) 12.7 cm^3, 2.11×10^{-29} m^3, 0.277 nm
7. (a) 4.00 u = 6.64×10^{-24} g (b) 55.9 u = 9.29×10^{-23} g (c) 207 u = 3.44×10^{-22} g
9. (a) 9.83×10^{-16} g (b) 1.06×10^7 atoms
11. (a) 4.01×10^{25} molecules (b) 3.65×10^4 molecules
13. no
15. (b) only
17. $0.579t$ ft^3/s + $1.19 \times 10^{-9} t^2$ ft^3/s^2
19. 1.39×10^3 m^2
21. (a) 0.071 4 gal/s (b) 2.70×10^{-4} m^3/s (c) 1.03 h
23. 4.05×10^3 m^2
25. 11.4×10^3 kg/m^3
27. 1.19×10^{57} atoms
29. (a) 190 y (b) 2.32×10^4 times
31. 151 μm
33. 1.00×10^{10} lb
35. 3.08×10^4 m^3
37. 5.0 m
39. 2.86 cm
41. $\sim 10^6$ balls
43. $\sim 10^7$ or 10^8 rev
45. $\sim 10^7$ or 10^8 blades
47. $\sim 10^2$ kg; $\sim 10^3$ kg
49. $\sim 10^2$ tuners
51. (a) (346 ± 13) m^2 (b) (66.0 ± 1.3) m
53. $(1.61 \pm 0.17) \times 10^3$ kg/m^3
55. 115.9 m
57. 316 m
59. 4.50 m^2
61. 3.41 m
63. 0.449%
65. (a) 0.529 cm/s (b) 11.5 cm/s
67. 1×10^{10} gal/yr
69. $\sim 10^{11}$ stars

Chapter 2

1. (a) 2.30 m/s (b) 16.1 m/s (c) 11.5 m/s
3. (a) 5 m/s (b) 1.2 m/s (c) − 2.5 m/s (d) − 3.3 m/s (e) 0
5. (a) 3.75 m/s (b) 0
7. (a)



(b) 1.60 m/s
9. (a) − 2.4 m/s (b) − 3.8 m/s (c) 4.0 s
11. (a) 5.0 m/s (b) − 2.5 m/s (c) 0 (d) 5.0 m/s
13. 1.34×10^4 m/s^2
15. (a)

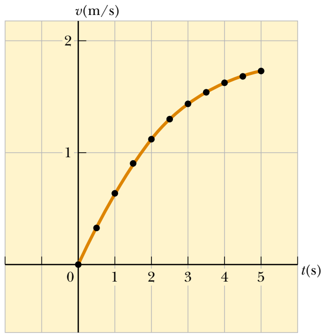


(b) 1.6 m/s^2 and 0.80 m/s^2
17. (a) 2.00 m (b) − 3.00 m/s (c) − 2.00 m/s^2
19. (a) 1.3 m/s^2 (b) 2.0 m/s^2 at 3 s (c) at $t = 6$ s and for $t > 10$ s (d) − 1.5 m/s^2 at 8 s
21. 2.74×10^5 m/s^2, which is 2.79×10^4 g
23. (a) 6.61 m/s (b) − 0.448 m/s^2
25. − 16.0 cm/s^2
27. (a) 2.56 m (b) − 3.00 m/s
29. (a) 8.94 s (b) 89.4 m/s
31. (a) 20.0 s (b) no
33. $x_f - x_i = v_{xf}\, t - a_x t^2/2$; $v_{xf} = 3.10$ m/s

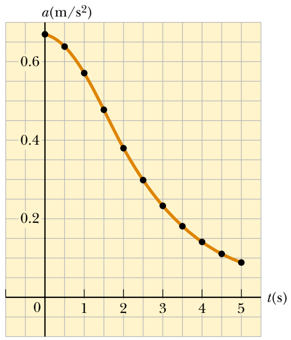
35. (a) 35.0 s (b) 15.7 m/s
37. (a) -202 m/s^2 (b) 198 m
39. (a) 3.00 m/s (b) 6.00 s (c) -0.300 m/s^2 (d) 2.05 m/s
41. (a) -4.90 m, -19.6 m, -44.1 m (b) -9.80 m/s, -19.6 m/s, -29.4 m/s
43. (a) 10.0 m/s up (b) 4.68 m/s down
45. No. In 0.2 s the bill falls out from between David's fingers.
47. (a) 29.4 m/s (b) 44.1 m
49. (a) 7.82 m (b) 0.782 s
51. (a) 1.53 s (b) 11.5 m (c) -4.60 m/s, -9.80 m/s^2
53. (a) $a_x = a_{xi} + Jt$, $v_x = v_{xi} + a_{xi}t + \frac{1}{2}Jt^2$, $x = x_i + v_{xi}t + \frac{1}{2}a_{xi}t^2 + \frac{1}{6}Jt^3$
55. 0.222 s
57. 0.509 s
59. (a) 41.0 s (b) 1.73 km (c) -184 m/s
61. $v_{xi}t + at^2/2$, in agreement with Equation 2.11
63. (a) 5.43 m/s^2 and 3.83 m/s^2 (b) 10.9 m/s and 11.5 m/s (c) Maggie by 2.62 m
65. (a) 45.7 s (b) 574 m (c) 12.6 m/s (d) 765 s
67. (a) 2.99 s (b) -15.4 m/s (c) 31.3 m/s down and 34.9 m/s down
69. (a) 5.46 s (b) 73.0 m (c) $v_{\text{Stan}} = 22.6$ m/s, $v_{\text{Kathy}} = 26.7$ m/s
71. (a) See top of next column. (b) See top of next column.
73. $0.577v$

Chapter 3

1. $(-2.75, -4.76)$ m
3. 1.15; 2.31
5. (a) 2.24 m (b) 2.24 m at 26.6° from the positive x axis.
7. (a) 484 m (b) 18.1° north of west
9. 70.0 m
11. (a) approximately 6.1 units at 112° (b) approximately 14.8 units at 22°
13. (a) 10.0 m (b) 15.7 m (c) 0
15. (a) 5.2 m at 60° (b) 3.0 m at 330° (c) 3.0 m at 150° (d) 5.2 m at 300°
17. approximately 420 ft at $-3°$
19. 5.83 m at 59.0° to the right of his initial direction
21. 1.31 km north and 2.81 km east
23. (a) 10.4 cm (b) 35.5°
25. 47.2 units at 122° from the positive x axis.
27. $(-25.0\mathbf{i})$m + $(43.3\mathbf{j})$m
29. 7.21 m at 56.3° from the positive x axis.
31. (a) $2.00\mathbf{i} - 6.00\mathbf{j}$ (b) $4.00\mathbf{i} + 2.00\mathbf{j}$ (c) 6.32 (d) 4.47 (e) 288°; 26.6° from the positive x axis.
33. (a) $(-11.1\mathbf{i} + 6.40\mathbf{j})$ m (b) $(1.65\mathbf{i} + 2.86\mathbf{j})$ cm (c) $(-18.0\mathbf{i} - 12.6\mathbf{j})$ in.
35. 9.48 m at 166°
37. (a) 185 N at 77.8° from the positive x axis (b) $(-39.3\mathbf{i} - 181\mathbf{j})$ N
39. $\mathbf{A} + \mathbf{B} = (2.60\mathbf{i} + 4.50\mathbf{j})$ m



Chapter 2, Problem 71(a)

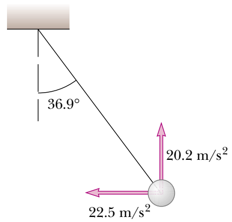


Chapter 2, Problem 71(b)

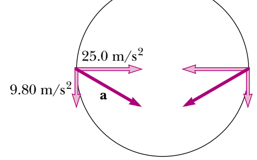
41. 196 cm at $-14.7°$ from the positive x axis.
43. (a) $8.00\mathbf{i} + 12.0\mathbf{j} - 4.00\mathbf{k}$ (b) $2.00\mathbf{i} + 3.00\mathbf{j} - 1.00\mathbf{k}$ (c) $-24.0\mathbf{i} - 36.0\mathbf{j} + 12.0\mathbf{k}$
45. (a) 5.92 m is the magnitude of $(5.00\mathbf{i} - 1.00\mathbf{j} - 3.00\mathbf{k})$ m (b) 19.0 m is the magnitude of $(4.00\mathbf{i} - 11.0\mathbf{j} + 15.0\mathbf{k})$ m
47. 157 km
49. (a) $-3.00\mathbf{i} + 2.00\mathbf{j}$ (b) 3.61 at 146° from the positive x axis. (c) $3.00\mathbf{i} - 6.00\mathbf{j}$
51. (a) $49.5\mathbf{i} + 27.1\mathbf{j}$ (b) 56.4 units at 28.7° from the positive x axis.
53. 1.15°
55. (a) 2.00, 1.00, 3.00 (b) 3.74 (c) $\theta_x = 57.7°$, $\theta_y = 74.5°$, $\theta_z = 36.7°$
57. 240 m at 237°
59. 390 mi/h at 7.37° north of east
61. $\mathbf{R}_1 = a\mathbf{i} + b\mathbf{j}$; $R_1 = \sqrt{a^2 + b^2}$ (b) $\mathbf{R}_2 = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Chapter 4

1. (a) 4.87 km at 209° from east (b) 23.3 m/s (c) 13.5 m/s at 209°
3. (a) $(18.0t)\mathbf{i} + (4.00t - 4.90t^2)\mathbf{j}$ (b) $18.0\mathbf{i} + (4.00 - 9.80t)\mathbf{j}$ (c) $-9.80\mathbf{j}$ (d) $(54.0\mathbf{i} - 32.1\mathbf{j})$ m (e) $(18.0\mathbf{i} - 25.4\mathbf{j})$ m/s (f) $(-9.80\mathbf{j})$ m/s^2
5. (a) $(2.00\mathbf{i} + 3.00\mathbf{j})$ m/s^2 (b) $(3.00t + t^2)\mathbf{i}$ m, $(1.50\,t^2 - 2.00t)\mathbf{j}$ m
7. (a) $(0.800\mathbf{i} - 0.300\mathbf{j})$ m/s^2 (b) 339° (c) $(360\mathbf{i} - 72.7\mathbf{j})$ m, $-15.2°$
9. (a) $(3.34\mathbf{i})$ m/s (b) $-50.9°$
11. (a) 20.0° (b) 3.05 s
13. $x = 7.23$ km $y = 1.68$ km
15. 53.1°
17. 22.4° or 89.4°
19. (a) The ball clears by 0.889 m (b) while descending
21. $d\tan\theta_i - gd^2/(2v_i^2\cos^2\theta_i)$
23. (a) 0.852 s (b) 3.29 m/s (c) 4.03 m/s (d) 50.8° (e) 1.12 s
25. 377 m/s^2
27. 10.5 m/s, 219 m/s^2
29. (a) 6.00 rev/s (b) 1.52 km/s^2 (c) 1.28 km/s^2
31. 1.48 m/s^2 inward at 29.9° behind the radius
33. (a) 13.0 m/s^2 (b) 5.70 m/s (c) 7.50 m/s^2
35. (a)



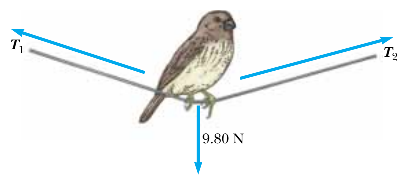
(b) 29.7 m/s^2 (c) 6.67 m/s at 36.9° above the horizontal
37. 2.02×10^3 s; 21.0% longer
39. 153 km/h at 11.3° north of west
41. (a) 36.9° (b) 41.6° (c) 3.00 min
43. 15.3 m
45. $2\,v_i t\cos\theta_i$
47. (b) $45° + \phi/2$; $v_i^2(1 - \sin\phi)/g\cos^2\phi$
49. (a) 41.7 m/s (b) 3.81 s (c) $(34.1\mathbf{i} - 13.4\mathbf{j})$ m/s; 36.6 m/s
51. (a) 25.0 m/s^2 (radial); 9.80 m/s^2 (tangential) (b)



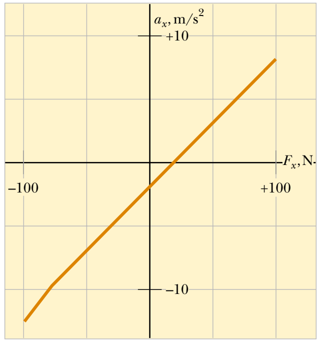
(c) 26.8 m/s^2 inward at 21.4° below the horizontal
53. 8.94 m/s at $-63.4°$ relative to the positive x axis.
55. 20.0 m
57. (a) 0.600 m (b) 0.402 m (c) 1.87 m/s^2 toward center (d) 9.80 m/s^2 down
59. (a) 6.80 km (b) 3.00 km vertically above the impact point (c) 66.2°
61. (a) 46.5 m/s (b) $-77.6°$ (c) 6.34 s
63. (a) 1.53 km (b) 36.2 s (c) 4.04 km
65. (a) 20.0 m/s, 5.00 s (b) $(16.0\mathbf{i} - 27.1\mathbf{j})$ m/s (c) 6.54 s (d) $(24.6\mathbf{i})$ m
67. (a) 43.2 m (b) $(9.66\mathbf{i} - 25.5\mathbf{j})$ m/s
69. Imagine you are shaking down the mercury in a fever thermometer. Starting with your hand at the level of your shoulder, move your hand down as fast as you can and snap it around an arc at the bottom. ~ 100 m/s$^2 \approx 10\,g$

Chapter 5

1. (a) 1/3 (b) 0.750 m/s^2
3. $(6.00\mathbf{i} + 15.0\mathbf{j})$ N; 16.2 N
5. 312 N
7. (a) $x = vt/2$ (b) $F_g v\mathbf{i}/gt + F_g\mathbf{j}$
9. (a) $(2.50\mathbf{i} + 5.00\mathbf{j})$ N (b) 5.59 N
11. (a) 3.64×10^{-18} N (b) 8.93×10^{-30} N is 408 billion times smaller.
13. 2.38 kN
15. (a) 5.00 m/s^2 at 36.9° (b) 6.08 m/s^2 at 25.3°
17. (a) $\sim 10^{-22}$ m/s^2 (b) $\sim 10^{-23}$ m
19. (a) 0.200 m/s^2 forward (b) 10.0 m (c) 2.00 m/s
21. (a) 15.0 lb up (b) 5.00 lb up (c) 0
23. 613 N

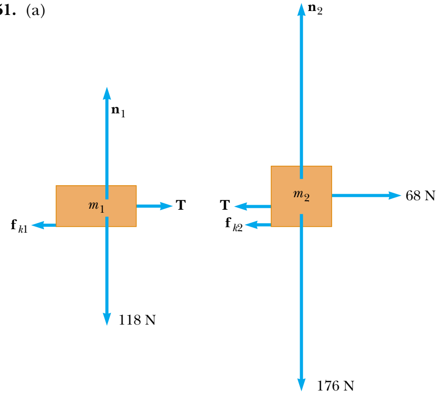


27. (a) 49.0 N (b) 98.0 N (c) 24.5 N
29. 8.66 N east
31. 100 N and 204 N
33. 3.73 m
35. $a = F/(m_1 + m_2)$; $T = F\,m_1/(m_1 + m_2)$
37. (a) $F_x > 19.6$ N (b) $F_x \leq -78.4$ N (c) See top of next page.
39. (a) 706 N (b) 814 N (c) 706 N (d) 648 N
41. $\mu_s = 0.306$; $\mu_k = 0.245$
43. (a) 256 m (b) 42.7 m
45. (a) 1.78 m/s^2 (b) 0.368 (c) 9.37 N (d) 2.67 m/s
47. (a) 0.161 (b) 1.01 m/s^2
49. 37.8 N



Chapter 5, Problem 37(c)

51. (a)

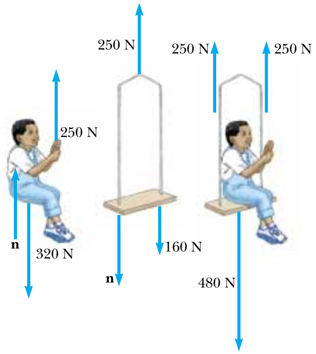


(b) 27.2 N, 1.29 m/s^2

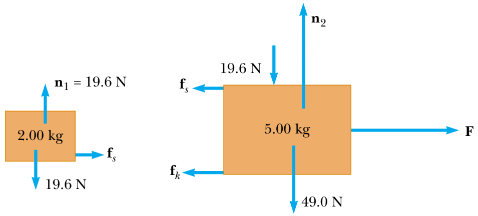
53. Any value between 31.7 N and 48.6 N
55. (a) See top of next column. (b) 0.408 m/s^2 (c) 83.3 N
57. 1.18 kN
59. (a) $Mg/2$, $Mg/2$, $Mg/2$, $3Mg/2$, Mg (b) $Mg/2$
61. (b)

θ	0	15.0°	30.0°	45.0°	60.0°
P(N)	40.0	46.4	60.1	94.3	260

63. (a) 19.3° (b) 4.21 N
65. (a) 2.13 s (b) 1.67 m
67. (a) See next column. Static friction between the two blocks accelerates the upper block. (b) 34.7 N (c) 0.306
69. $(M + m_1 + m_2)(m_2 g/m_1)$

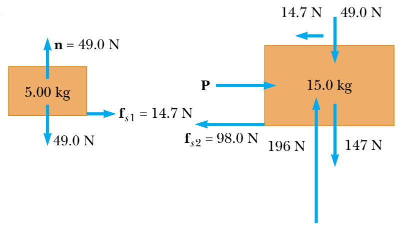


Chapter 5, Problem 55(a)



Chapter 5, Problem 67(a)

71. (a)



(b) 113 N (c) 0.980 m/s^2 and 1.96 m/s^2

73. (a) 0.087 1 (b) 27.4 N
75. (a) 30.7° (b) 0.843 N
77. (a) 3.34 (b) Either the car would flip over backwards, or the wheels would skid, spinning in place, and the time would increase.

Chapter 6

1. (a) 8.00 m/s (b) 3.02 N
3. Any speed up to 8.08 m/s

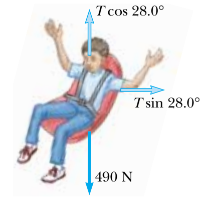
5. 6.22×10^{-12} N
7. (a) 1.52 m/s^2 (b) 1.66 km/s (c) 6 820 s
9. (a) static friction (b) 0.085 0
11. $v \leq 14.3$ m/s
13. (a) 68.6 N toward the center of the circle and 784 N up (b) 0.857 m/s^2
15. No. The jungle lord needs a vine of tensile strength 1.38 kN.
17. (a) 4.81 m/s (b) 700 N up
19. 3.13 m/s
21. (a) 2.49×10^4 N up (b) 12.1 m/s
23. (a) 0.822 m/s^2 (b) 37.0 N (c) 0.0839
25. (a) 17.0° (b) 5.12 N
27. (a) 491 N (b) 50.1 kg (c) 2.00 m/s^2
29. 0.0927°
31. (a) 32.7 s^{-1} (b) 9.80 m/s^2 (c) 4.90 m/s^2
33. 3.01 N
35. (a) 1.47 N·s/m (b) 2.04×10^{-3} s (c) 2.94×10^{-2} N
37. (a) 0.0347 s^{-1} (b) 2.50 m/s (c) $a = -cv$
39. ~ 10^1 N
41. (a) 13.7 m/s down
(b)

t (s)	x (m)	v (m/s)
0	0	0
0.2	0	−1.96
0.4	−0.392	−3.88
. . .	. . .	. . .
1.0	−3.77	−8.71
. . . 2.0	−14.4	−12.56
. . . 4.0	−41.0	−13.67

43. (a) 49.5 m/s and 4.95 m/s
(b)

t (s)	y (m)	v (m/s)
0	1 000	0
. . . 1	995	−9.7
. . . 2	980	−18.6
. . . 10	674	−47.7
. . . 10.1	671	−16.7
. . . 12	659	−4.95
. . . 145	0	−4.95

45. (a) 2.33×10^{-4} kg/m (b) 53 m/s (c) 42 m/s. The second trajectory is higher and shorter. In both, the ball attains maximum height when it has covered about 57% of its horizontal range, and it attains minimum speed somewhat later. The impact speeds also are both about 30 m/s.
47. (a) $mg - mv^2/R$ (b) $\sqrt{gR}$
49. (a) 2.63 m/s^2 (b) 201 m (c) 17.7 m/s
51. (a) 9.80 N (b) 9.80 N (c) 6.26 m/s
53. (b) 732 N down at the equator and 735 N down at the poles
59. (a) 1.58 m/s^2 (b) 455 N (c) 329 N (d) 397 N upward and 9.15° inward
61. (a) 5.19 m/s (b) Child + seat:

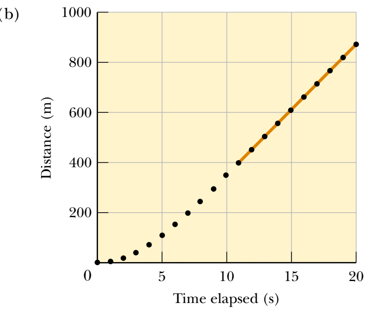


$T = 555$ N

63. (b) 2.54 s; 23.6 rev/min
65. 215 N horizontally inward
67. (a) either 70.4° or 0° (b) 0°
69. 12.8 N
71. (a)

t (s)	d (m)
0	0
1	4.9
2	18.9
. . . 5	112.6
. . . 10	347.0
11	399.1
. . . 15	611.3
. . . 20	876.5

(b)



(c) The graph is straight for 11 s < t < 20 s, with slope 53.0 m/s.

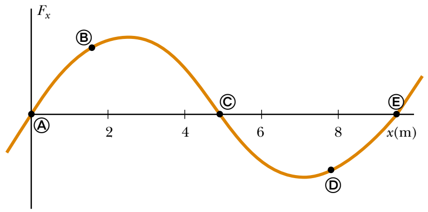
Chapter 7

1. 15.0 MJ
3. (a) 32.8 mJ (b) −32.8 mJ
5. (a) 31.9 J (b) 0 (c) 0 (d) 31.9 J
7. 4.70 kJ
9. 14.0
11. (a) 16.0 J (b) 36.9°
13. (a) 11.3° (b) 156° (c) 82.3°

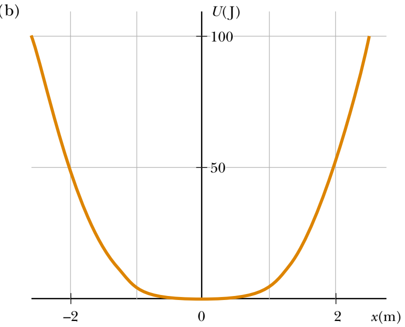
15. (a) 24.0 J (b) − 3.00 J (c) 21.0 J
17. (a) 7.50 J (b) 15.0 J (c) 7.50 J (d) 30.0 J
19. (a) 0.938 cm (b) 1.25 J
21. 0.299 m/s
23. 12.0 J
25. (b) mgR
27. (a) 1.20 J (b) 5.00 m/s (c) 6.30 J
29. (a) 60.0 J (b) 60.0 J
31. (a) $\sqrt{2W/m}$ (b) W/d
33. (a) 650 J (b) − 588 J (c) 0 (d) 0 (e) 62.0 J (f) 1.76 m/s
35. (a) − 168 J (b) − 184 J (c) 500 J (d) 148 J (e) 5.64 m/s
37. 2.04 m
39. (a) 22 500 N (b) 1.33×10^{-4} s
41. (a) 0.791 m/s (b) 0.531 m/s
43. 875 W
45. 830 N
47. (a) 5 910 W (b) It is 53.0% of 11 100 W
49. (a) 0.013 5 gal (b) 73.8 (c) 8.08 kW
51. 5.90 km/L
53. (a) 5.37×10^{-11} J (b) 1.33×10^{-9} J
55. 90.0 J
59. (a) $(2 + 24t^2 + 72t^4)$ J (b) $12t$ m/s^2; $48t$ N (c) $(48t + 288t^3)$ W (d) 1 250 J
61. − 0.047 5 J
63. 878 kN
65. (b) 240 W
67. (a) $\mathbf{F}_1 = (20.5\mathbf{i} + 14.3\mathbf{j})$ N; $\mathbf{F}_2 = (-36.4\mathbf{i} + 21.0\mathbf{j})$ N (b) $(-15.9\mathbf{i} + 35.3\mathbf{j})$ N (c) $(-3.18\mathbf{i} + 7.07\mathbf{j})$ m/s^2 (d) $(-5.54\mathbf{i} + 23.7\mathbf{j})$ m/s (e) $(-2.30\mathbf{i} + 39.3\mathbf{j})$ m (f) 1 480 J (g) 1 480 J
69. (a) 4.12 m (b) 3.35 m
71. 1.68 m/s
73. (a) 14.5 m/s (b) 1.75 kg (c) 0.350 kg
75. 0.799 J

Chapter 8

1. (a) 259 kJ, 0, − 259 kJ (b) 0, − 259 kJ, − 259 kJ
3. (a) − 196 J (b) − 196 J (c) − 196 J. The force is conservative.
5. (a) 125 J (b) 50.0 J (c) 66.7 J (d) Nonconservative. The results differ.
7. (a) 40.0 J (b) − 40.0 J (c) 62.5 J
9. (a) $Ax^2/2 - Bx^3/3$ (b) $\Delta U = 5A/2 - 19B/3$; $\Delta K = -5A/2 + 19B/3$
11. 0.344 m
13. (a) $v_B = 5.94$ m/s; $v_C = 7.67$ m/s (b) 147 J
15. $v = (3gR)^{1/2}$, 0.098 0 N down
17. 10.2 m
19. (a) 19.8 m/s (b) 78.4 J (c) 1.00
21. (a) 4.43 m/s (b) 5.00 m
23. (a) 18.5 km, 51.0 km (b) 10.0 MJ
25. (b) 60.0°
27. 5.49 m/s
29. 2.00 m/s, 2.79 m/s, 3.19 m/s
31. 3.74 m/s
33. (a) − 160 J (b) 73.5 J (c) 28.8 N (d) 0.679
35. 489 kJ
37. (a) 1.40 m/s (b) 4.60 cm after release (c) 1.79 m/s
39. 1.96 m
41. (A/r^2) away from the other particle
43. (a) $r = 1.5$ mm, stable; 2.3 mm, unstable; 3.2 mm, stable; $r \to \infty$ neutral (b) $-5.6\text{ J} < E < 1$ J (c) 0.6 mm $< r <$ 3.6 mm (d) 2.6 J (e) 1.5 mm (f) 4 J
45. (a) + at Ⓑ, − at Ⓓ, 0 at Ⓐ, Ⓒ, and Ⓔ (b) Ⓒ stable; Ⓐ and Ⓔ unstable
(c)



47. (b)



Equilibrium at $x = 0$ (c) $v = \sqrt{0.800\text{ J}/m}$
49. (a) 1.50×10^{-10} J (b) 1.07×10^{-9} J (c) 9.15×10^{-10} J
51. 48.2° Note that the answer is independent of the pumpkin's mass and of the radius of the dome.
53. (a) 0.225 J (b) $\Delta E_f = -0.363$ J (c) No; the normal force changes in a complicated way.
55. ~ 10^2 W sustainable power
57. 0.327
59. (a) 23.6 cm (b) 5.90 m/s^2 up the incline; no. (c) Gravitational potential energy turns into kinetic energy plus elastic potential energy and then entirely into elastic potential energy.
61. 1.25 m/s

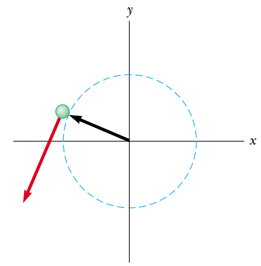
63. (a) 0.400 m (b) 4.10 m/s (c) The block stays on the track.
65. (b) 2.06 m/s
67. (b) 1.44 m (c) 0.400 m (d) No. A very strong wind pulls the string out horizontally (parallel to the ground). The largest possible equilibrium height is equal to L.
71. (a) 6.15 m/s (b) 9.87 m/s
73. 0.923 m/s

Chapter 9

1. (a) $(9.00\mathbf{i} - 12.0\mathbf{j})$ kg·m/s (b) 15.0 kg·m/s at 307°
3. 6.25 cm/s west
5. ~ 10^{-23} m/s
7. (b) $p = \sqrt{2mK}$
9. (a) 13.5 N·s (b) 9.00 kN (c) 18.0 kN
11. 260 N normal to the wall
13. 15.0 N in the direction of the initial velocity of the exiting water stream
15. 65.2 m/s
17. 301 m/s
19. (a) $v_{gx} = 1.15$ m/s (b) $v_{px} = -0.346$ m/s
21. (a) 20.9 m/s east (b) 8.68 kJ into internal energy
23. (a) 2.50 m/s (b) 37.5 kJ (c) Each process is the time-reversal of the other. The same momentum conservation equation describes both.
25. (a) 0.284 (b) 115 fJ and 45.4 fJ
27. 91.2 m/s
29. (a) 2.88 m/s at 32.3° north of east (b) 783 J into internal energy
31. No; his speed was 41.5 mi/h.
33. 2.50 m/s at − 60.0°
35. $(3.00\mathbf{i} - 1.20\mathbf{j})$ m/s
37. Orange: $v_i \cos\theta$; yellow: $v_i \sin\theta$
39. (a) $(-9.33\mathbf{i} - 8.33\mathbf{j})$ Mm/s (b) 439 fJ
41. $\mathbf{r}_{CM} = (11.7\mathbf{i} + 13.3\mathbf{j})$ cm
43. 0.006 73 nm from the oxygen nucleus along the bisector of the angle
45. (a) 15.9 g (b) 0.153 m
47. 0.700 m
49. (a) $(1.40\mathbf{i} + 2.40\mathbf{j})$ m/s (b) $(7.00\mathbf{i} + 12.0\mathbf{j})$ kg·m/s
51. (a) 39.0 MN up (b) 3.20 m/s^2 up
53. (a) 442 metric tons (b) 19.2 metric tons
55. (a) $(1.33\mathbf{i})$ m/s (b) $(-235\mathbf{i})$ N (c) 0.680 s (d) $(-160\mathbf{i})$ N·s and $(+160\mathbf{i})$ N·s (e) 1.81 m (f) 0.454 m (g) − 427 J (h) + 107 J (i) Equal friction forces act through different distances on person and cart to do different amounts of work on them. The total work on both together, − 320 J, becomes + 320 J of internal energy in this perfectly inelastic collision.
57. 1.39 km/s
59. 240 s
61. 0.980 m
63. (a) 6.81 m/s (b) 1.00 m
65. $(3Mgx/L)\mathbf{j}$
67. (a) 3.75 kg·m/s^2 (b) 3.75 N (c) 3.75 N (d) 2.81 J (e) 1.41 J (f) Friction between sand and belt converts half of the input work into internal energy.
69. (a) As the child walks to the right, the boat moves to the left and the center of mass remains fixed. (b) 5.55 m from the pier (c) No, since 6.55 m is less than 7.00 m.
71. (a) 100 m/s (b) 374 J
73. (a) $\sqrt{2}\,v_i$ for m and $\sqrt{2/3}\,v_i$ for $3m$ (b) 35.3°
75. (a) 3.73 km/s (b) 153 km

Chapter 10

1. (a) 4.00 rad/s^2 (b) 18.0 rad
3. (a) 1 200 rad/s (b) 25.0 s
5. (a) 5.24 s (b) 27.4 rad
7. (a) 5.00 rad, 10.0 rad/s, 4.00 rad/s^2 (b) 53.0 rad, 22.0 rad/s, 4.00 rad/s^2
9. 13.7 rad/s^2
11. ~ 10^7 rev/y
13. (a) 0.180 rad/s (b) 8.10 m/s^2 toward the center of the track
15. (a) 8.00 rad/s (b) 8.00 m/s, $a_r = -64.0$ m/s^2, $a_t = 4.00$ m/s^2 (c) 9.00 rad
17. (a) 54.3 rev (b) 12.1 rev/s
19. (a) 126 rad/s (b) 3.78 m/s (c) 1.27 km/s^2 (d) 20.2 m
21. (a) $-2.73\mathbf{i}$ m $+ 1.24\mathbf{j}$ m (b) second quadrant, 156° (c) $-1.85\mathbf{i}$ m/s $- 4.10\mathbf{j}$ m/s (d) into the third quadrant at 246°



(e) $6.15\mathbf{i}$ m/s^2 $- 2.78\mathbf{j}$ m/s^2
(f) $24.6\mathbf{i}$ N $- 11.1\mathbf{j}$ N
23. (a) 92.0 kg·m^2, 184 J (b) 6.00 m/s, 4.00 m/s, 8.00 m/s, 184 J
25. (a) 143 kg·m^2 (b) 2.57 kJ
29. 1.28 kg·m^2
31. ~ 10^0 = 1 kg·m^2
33. − 3.55 N·m
35. 882 N·m
37. (a) 24.0 N·m (b) 0.035 6 rad/s^2 (c) 1.07 m/s^2
39. (a) 0.309 m/s^2 (b) 7.67 N and 9.22 N
41. (a) 872 N (b) 1.40 kN

43. 2.36 m/s
45. (a) 11.4 N, 7.57 m/s^2, 9.53 m/s down (b) 9.53 m/s
49. (a) $2(Rg/3)^{1/2}$ (b) $4(Rg/3)^{1/2}$ (c) $(Rg)^{1/2}$
51. $\frac{1}{3}\ell$
53. (a) 1.03 s (b) 10.3 rev
55. (a) 4.00 J (b) 1.60 s (c) yes
57. (a) 12.5 rad/s (b) 128 rad
59. (a) $(3g/L)^{1/2}$ (b) $3g/2L$ (c) $-\frac{3}{2}g\mathbf{i} - \frac{3}{4}g\mathbf{j}$
(d) $-\frac{3}{2}Mg\mathbf{i} + \frac{1}{4}Mg\mathbf{j}$
61. $\alpha = g(h_2 - h_1)/2\pi R^2$
63. (b) $2gM(\sin\theta - \mu\cos\theta)(m + 2M)^{-1}$
65. 139 m/s
67. 5.80 $kg\cdot m^2$; the height makes no difference.
69. (a) 2 160 N·m (b) 439 W
71. (a) 118 N and 156 N (b) 1.19 $kg\cdot m^2$
73. (a) $\alpha = -0.176$ rad/s^2 (b) 1.29 rev (c) 9.26 rev

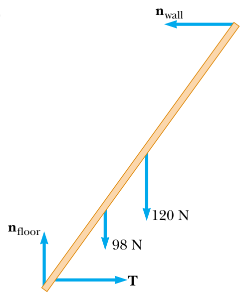
Chapter 11

1. (a) 500 J (b) 250 J (c) 750 J
3. $\frac{7}{10}Mv^2$
5. (a) $\frac{2}{3}g\sin\theta$ for the disk, larger than $\frac{1}{2}g\sin\theta$ for the hoop
(b) $\frac{1}{3}\tan\theta$
7. 1.21×10^{-4} $kg\cdot m^2$. The height is unnecessary.
9. $-7.00\mathbf{i} + 16.0\mathbf{j} - 10.0\mathbf{k}$
11. (a) $-17.0\mathbf{k}$ (b) 70.5°
13. (a) 2.00 N·m (b) $\mathbf{k}$
15. (a) negative z direction (b) positive z direction
17. 45.0°
19. $(17.5\mathbf{k})$ $kg\cdot m^2/s$
21. $(60.0\mathbf{k})$ $kg\cdot m^2/s$
23. $mvR[\cos(vt/R) + 1]\mathbf{k}$
25. (a) zero (b) $(-mv_i^3\sin^2\theta\cos\theta/2g)\mathbf{k}$
(c) $(-2mv_i^3\sin^2\theta\cos\theta/g)\mathbf{k}$ (d) The downward force of gravity exerts a torque in the $-z$ direction.
27. $-m\ell\, gt\cos\theta\,\mathbf{k}$
29. 4.50 $kg\cdot m^2/s$ up
31. (a) 0.433 $kg\cdot m^2/s$ (b) 1.73 $kg\cdot m^2/s$
33. (a) $\omega_f = \omega_i I_1/(I_1 + I_2)$ (b) $I_1/(I_1 + I_2)$
35. (a) 1.91 rad/s (b) 2.53 J, 6.44 J
37. (a) 0.360 rad/s counterclockwise (b) 99.9 J
39. (a) $mv\ell$ down (b) $M/(M + m)$
41. (a) $\omega = 2mv_i d/(M + 2m)R^2$ (b) No; some mechanical energy changes into internal energy.
43. (a) 2.19×10^6 m/s (b) 2.18×10^{-18} J
(c) 4.13×10^{16} rad/s
45. $[10Rg(1 - \cos\theta)/7r^2]^{1/2}$
51. (a) $2.70R$ (b) $F_x = -\frac{20}{7}mg$, $F_y = -mg$
53. 0.632
55. (a) $v_i r_i/r$ (b) $T = (mv_i^2 r_i^2)r^{-3}$ (c) $\frac{1}{2}mv_i^2(r_i^2/r^2 - 1)$
(d) 4.50 m/s, 10.1 N, 0.450 J
57. 54.0°
59. (a) 3 750 $kg\cdot m^2/s$ (b) 1.88 kJ (c) 3 750 $kg\cdot m^2/s$
(d) 10.0 m/s (e) 7.50 kJ (f) 5.62 kJ
61. $(M/m)[3ga(\sqrt{2} - 1)]^{1/2}$
63. (c) $(8Fd/3M)^{1/2}$

67. (a) 0.800 m/s^2, 0.400 m/s^2 (b) 0.600 N backward on the plank and forward on the roller, at the top of each roller; 0.200 N forward on each roller and backward on the floor, at the bottom of each roller.

Chapter 12

1. 10.0 N up; 6.00 N·m counterclockwise
3. $[(m_1 + m_b)d + m_1\ell/2]/m_2$
5. -0.429 m
7. (3.85 cm, 6.85 cm)
9. (-1.50 m, -1.50 m)
11. (a) 859 N (b) 1 040 N left and upward at 36.9°
13. (a) $f_s = 268$ N, $n = 1\,300$ N (b) 0.324
15. (a) 1.04 kN at 60.0° (b) $(370\mathbf{i} + 900\mathbf{j})$ N
17. 2.94 kN on each rear wheel and 4.41 kN on each front wheel
19. (a) 29.9 N (b) 22.2 N
21. (a) 35.5 kN (b) 11.5 kN (c) -4.19 kN
23. 88.2 N and 58.8 N
25. 4.90 mm
27. 0.023 8 mm
29. 0.912 mm
31. $\dfrac{8m_1 m_2 g L_i}{\pi d^2 Y(m_1 + m_2)}$
33. (a) 3.14×10^4 N (b) 6.28×10^4 N
35. 1.80×10^8 N/m^2
37. $n_A = 5.98 \times 10^5$ N, $n_B = 4.80 \times 10^5$ N
39. (a) 0.400 mm (b) 40.0 kN (c) 2.00 mm (d) 2.40 mm
(e) 48.0 kN
41. (a)

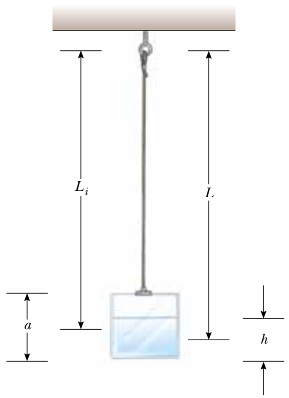


(b) 69.8 N (c) $0.877L$
43. (a) 160 N right (b) 13.2 N right (c) 292 N up
(d) 192 N
45. (a) $T = F_g(L + d)/\sin\theta(2L + d)$
(b) $R_x = F_g(L + d)\cot\theta/(2L + d)$; $R_y = F_g L/(2L + d)$
47. 0.789 L
49. 5.08 kN, $R_x = 4.77$ kN, $R_y = 8.26$ kN
51. $T = 2.71$ kN, $R_x = 2.65$ kN
53. (a) $\mu_k = 0.571$; the normal force acts 20.1 cm to the left of the front edge of the sliding cabinet. (b) 0.501 m

55. (b) 60.0°
57. (a) $M = (m/2)(2\mu_s\sin\theta - \cos\theta)(\cos\theta - \mu_s\sin\theta)^{-1}$
(b) $R = (m + M)g(1 + \mu_s^2)^{1/2}$,
$F = g[M^2 + \mu_s^2(m + M)^2]^{1/2}$
59. (a) 133 N (b) $n_A = 429$ N and $n_B = 257$ N
(c) $R_x = 133$ N and $R_y = -257$ N
61. 66.7 N
65. 1.09 m
67. (a) 4 500 N (b) 4.50×10^6 N/m^2 (c) yes.
69. (a) $P_y = (F_g/L)(d - ah/g)$ (b) 0.306 m
(c) $\mathbf{P} = (-306\mathbf{i} + 553\mathbf{j})$ N
71. $n_A = n_E = 6.66$ kN; $F_{AB} = 10.4$ kN $= F_{BC} = F_{DC} = F_{DE}$; $F_{AC} = 7.94$ kN $= F_{CE}$; $F_{BD} = 15.9$ kN

Chapter 13

1. (a) 1.50 Hz, 0.667 s (b) 4.00 m (c) π rad (d) 2.83 m
3. (a) 20.0 cm (b) 94.2 cm/s as the particle passes through equilibrium (c) 17.8 m/s^2 at the maximum displacement from equilibrium
5. (b) 18.8 cm/s, 0.333 s (c) 178 cm/s^2, 0.500 s
(d) 12.0 cm
7. 0.627 s
9. (a) 40.0 cm/s, 160 cm/s^2 (b) 32.0 cm/s, -96.0 cm/s^2
(c) 0.232 s
11. 40.9 N/m
13. (a) 0.750 m (b) $x = -(0.750\text{ m})\sin(2.00t/\text{s})$
15. 0.628 m/s
17. 2.23 m/s
19. (a) 28.0 mJ (b) 1.02 m/s (c) 12.2 mJ (d) 15.8 mJ
21. (a) 2.61 m/s (b) 2.38 m/s
23. 2.60 cm and -2.60 cm
25. (a) 35.7 m (b) 29.1 s

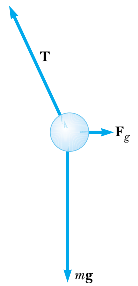


Chapter 13, Problem 57(a)

27. $\sim 10^0$ s
29. (a) 0.817 m/s (b) 2.54 rad/s^2 (c) 0.634 N
33. 0.944 $kg\cdot m^2$
37. (a) 5.00×10^{-7} $kg\cdot m^2$ (b) 3.16×10^{-4} N·m/rad
39. The x coordinate of the crank pin is $A\cos\omega t$.
41. 1.00×10^{-3} s^{-1}
43. (a) 2.95 Hz (b) 2.85 cm
47. Either 1.31 Hz or 0.641 Hz
49. 6.58 kN/m
51. (a) $2Mg$; $Mg(1 + y/L)$ (b) $T = (4\pi/3)(2L/g)^{1/2}$; 2.68 s
53. 6.62 cm
55. 9.19×10^{13} Hz
57. (a) See bottom of preceding column.
(b) $\dfrac{dT}{dt} = \dfrac{\pi(dM/dt)}{2\rho a^2 g^{1/2}[L_i + (dM/dt)t/2\rho a^2]^{1/2}}$
(c) $T = 2\pi g^{-1/2}[L_i + (dM/dt)t/2\rho a^2]^{1/2}$
59. $f = (2\pi L)^{-1}(gL + kh^2/M)^{1/2}$
61. (a) 3.56 Hz (b) 2.79 Hz (c) 2.10 Hz
63. (a) 3.00 s (b) 14.3 J (c) 25.5°
65. 0.224 rad/s

Chapter 14

1. $\sim 10^{-7}$ N toward you
3. $\mathbf{g} = (Gm/\ell^2)(\frac{1}{2} + \sqrt{2})$ toward the opposite corner
5. $(-100\mathbf{i} + 59.3\mathbf{j})$ pN
7. (a) 4.39×10^{20} N (b) 1.99×10^{20} N (c) 3.55×10^{22} N
9. 0.613 m/s^2 toward the Earth
11.

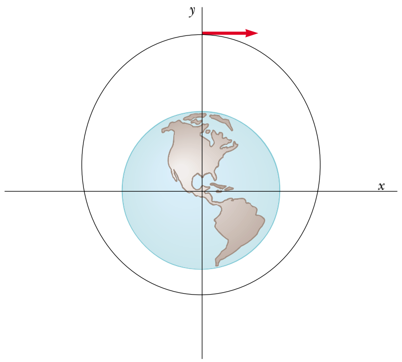


Either (1.000 m $-$ 61.3 nm) or, if the objects have very high density, 247 mm.
15. 12.6×10^{31} kg
17. 1.27
19. 1.90×10^{27} kg
21. 8.92×10^7 m
25. $g = 2MGr(r^2 + a^2)^{-3/2}$ toward the center of mass
27. (a) -4.77×10^9 J (b) 569 N (c) 569 N up
29. (a) 1.84×10^9 kg/m^3 (b) 3.27×10^6 m/s^2
(c) -2.08×10^{13} J
31. (a) -1.67×10^{-14} J (b) At the center
33. 1.58×10^{10} J
35. (a) 1.48 h (b) 7.79 km/s (c) 6.43×10^9 J

37. 1.66×10^4 m/s
41. 15.6 km/s
43. $GM_E m/12R_E$
45. $2GmM/\pi R^2$ straight up in the picture
47. (a) 7.41×10^{-10} N (b) 1.04×10^{-8} N (c) 5.21×10^{-9} N
49. 2.26×10^{-7}
51. (b) 1.10×10^{32} kg
53. (b) $GMm/2R$
55. 7.79×10^{14} kg
57. 7.41×10^{-10} N
59. $v_{esc} = (8\pi G\rho/3)^{1/2} R$
61. (a) $v_1 = m_2(2G/d)^{1/2}(m_1 + m_2)^{-1/2}$
$v_2 = m_1(2G/d)^{1/2}(m_1 + m_2)^{-1/2}$
$v_{rel} = (2G/d)^{1/2}(m_1 + m_2)^{1/2}$
(b) $K_1 = 1.07 \times 10^{32}$ J, $K_2 = 2.67 \times 10^{31}$ J
63. (a) $A = M/\pi R^4$ (b) $F = GmM/r^2$ toward the center (c) $F = GmMr^2/R^4$ toward the center
65. 119 km
67. (a) −36.7 MJ (b) 9.24×10^{10} kg·m²/s (c) 5.58 km/s, 10.4 Mm (d) 8.69 Mm (e) 134 min
71.

t (s)	x (m)	y (m)	v_x (m/s)	v_y (m/s)
0	0	12 740 000	5 000	0
10	50 000	12 740 000	4 999.9	−24.6
20	99 999	12 739 754	4 999.7	−49.1
30	149 996	12 739 263	4 999.4	−73.7 . . .

The object does not hit the Earth; its minimum radius is $1.33R_E$. Its period is 1.09×10^4 s. A circular orbit would require speed 5.60 km/s.

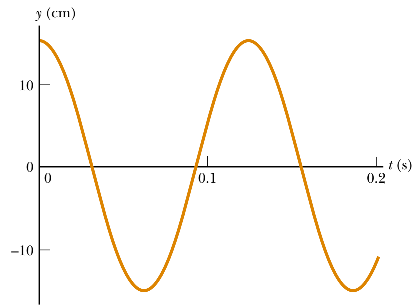


Chapter 15

1. 0.111 kg
3. 6.24 MPa
5. 5.27×10^{18} kg
7. 1.62 m
9. 7.74×10^{-3} m²
11. 271 kN horizontally backward
13. $P_0 + (\rho d/2)(g^2 + a^2)^{1/2}$
15. 0.722 mm
17. 10.5 m; no, some alcohol and water evaporate.
19. 12.6 cm
21. 1.07 m²
23. (a) 9.80 N (b) 6.17 N
25. (a) 7.00 cm (b) 2.80 kg
27. $\rho_{oil} = 1\,250$ kg/m³; $\rho_{sphere} = 500$ kg/m³
29. 1 430 m³
31. 2.67×10^3 kg
33. (a) 1.06 m/s (b) 4.24 m/s
35. (a) 17.7 m/s (b) 1.73 mm
37. 31.6 m/s
39. 68.0 kPa
41. 103 m/s
43. (a) 4.43 m/s (b) The siphon can be no higher than 10.3 m.
45. $2\sqrt{h(h_0 - h)}$
47. 0.258 N
49. 1.91 m
53. 709 kg/m³
55. top scale 17.3 N; bottom scale 31.7 N
59. 90.04%
61. 4.43 m/s
63. (a) 10.3 m (b) 0
65. (a) 18.3 mm (b) 14.3 mm (c) 8.56 mm
67. (a) 2.65 m/s (b) 2.31×10^4 Pa
69. (a) 1.25 cm (b) 13.8 m/s

Chapter 16

1. $y = 6\,[(x - 4.5t)^2 + 3]^{-1}$
3. (a) left (b) 5.00 m/s
5. (a) longitudinal (b) 665 s
7. (a) 156° (b) 0.058 4 cm
9. (a) y_1 in $+x$ direction, y_2 in $-x$ direction (b) 0.750 s (c) 1.00 m
11. 30.0 N
13. 1.64 m/s²
15. 13.5 N
17. 586 m/s
19. 32.9 ms
21. 0.329 s
23. (a) See top of next page (b) 0.125 s
25. 0.319 m
27. 2.40 m/s
29. (a) 0.250 m (b) 40.0 rad/s (c) 0.300 rad/m (d) 20.9 m (e) 133 m/s (f) $+x$
31. (a) $y = (8.00 \text{ cm}) \sin(7.85x + 6\pi t)$
(b) $y = (8.00 \text{ cm}) \sin(7.85x + 6\pi t - 0.785)$
33. (a) 0.500 Hz, 3.14 rad/s (b) 3.14 rad/m (c) $(0.100 \text{ m}) \sin(3.14x/\text{m} - 3.14t/\text{s})$



Chapter 16, Problem 23(a)

(d) $(0.100 \text{ m}) \sin(-3.14t/\text{s})$
(e) $(0.100 \text{ m}) \sin(4.71 \text{ rad} - 3.14t/\text{s})$ (f) 0.314 m/s
35. 2.00 cm, 2.98 m, 0.576 Hz, 1.72 m/s
37. (b) 3.18 Hz
41. 55.1 Hz
43. (a) 62.5 m/s (b) 7.85 m (c) 7.96 Hz (d) 21.1 W
45. (a) $A = 40.0$ (b) $A = 7.00$, $B = 0$, $C = 3.00$. One can take the dot product of the given equation with each one of **i**, **j**, and **k**. (c) By inspection, $A = 0$, $B = 7.00$ mm, $C = 3.00$/m, $D = 4.00$/s, $E = 2.00$. Consider the average value of both sides of the given equation to find A. Then consider the maximum value of both sides to find B. You can evaluate the partial derivative of both sides of the given equation with respect to x and separately with respect to t to obtain equations yielding C and D upon chosen substitutions for x and t. Then substitute $x = 0$ and $t = 0$ to obtain E.
47. It is if $v = (T/\mu)^{1/2}$
49. ~1 min
51. (a) 3.33**i** m/s (b) −5.48 cm (c) 0.667 m, 5.00 Hz (d) 11.0 m/s
53. $(Lm/Mg \sin\theta)^{1/2}$
55. (a) 39.2 N (b) 0.892 m (c) 83.6 m/s
57. 14.7 kg
61. (a) $(0.707)2(L/g)^{1/2}$ (b) $L/4$
63. 3.86×10^{-4}
65. (a) $v = (2T_0/\mu_0)^{1/2} = v_0 2^{1/2}$
$v' = (2T_0/3\mu_0)^{1/2} = v_0\,(2/3)^{1/2}$
(b) $0.966t_0$
67. 130 m/s, 1.73 km

Chapter 17

1. 5.56 km
3. 7.82 m
5. (a) 27.2 s (b) 25.7 s; the interval in (a) is longer
7. (a) 153 m/s (b) 614 m
9. (a) amplitude 2.00 μm, wavelength 40.0 cm, speed 54.6 m/s (b) −0.433 μm (c) 1.72 mm/s
11. $\Delta P = (0.2 \text{ Pa}) \sin(62.8x/\text{m} - 2.16 \times 10^4\, t/\text{s})$
13. (a) 6.52 mm (b) 20.5 m/s
15. 5.81 m
17. 66.0 dB
19. (a) 3.75 W/m² (b) 0.600 W/m²
21. (a) 1.32×10^{-4} W/m² (b) 81.2 dB
23. 65.6 dB
25. (a) 65.0 dB (b) 67.8 dB (c) 69.6 dB
27. 1.13 μW
29. (a) 30.0 m (b) 9.49×10^5 m
31. (a) 332 J (b) 46.4 dB
33. (a) 75.7-Hz drop (b) 0.948 m
35. 26.4 m/s
37. 19.3 m
39. (a) 338 Hz (b) 483 Hz
41. 56.4°
43. (a) 56.3 s (b) 56.6 km farther along
45. 400 m; 27.5%
47. (a) 23.2 cm (b) 8.41×10^{-8} m (c) 1.38 cm
49. (a) 0.515/min (b) 0.614/min
51. 7.94 km
53. (a) 55.8 m/s (b) 2 500 Hz
55. Bat is gaining on the insect at the rate of 1.69 m/s.
57. (a)

(b) 0.343 m (c) 0.303 m (d) 0.383 m (e) 1.03 kHz
59. (a) 0.691 m (b) 691 km
61. 1204.2 Hz
63. (a) 0.948° (b) 4.40°
65. 1.34×10^4 N
67. 95.5 s
69. (b) 531 Hz
71. (a) 6.45 (b) 0
73. ~10^{11} Hz

Chapter 18

1. (a) 9.24 m (b) 600 Hz
3. 5.66 cm
5. 91.3°
7. (a) 2 (b) 9.28 m and 1.99 m
9. 15.7 m, 31.8 Hz, 500 m/s
11. At 0.089 1 m, 0.303 m, 0.518 m, 0.732 m, 0.947 m, and 1.16 m from one speaker
13. (a) 4.24 cm (b) 6.00 cm (c) 6.00 cm (d) 0.500 cm, 1.50 cm, and 2.50 cm
17. 0.786 Hz, 1.57 Hz, 2.36 Hz, and 3.14 Hz
19. (a) 163 N (b) 660 Hz
21. 19.976 kHz

23. 31.2 cm from the bridge; 3.84%
25. (a) 350 Hz (b) 400 kg
27. 0.352 Hz
29. (a) 3.66 m/s (b) 0.200 Hz
31. (a) 0.357 m (b) 0.715 m
33. (a) 531 Hz (b) 42.5 mm
35. around 3 kHz
37. n(206 Hz) for $n = 1$ to 9, and n(84.5 Hz) for $n = 2$ to 23
39. 239 s
41. 0.502 m and 0.837 m
43. (a) 350 m/s (b) 1.14 m
45. (a) 19.5 cm (b) 841 Hz
47. (a) 1.59 kHz (b) odd-numbered harmonics (c) 1.11 kHz
49. 5.64 beats/s
51. (a) 1.99 beats/s (b) 3.38 m/s
53. The second harmonic of E is close to the third harmonic of A, and the fourth harmonic of C# is close to the fifth harmonic of A.
55. (a) 3.33 rad (b) 283 Hz
57. 3.85 m/s away from the station or 3.77 m/s toward the station
59. 85.7 Hz
61. 31.1 N
63. (a) 59.9 Hz (b) 20.0 cm
65. (a) 1/2 (b) $[n/(n+1)]^2\,T$ (c) 9/16
67. 50.0 Hz, 1.70 m
69. (a) $2A\sin(2\pi x/\lambda)\cos(2\pi vt/\lambda)$
(b) $2A\sin(\pi x/L)\cos(\pi vt/L)$
(c) $2A\sin(2\pi x/L)\cos(2\pi vt/L)$
(d) $2A\sin(n\pi x/L)\cos(n\pi vt/L)$

Chapter 19

1. (a) 37.0°C = 310 K (b) − 20.6°C = 253 K
3. (a) − 274°C (b) 1.27 atm (c) 1.74 atm
5. (a) − 320°F (b) 77.3 K
7. (a) 810°F (b) 450 K
9. 3.27 cm
11. (a) 3.005 8 m (b) 2.998 6 m
13. 55.0°C
15. (a) 0.109 cm^2 (b) increase
17. (a) 0.176 mm (b) 8.78 μm (c) 0.093 0 cm^3
19. (a) 2.52 MN/m^2 (b) It will not break.
21. 1.14°C
23. (a) 99.4 cm^3 (b) 0.943 cm
25. (a) 3.00 mol (b) 1.80×10^{24} molecules
27. 1.50×10^{29} molecules
29. 472 K
31. (a) 41.6 mol (b) 1.20 kg, in agreement with the tabulated density
33. (a) 400 kPa (b) 449 kPa
35. 2.27 kg
37. 3.67 cm^3
39. 4.39 kg
43. (a) 94.97 cm (b) 95.03 cm
45. 208°C
47. 3.55 cm
49. (a) Expansion makes density drop. (b) $5 \times 10^{-5}\ (°\text{C})^{-1}$
51. (a) $h = nRT/(mg + P_0A)$ (b) 0.661 m
53. $\alpha\,\Delta T$ is much less than 1.
55. (a) 9.49×10^{-5} s (b) 57.4 s lost
57. (a) $\rho g P_0 V_i (P_0 + \rho g d)^{-1}$ (b) decrease (c) 10.3 m
61. (a) 5.00 MPa (b) 9.58×10^{-3}
63. 2.74 m
65. $L_c = 9.17$ cm, $L_s = 14.2$ cm
67. (a) $L_f = L_i e^{\alpha \Delta T}$ (b) 2.00×10^{-4}%; 59.4%
69. (a) 6.17×10^{-3} kg/m (b) 632 N (c) 580 N; 192 Hz

Chapter 20

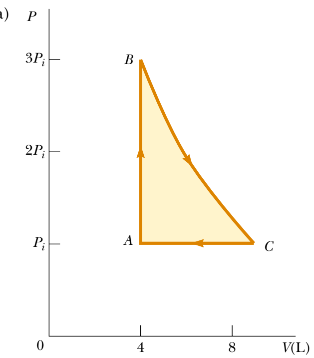
1. (10.0 + 0.117)°C
3. 0.234 kJ/kg · °C
5. 29.6°C
7. (a) 0.435 cal/g · °C (b) beryllium
9. (a) 25.8°C (b) No
11. 50.7 ks
13. 0.294 g
15. 0.414 kg
17. (a) 0°C (b) 114 g
19. 59.4°C
21. 1.18 MJ
23. (a) $4P_iV_i$ (b) $T = (P_i/nRV_i)V^2$
25. 466 J
27. 810 J, 506 J, 203 J
29. $Q = -720$ J
31.

	Q	W	ΔE_{int}
BC	−	0	−
CA	−	−	−
AB	+	+	+

33. (a) 7.50 kJ (b) 900 K
35. 3.10 kJ; 37.6 kJ
37. (a) 0.041 0 m^3 (b) − 5.48 kJ (c) − 5.48 kJ
41. 2.40×10^6 cal/s
43. 10.0 kW
45. 51.2°C
47. (a) 0.89 ft^2 · °F · h/Btu (b) 1.85 ft^2 · °F · h/Btu (c) 2.08
49. (a) $\sim 10^3$ W (b) decreasing at $\sim 10^{-1}$ K/s
51. 364 K
53. 47.7 g
55. (a) 16.8 L (b) 0.351 L/s
57. 2.00 kJ/kg · °C
59. 1.87 kJ
61. (a) $4P_iV_i$ (b) $4P_iV_i$ (c) 9.08 kJ
63. 5.31 h
65. 872 g
67. (a) 15.0 mg. Block: $Q = 0$, $W = +5.00$ J, $\Delta E_{int} = 0$, $\Delta K = -5.00$ J; Ice: $Q = 0$, $W = -5.00$ J, $\Delta E_{int} = 5.00$ J, $\Delta K = 0$.
(b) 15.0 mg. Block: $Q = 0$, $W = 0$, $\Delta E_{int} = 5.00$ J, $\Delta K = -5.00$ J; Metal: $Q = 0$, $W = 0$, $\Delta E_{int} = 0$, $\Delta K = 0$.
(c) 0.004 04°C. Moving slab: $Q = 0$, $W = +2.50$ J, $\Delta E_{int} = 2.50$ J, $\Delta K = -5.00$ J; Stationary slab: $Q = 0$, $W = -2.50$ J, $\Delta E_{int} = 2.50$ J, $\Delta K = 0$
69. 10.2 h
71. 9.32 kW

Chapter 21

1. 6.64×10^{-27} kg
3. 0.943 N; 1.57 Pa
5. 17.6 kPa
7. 3.32 mol
9. (a) 3.53×10^{23} atoms (b) 6.07×10^{-21} J (c) 1.35 km/s
11. (a) 8.76×10^{-21} J for both (b) 1.62 km/s for helium; 514 m/s for argon
13. 75.0 J
15. (a) 3.46 kJ (b) 2.45 kJ (c) 1.01 kJ
17. (a) 118 kJ (b) 6.03×10^3 kg
19. Between 10^{-2}°C and 10^{-3}°C
21. (a) 316 K (b) 200 J
23. $9\,P_iV_i$
25. (a) 1.39 atm (b) 366 K, 253 K (c) 0, 4.66 kJ, − 4.66 kJ
27. 227 K
29. (a)



(b) 8.79 L (c) 900 K (d) 300 K (e) 336 J
31. 25.0 kW
33. (a) 9.95 cal/K, 13.9 cal/K (b) 13.9 cal/K, 17.9 cal/K
35. 2.33×10^{-21} J
37. The ratio of oxygen to nitrogen molecules decreases to 85.5% of its sea-level value.
39. (a) 6.80 m/s (b) 7.41 m/s (c) 7.00 m/s
43. 819°C
45. (a) 3.21×10^{12} molecules (b) 778 km (c) 6.42×10^{-4} s^{-1}
49. (a) 9.36×10^{-8} m (b) 9.36×10^{-8} atm (c) 302 atm
51. (a) 100 kPa, 66.5 L, 400 K, 5.82 kJ, 7.48 kJ, 1.66 kJ
(b) 133 kPa, 49.9 L, 400 K, 5.82 kJ, 5.82 kJ, 0
(c) 120 kPa, 41.6 L, 300 K, 0, − 910 J, − 910 J
(d) 120 kPa, 43.3 L, 312 K, 722 J, 0, − 722 J
55. 0.625
57. (a) Pressure increases as volume decreases.
(d) 0.500 atm^{-1}, 0.300 atm^{-1}
59. 1.09×10^{-3}; 2.69×10^{-2}; 0.529; 1.00; 0.199; 1.01×10^{-41}; 1.25×10^{-1082}
61. (a) Larger-mass molecules settle to the outside.
63. (a) 0.203 mol (b) $T_B = T_C = 900$ K; $V_C = 15.0$ L

(c, d)	P (atm)	V (L)	T (K)	E_{int} (kJ)
A	1	5	300	0.760
B	3	5	900	2.28
C	1	15	900	2.28

(e) For $A \to B$, lock the piston in place and put the cylinder into an oven at 900 K. For $B \to C$, keep the gas in the oven while gradually letting the gas expand to lift a load on the piston as far as it can. For $C \to A$, move the cylinder from the oven back to the 300-K room and let the gas cool and contract.

(f, g)	Q (kJ)	W (kJ)	ΔE_{int} (kJ)
$A \to B$	1.52	0	1.52
$B \to C$	1.67	1.67	0
$C \to A$	− 2.53	− 1.01	− 1.52
$ABCA$	0.656	0.656	0

65. (a) 3.34×10^{26} molecules (b) during the 27th day (c) 2.53×10^6
67. (a) 0.510 m/s (b) 20 ms

Chapter 22

1. (a) 6.94% (b) 335 J
3. (a) 10.7 kJ (b) 0.533 s
5. (a) 1.00 kJ (b) 0
7. (a) 67.2% (b) 58.8 kW
9. (a) 869 MJ (b) 330 MJ
11. (a) 741 J (b) 459 J
13. 0.330 or 33.0%
15. (a) 5.12% (b) 5.27 TJ/h (c) As conventional energy sources become more expensive, or as their true costs are recognized, alternative sources become economically viable.
17. (a) 214 J, 64.3 J
(b) − 35.7 J, − 35.7 J. The net effect is the transport of energy from the cold to the hot reservoir without expenditure of external work.
(c) 333 J, 233 J
(d) 83.3 J, 83.3 J, 0. The net effect is the expulsion of the energy entering the system by heat, entirely by work, in a cyclic process.
(e) − 0.111 J/K. The entropy of the Universe has decreased.

19. (a) 244 kPa (b) 192 J
21. 146 kW, 70.8 kW
23. 9.00
27. 72.2 J
29. (a) 24.0 J (b) 144 J
31. − 610 J/K
33. 195 J/K
35. 3.27 J/K
37. 1.02 kJ/K
39. 5.76 J/K. Temperature is constant if the gas is ideal.
41. 0.507 J/K
43. 18.4 J/K
45. (a) 1 (b) 6
47. (a)

Macrostate	Possible Microstates	Total Number of Microstates
All R	RRR	1
2R, 1G	RRG, RGR, GRR	3
1R, 2G	GRR, GRG, RGG	3
All G	GGG	1

(b)

Macrostate	Possible Microstates	Total Number of Microstates
All R	RRRRR	1
4R, 1G	RRRRG, RRRGR, RRGRR, RGRRR, GRRRR	5
3R, 2G	RRRGG, RRGRG, RGRRG, GRRRG, RRGGR, RGRGR, GRRGR, RGGRR, GRGRR, GGRRR	10
2R, 3G	GGGRR, GGRGR, GRGGR, RGGGR, GGRRG, GRGRG, RGGRG, GRRGG, RGRGG, RRGGG	10
1R, 4G	GGGGR, GGGRG, GGRGG, GRGGG, RGGGG	5
All G	GGGGG	1

49. 1.86
51. (a) 5.00 kW (b) 763 W
53. (a) $2nRT_i \ln 2$ (b) 0.273
55. 23.1 mW
57. 5.97×10^4 kg/s
59. (a) 3.19 cal/K (b) 98.19°F, 2.59 cal/K
61. 1.18 J/K
63. (a) $10.5nRT_i$ (b) $8.50nRT_i$ (c) 0.190 (d) 0.833
65. $nC_P \ln 3$
69. (a) 96.9 W = 8.33×10^4 cal/hr
(b) 1.19°C/h = 2.14°F/h

APPENDIX A • Tables

TABLE A.1 Conversion Factors

Length

	m	cm	km	in.	ft	mi
1 meter	1	10^2	10^{-3}	39.37	3.281	6.214×10^{-4}
1 centimeter	10^{-2}	1	10^{-5}	0.393 7	3.281×10^{-2}	6.214×10^{-6}
1 kilometer	10^3	10^5	1	3.937×10^4	3.281×10^3	0.621 4
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	0.304 8	30.48	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile	1 609	1.609×10^5	1.609	6.336×10^4	5 280	1

Mass

	kg	g	slug	u
1 kilogram	1	10^3	6.852×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^4	1	8.789×10^{27}
1 atomic mass unit	1.660×10^{-27}	1.660×10^{-24}	1.137×10^{-28}	1

Note: 1 metric ton = 1 000 kg.

Time

	s	min	h	day	yr
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{-2}	6.994×10^{-4}	1.901×10^{-6}
1 hour	3 600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^4	1 440	24	1	2.738×10^{-5}
1 year	3.156×10^7	5.259×10^5	8.766×10^3	365.2	1

Speed

	m/s	cm/s	ft/s	mi/h
1 meter per second	1	10^2	3.281	2.237
1 centimeter per second	10^{-2}	1	3.281×10^{-2}	2.237×10^{-2}
1 foot per second	0.304 8	30.48	1	0.681 8
1 mile per hour	0.447 0	44.70	1.467	1

Note: 1 mi/min = 60 mi/h = 88 ft/s.

continued

TABLE A.1 Continued

Force

	N	lb
1 newton	1	0.224 8
1 pound	4.448	1

Work, Energy, Heat

	J	ft·lb	eV
1 joule	1	0.737 6	6.242×10^{18}
1 ft·lb	1.356	1	8.464×10^{18}
1 eV	1.602×10^{-19}	1.182×10^{-19}	1
1 cal	4.186	3.087	2.613×10^{19}
1 Btu	1.055×10^{3}	7.779×10^{2}	6.585×10^{21}
1 kWh	3.600×10^{6}	2.655×10^{6}	2.247×10^{25}

	cal	Btu	kWh
1 joule	0.238 9	9.481×10^{-4}	2.778×10^{-7}
1 ft·lb	0.323 9	1.285×10^{-3}	3.766×10^{-7}
1 eV	3.827×10^{-20}	1.519×10^{-22}	4.450×10^{-26}
1 cal	1	3.968×10^{-3}	1.163×10^{-6}
1 Btu	2.520×10^{2}	1	2.930×10^{-4}
1 kWh	8.601×10^{5}	3.413×10^{2}	1

Pressure

	Pa	atm
1 pascal	1	9.869×10^{-6}
1 atmosphere	1.013×10^{5}	1
1 centimeter mercury[a]	1.333×10^{3}	1.316×10^{-2}
1 pound per inch^2	6.895×10^{3}	6.805×10^{-2}
1 pound per foot^2	47.88	4.725×10^{-4}

	cm Hg	lb/in.^{2}	lb/ft^2
1 newton per meter^2	7.501×10^{-4}	1.450×10^{-4}	2.089×10^{-2}
1 atmosphere	76	14.70	2.116×10^{3}
1 centimeter mercury[a]	1	0.194 3	27.85
1 pound per inch^2	5.171	1	144
1 pound per foot^2	3.591×10^{-2}	6.944×10^{-3}	1

[a] At 0°C and at a location where the acceleration due to gravity has its "standard" value, 9.806 65 m/s^2.

TABLE A.2 Symbols, Dimensions, and Units of Physical Quantities

Quantity	Common Symbol	Unit[a]	Dimensions[b]	Unit in Terms of Base SI Units
Acceleration	**a**	m/s^2	L/T^2	m/s^2
Amount of substance	n	mole		mol
Angle	θ, ϕ	radian (rad)	1	
Angular acceleration	$\boldsymbol{\alpha}$	rad/s^2	T^{-2}	s^{-2}
Angular frequency	ω	rad/s	T^{-1}	s^{-1}
Angular momentum	**L**	kg·m^2/s	ML^2/T	kg·m^2/s
Angular velocity	$\boldsymbol{\omega}$	rad/s	T^{-1}	s^{-1}
Area	A	m^2	L^2	m^2
Atomic number	Z			
Capacitance	C	farad (F)	Q^{2}T^2/ML^2	A^2·s^4/kg·m^2
Charge	q, Q, e	coulomb (C)	Q	A·s
Charge density				
Line	λ	C/m	Q/L	A·s/m
Surface	σ	C/m^2	Q/L^2	A·s/m^2
Volume	ρ	C/m^3	Q/L^3	A·s/m^3
Conductivity	σ	1/Ω·m	Q^{2}T/ML^3	A^2·s^3/kg·m^3
Current	I	AMPERE	Q/T	A
Current density	**J**	A/m^2	Q/T^2	A/m^2
Density	ρ	kg/m^3	M/L^3	kg/m^3
Dielectric constant	κ			
Displacement	**r**, **s**	METER	L	m
Distance	d, h			
Length	ℓ, L			
Electric dipole moment	**p**	C·m	QL	A·s·m
Electric field	**E**	V/m	ML/QT^2	kg·m/A·s^3
Electric flux	Φ_E	V·m	ML^3/QT^2	kg·m^3/A·s^3
Electromotive force	$\mathcal{E}$	volt (V)	ML^2/QT^2	kg·m^2/A·s^3
Energy	E, U, K	joule (J)	ML^2/T^2	kg·m^2/s^2
Entropy	S	J/K	ML^2/T^2·K	kg·m^2/s^2·K
Force	**F**	newton (N)	ML/T^2	kg·m/s^2
Frequency	f	hertz (Hz)	T^{-1}	s^{-1}
Heat	Q	joule (J)	ML^2/T^2	kg·m^2/s^2
Inductance	L	henry (H)	ML^2/Q^2	kg·m^2/A^2·s^2
Magnetic dipole moment	$\boldsymbol{\mu}$	N·m/T	QL^2/T	A·m^2
Magnetic field	**B**	tesla (T) (= Wb/m^2)	M/QT	kg/A·s^2
Magnetic flux	Φ_B	weber (Wb)	ML^2/QT	kg·m^2/A·s^2
Mass	m, M	KILOGRAM	M	kg
Molar specific heat	C	J/mol·K		kg·m^2/s^2·mol·K
Moment of inertia	I	kg·m^2	ML^2	kg·m^2
Momentum	**p**	kg·m/s	ML/T	kg·m/s
Period	T	s	T	s
Permeability of space	μ_0	N/A^2 (= H/m)	ML/Q^{2}T	kg·m/A^2·s^2
Permittivity of space	ϵ_0	C^2/N·m^2 (= F/m)	Q^{2}T^2/ML^3	A^2·s^4/kg·m^3
Potential	V	volt (V) (= J/C)	ML^2/QT^2	kg·m^2/A·s^3
Power	$\mathcal{P}$	watt (W) (= J/s)	ML^2/T^3	kg·m^2/s^3

continued

TABLE A.2 Continued

Quantity	Common Symbol	Unit[a]	Dimensions[b]	Unit in Terms of Base SI Units
Pressure	P	pascal (Pa) = (N/m^2)	M/LT^2	kg/m·s^2
Resistance	R	ohm (Ω)(=V/A)	ML^2/Q^{2}T	kg·m^2/A^2·s^3
Specific heat	c	J/kg·K	L^2/T^2·K	m^2/s^2·K
Speed	v	m/s	L/T	m/s
Temperature	T	KELVIN	K	K
Time	t	SECOND	T	s
Torque	$\boldsymbol{\tau}$	N·m	ML^2/T^2	kg·m^2/s^2
Volume	V	m^3	L^3	m^3
Wavelength	λ	m	L	m
Work	W	joule (J)(=N·m)	ML^2/T^2	kg·m^2/s^2

[a] The base SI units are given in uppercase letters.

[b] The symbols M, L, T, and Q denote mass, length, time, and charge, respectively.

TABLE A.3 Table of Atomic Masses[a]

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
0	(Neutron)	n		1*	1.008 665		10.4 min
1	Hydrogen	H	1.007 9	1	1.007 825	99.985	
	Deuterium	D		2	2.014 102	0.015	
	Tritium	T		3*	3.016 049		12.33 yr
2	Helium	He	4.002 60	3	3.016 029	0.000 14	
				4	4.002 602	99.999 86	
				6*	6.018 886		0.81 s
3	Lithium	Li	6.941	6	6.015 121	7.5	
				7	7.016 003	92.5	
				8*	8.022 486		0.84 s
4	Beryllium	Be	9.012 2	7*	7.016 928		53.3 days
				9	9.012 174	100	
				10*	10.013 534		1.5×10^6 yr
5	Boron	B	10.81	10	10.012 936	19.9	
				11	11.009 305	80.1	
				12*	12.014 352		0.020 2 s
6	Carbon	C	12.011	10*	10.016 854		19.3 s
				11*	11.011 433		20.4 min
				12	12.000 000	98.90	
				13	13.003 355	1.10	
				14*	14.003 242		5 730 yr
				15*	15.010 599		2.45 s
7	Nitrogen	N	14.006 7	12*	12.018 613		0.011 0 s
				13*	13.005 738		9.96 min
				14	14.003 074	99.63	
				15	15.000 108	0.37	
				16*	16.006 100		7.13 s
				17*	17.008 450		4.17 s

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
8	Oxygen	O	15.999 4	14*	14.008 595		70.6 s
				15*	15.003 065		122 s
				16	15.994 915	99.761	
				17	16.999 132	0.039	
				18	17.999 160	0.20	
				19*	19.003 577		26.9 s
9	Fluorine	F	18.998 40	17*	17.002 094		64.5 s
				18*	18.000 937		109.8 min
				19	18.998 404	100	
				20*	19.999 982		11.0 s
				21*	20.999.950		4.2 s
10	Neon	Ne	20.180	18*	18.005 710		1.67 s
				19*	19.001 880		17.2 s
				20	19.992 435	90.48	
				21	20.993 841	0.27	
				22	21.991 383	9.25	
				23*	22.994 465		37.2 s
11	Sodium	Na	22.989 87	21*	20.997 650		22.5 s
				22*	21.994 434		2.61 yr
				23	22.989 770	100	
				24*	23.990 961		14.96 h
12	Magnesium	Mg	24.305	23*	22.994 124		11.3 s
				24	23.985 042	78.99	
				25	24.985 838	10.00	
				26	25.982 594	11.01	
				27*	26.984 341		9.46 min
13	Aluminum	Al	26.981 54	26*	25.986 892		7.4×10^5 yr
				27	26.981 538	100	
				28*	27.981 910		2.24 min
14	Silicon	Si	28.086	28	27.976 927	92.23	
				29	28.976 495	4.67	
				30	29.973 770	3.10	
				31*	30.975 362		2.62 h
				32*	31.974 148		172 yr
15	Phosphorus	P	30.973 76	30*	29.978 307		2.50 min
				31	30.973 762	100	
				32*	31.973 908		14.26 days
				33*	32.971 725		25.3 days
16	Sulfur	S	32.066	32	31.972 071	95.02	
				33	32.971 459	0.75	
				34	33.967 867	4.21	
				35*	34.969 033		87.5 days
				36	35.967 081	0.02	
17	Chlorine	Cl	35.453	35	34.968 853	75.77	
				36*	35.968 307		3.0×10^5 yr
				37	36.965 903	24.23	

continued

TABLE A.3 *Continued*

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
18	Argon	Ar	39.948	36	35.967 547	0.337	
				37*	36.966 776		35.04 days
				38	37.962 732	0.063	
				39*	38.964 314		269 yr
				40	39.962 384	99.600	
				42*	41.963 049		33 yr
19	Potassium	K	39.098 3	39	38.963 708	93.258 1	
				40*	39.964 000	0.011 7	1.28×10^{9} yr
				41	40.961 827	6.730 2	
20	Calcium	Ca	40.08	40	39.962 591	96.941	
				41*	40.962 279		1.0×10^{5} yr
				42	41.958 618	0.647	
				43	42.958 767	0.135	
				44	43.955 481	2.086	
				46	45.953 687	0.004	
				48	47.952 534	0.187	
21	Scandium	Sc	44.955 9	41*	40.969 250		0.596 s
				45	44.955 911	100	
22	Titanium	Ti	47.88	44*	43.959 691		49 yr
				46	45.952 630	8.0	
				47	46.951 765	7.3	
				48	47.947 947	73.8	
				49	48.947 871	5.5	
				50	49.944 792	5.4	
23	Vanadium	V	50.941 5	48*	47.952 255		15.97 days
				50*	49.947 161	0.25	1.5×10^{17} yr
				51	50.943 962	99.75	
24	Chromium	Cr	51.996	48*	47.954 033		21.6 h
				50	49.946 047	4.345	
				52	51.940 511	83.79	
				53	52.940 652	9.50	
				54	53.938 883	2.365	
25	Manganese	Mn	54.938 05	54*	53.940 361		312.1 days
				55	54.938 048	100	
26	Iron	Fe	55.847	54	53.939 613	5.9	
				55*	54.938 297		2.7 yr
				56	55.934 940	91.72	
				57	56.935 396	2.1	
				58	57.933 278	0.28	
				60*	59.934 078		1.5×10^{6} yr
27	Cobalt	Co	58.933 20	59	58.933 198	100	
				60*	59.933 820		5.27 yr
28	Nickel	Ni	58.693	58	57.935 346	68.077	
				59*	58.934 350		7.5×10^{4} yr
				60	59.930 789	26.223	
				61	60.931 058	1.140	
				62	61.928 346	3.634	
				63*	62.929 670		100 yr
				64	63.927 967	0.926	

TABLE A.3 *Continued*

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
29	Copper	Cu	63.54	63	62.929 599	69.17	
				65	64.927 791	30.83	
30	Zinc	Zn	65.39	64	63.929 144	48.6	
				66	65.926 035	27.9	
				67	66.927 129	4.1	
				68	67.924 845	18.8	
				70	69.925 323	0.6	
31	Gallium	Ga	69.723	69	68.925 580	60.108	
				71	70.924 703	39.892	
32	Germanium	Ge	72.61	70	69.924 250	21.23	
				72	71.922 079	27.66	
				73	72.923 462	7.73	
				74	73.921 177	35.94	
				76	75.921 402	7.44	
33	Arsenic	As	74.921 6	75	74.921 594	100	
34	Selenium	Se	78.96	74	73.922 474	0.89	
				76	75.919 212	9.36	
				77	76.919 913	7.63	
				78	77.917 307	23.78	
				79*	78.918 497		$\leqslant 6.5 \times 10^{4}$ yr
				80	79.916 519	49.61	
				82*	81.916 697	8.73	1.4×10^{20} yr
35	Bromine	Br	79.904	79	78.918 336	50.69	
				81	80.916 287	49.31	
36	Krypton	Kr	83.80	78	77.920 400	0.35	
				80	79.916 377	2.25	
				81*	80.916 589		2.1×10^{5} yr
				82	81.913 481	11.6	
				83	82.914 136	11.5	
				84	83.911 508	57.0	
				85*	84.912 531		10.76 yr
				86	85.910 615	17.3	
37	Rubidium	Rb	85.468	85	84.911 793	72.17	
				87*	86.909 186	27.83	4.75×10^{10} yr
38	Strontium	Sr	87.62	84	83.913 428	0.56	
				86	85.909 266	9.86	
				87	86.908 883	7.00	
				88	87.905 618	82.58	
				90*	89.907 737		29.1 yr
39	Yttrium	Y	88.905 8	89	88.905 847	100	
40	Zirconium	Zr	91.224	90	89.904 702	51.45	
				91	90.905 643	11.22	
				92	91.905 038	17.15	
				93*	92.906 473		1.5×10^{6} yr
				94	93.906 314	17.38	
				96	95.908 274	2.80	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
41	Niobium	Nb	92.906 4	91*	90.906 988		6.8×10^2 yr
				92*	91.907 191		3.5×10^7 yr
				93	92.906 376	100	
				94*	93.907 280		2×10^4 yr
42	Molybdenum	Mo	95.94	92	91.906 807	14.84	
				93*	92.906 811		3.5×10^3 yr
				94	93.905 085	9.25	
				95	94.905 841	15.92	
				96	95.904 678	16.68	
				97	96.906 020	9.55	
				98	97.905 407	24.13	
				100	99.907 476	9.63	
43	Technetium	Tc		97*	96.906 363		2.6×10^6 yr
				98*	97.907 215		4.2×10^6 yr
				99*	98.906 254		2.1×10^5 yr
44	Ruthenium	Ru	101.07	96	95.907 597	5.54	
				98	97.905 287	1.86	
				99	98.905 939	12.7	
				100	99.904 219	12.6	
				101	100.905 558	17.1	
				102	101.904 348	31.6	
				104	103.905 428	18.6	
45	Rhodium	Rh	102.905 5	103	102.905 502	100	
46	Palladium	Pd	106.42	102	101.905 616	1.02	
				104	103.904 033	11.14	
				105	104.905 082	22.33	
				106	105.903 481	27.33	
				107*	106.905 126		6.5×10^6 yr
				108	107.903 893	26.46	
				110	109.905 158	11.72	
47	Silver	Ag	107.868	107	106.905 091	51.84	
				109	108.904 754	48.16	
48	Cadmium	Cd	112.41	106	105.906 457	1.25	
				108	107.904 183	0.89	
				109*	108.904 984		462 days
				110	109.903 004	12.49	
				111	110.904 182	12.80	
				112	111.902 760	24.13	
				113*	112.904 401	12.22	9.3×10^{15} yr
				114	113.903 359	28.73	
				116	115.904 755	7.49	
49	Indium	In	114.82	113	112.904 060	4.3	
				115*	114.903 876	95.7	4.4×10^{14} yr
50	Tin	Sn	118.71	112	111.904 822	0.97	
				114	113.902 780	0.65	
				115	114.903 345	0.36	
				116	115.901 743	14.53	
				117	116.902 953	7.68	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
(50)	(Tin)			118	117.901 605	24.22	
				119	118.903 308	8.58	
				120	119.902 197	32.59	
				121*	120.904 237		55 yr
				122	121.903 439	4.63	
				124	123.905 274	5.79	
51	Antimony	Sb	121.76	121	120.903 820	57.36	
				123	122.904 215	42.64	
				125*	124.905 251		2.7 yr
52	Tellurium	Te	127.60	120	119.904 040	0.095	
				122	121.903 052	2.59	
				123*	122.904 271	0.905	1.3×10^{13} yr
				124	123.902 817	4.79	
				125	124.904 429	7.12	
				126	125.903 309	18.93	
				128*	127.904 463	31.70	$> 8 \times 10^{24}$ yr
				130*	129.906 228	33.87	$\leqslant 1.25 \times 10^{21}$ yr
53	Iodine	I	126.904 5	127	126.904 474	100	
				129*	128.904 984		1.6×10^7 yr
54	Xenon	Xe	131.29	124	123.905 894	0.10	
				126	125.904 268	0.09	
				128	127.903 531	1.91	
				129	128.904 779	26.4	
				130	129.903 509	4.1	
				131	130.905 069	21.2	
				132	131.904 141	26.9	
				134	133.905 394	10.4	
				136*	135.907 215	8.9	$\geqslant 2.36 \times 10^{21}$ yr
55	Cesium	Cs	132.905 4	133	132.905 436	100	
				134*	133.906 703		2.1 yr
				135*	134.905 891		2×10^6 yr
				137*	136.907 078		30 yr
56	Barium	Ba	137.33	130	129.906 289	0.106	
				132	131.905 048	0.101	
				133*	132.905 990		10.5 yr
				134	133.904 492	2.42	
				135	134.905 671	6.593	
				136	135.904 559	7.85	
				137	136.905 816	11.23	
				138	137.905 236	71.70	
57	Lanthanum	La	138.905	137*	136.906 462		6×10^4 yr
				138*	137.907 105	0.090 2	1.05×10^{11} yr
				139	138.906 346	99.909 8	
58	Cerium	Ce	140.12	136	135.907 139	0.19	
				138	137.905 986	0.25	
				140	139.905 434	88.43	
				142*	141.909 241	11.13	$> 5 \times 10^{16}$ yr
59	Praseodymium	Pr	140.907 6	141	140.907 647	100	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
60	Neodymium	Nd	144.24	142	141.907 718	27.13	
				143	142.909 809	12.18	
				144*	143.910 082	23.80	2.3×10^{15} yr
				145	144.912 568	8.30	
				146	145.913 113	17.19	
				148	147.916 888	5.76	
				150*	149.920 887	5.64	$>1 \times 10^{18}$ yr
61	Promethium	Pm		143*	142.910 928		265 days
				145*	144.912 745		17.7 yr
				146*	145.914 698		5.5 yr
				147*	146.915 134		2.623 yr
62	Samarium	Sm	150.36	144	143.911 996	3.1	
				146*	145.913 043		1.0×10^{8} yr
				147*	146.914 894	15.0	1.06×10^{11} yr
				148*	147.914 819	11.3	7×10^{15} yr
				149*	148.917 180	13.8	$>2 \times 10^{15}$ yr
				150	149.917 273	7.4	
				151*	150.919 928		90 yr
				152	151.919 728	26.7	
				154	153.922 206	22.7	
63	Europium	Eu	151.96	151	150.919 846	47.8	
				152*	151.921 740		13.5 yr
				153	152.921 226	52.2	
				154*	153.922 975		8.59 yr
				155*	154.922 888		4.7 yr
64	Gadolinium	Gd	157.25	148*	147.918 112		75 yr
				150*	149.918 657		1.8×10^{6} yr
				152*	151.919 787	0.20	1.1×10^{14} yr
				154	153.920 862	2.18	
				155	154.922 618	14.80	
				156	155.922 119	20.47	
				157	156.923 957	15.65	
				158	157.924 099	24.84	
				160	159.927 050	21.86	
65	Terbium	Tb	158.925 3	159	158.925 345	100	
66	Dysprosium	Dy	162.50	156	155.924 277	0.06	
				158	157.924 403	0.10	
				160	159.925 193	2.34	
				161	160.926 930	18.9	
				162	161.926 796	25.5	
				163	162.928 729	24.9	
				164	163.929 172	28.2	
67	Holmium	Ho	164.930 3	165	164.930 316	100	
				166*	165.932 282		1.2×10^{3} yr
68	Erbium	Er	167.26	162	161.928 775	0.14	
				164	163.929 198	1.61	
				166	165.930 292	33.6	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
(68)	(Erbium)			167	166.932 047	22.95	
				168	167.932 369	27.8	
				170	169.935 462	14.9	
69	Thulium	Tm	168.934 2	169	168.934 213	100	
				171*	170.936 428		1.92 yr
70	Ytterbium	Yb	173.04	168	167.933 897	0.13	
				170	169.934 761	3.05	
				171	170.936 324	14.3	
				172	171.936 380	21.9	
				173	172.938 209	16.12	
				174	173.938 861	31.8	
				176	175.942 564	12.7	
71	Lutecium	Lu	174.967	173*	172.938 930		1.37 yr
				175	174.940 772	97.41	
				176*	175.942 679	2.59	3.78×10^{10} yr
72	Hafnium	Hf	178.49	174*	173.940 042	0.162	2.0×10^{15} yr
				176	175.941 404	5.206	
				177	176.943 218	18.606	
				178	177.943 697	27.297	
				179	178.945 813	13.629	
				180	179.946 547	35.100	
73	Tantalum	Ta	180.947 9	180	179.947 542	0.012	
				181	180.947 993	99.988	
74	Tungsten (Wolfram)	W	183.85	180	179.946 702	0.12	
				182	181.948 202	26.3	
				183	182.950 221	14.28	
				184	183.950 929	30.7	
				186	185.954 358	28.6	
75	Rhenium	Re	186.207	185	184.952 951	37.40	
				187*	186.955 746	62.60	4.4×10^{10} yr
76	Osmium	Os	190.2	184	183.952 486	0.02	
				186*	185.953 834	1.58	2.0×10^{15} yr
				187	186.955 744	1.6	
				188	187.955 832	13.3	
				189	188.958 139	16.1	
				190	189.958 439	26.4	
				192	191.961 468	41.0	
				194*	193.965 172		6.0 yr
77	Iridium	Ir	192.2	191	190.960 585	37.3	
				193	192.962 916	62.7	
78	Platinum	Pt	195.08	190*	189.959 926	0.01	6.5×10^{11} yr
				192	191.961 027	0.79	
				194	193.962 655	32.9	
				195	194.964 765	33.8	
				196	195.964 926	25.3	
				198	197.967 867	7.2	
79	Gold	Au	196.966 5	197	196.966 543	100	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
80	Mercury	Hg	200.59	196	195.965 806	0.15	
				198	197.966 743	9.97	
				199	198.968 253	16.87	
				200	199.968 299	23.10	
				201	200.970 276	13.10	
				202	201.970 617	29.86	
				204	203.973 466	6.87	
81	Thallium	Tl	204.383	203	202.972 320	29.524	
				204*	203.973 839		3.78 yr
				205	204.974 400	70.476	
		(Ra E″)		206*	205.976 084		4.2 min
		(Ac C″)		207*	206.977 403		4.77 min
		(Th C″)		208*	207.981 992		3.053 min
		(Ra C″)		210*	209.990 057		1.30 min
82	Lead	Pb	207.2	202*	201.972 134		5×10^4 yr
				204*	203.973 020	1.4	$\geqslant 1.4 \times 10^{17}$ yr
				205*	204.974 457		1.5×10^7 yr
				206	205.974 440	24.1	
				207	206.975 871	22.1	
				208	207.976 627	52.4	
		(Ra D)		210*	209.984 163		22.3 yr
		(Ac B)		211*	210.988 734		36.1 min
		(Th B)		212*	211.991 872		10.64 h
		(Ra B)		214*	213.999 798		26.8 min
83	Bismuth	Bi	208.980 3	207*	206.978 444		32.2 yr
				208*	207.979 717		3.7×10^5 yr
				209	208.980 374	100	
		(Ra E)		210*	209.984 096		5.01 days
		(Th C)		211*	210.987 254		2.14 min
				212*	211.991 259		60.6 min
		(Ra C)		214*	213.998 692		19.9 min
				215*	215.001 836		7.4 min
84	Polonium	Po		209*	208.982 405		102 yr
		(Ra F)		210*	209.982 848		138.38 days
		(Ac C′)		211*	210.986 627		0.52 s
		(Th C′)		212*	211.988 842		0.30 μs
		(Ra C′)		214*	213.995 177		164 μs
		(Ac A)		215*	214.999 418		0.001 8 s
		(Th A)		216*	216.001 889		0.145 s
		(Ra A)		218*	218.008 965		3.10 min
85	Astatine	At		215*	214.998 638		≈ 100 μs
				218*	218.008 685		1.6 s
				219*	219.011 294		0.9 min
86	Radon	Rn					
		(An)		219*	219.009 477		3.96 s
		(Tn)		220*	220.011 369		55.6 s
		(Rn)		222*	222.017 571		3.823 days
87	Francium	Fr					
		(Ac K)		223*	223.019 733		22 min

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
88	Radium	Ra					
		(Ac X)		223*	223.018 499		11.43 days
		(Th X)		224*	224.020 187		3.66 days
		(Ra)		226*	226.025 402		1 600 yr
		(Ms Th_1)		228*	228.031 064		5.75 yr
89	Actinium	Ac		227*	227.027 749		21.77 yr
		(Ms Th_2)		228*	228.031 015		6.15 h
90	Thorium	Th	232.038 1				
		(Rd Ac)		227*	227.027 701		18.72 days
		(Rd Th)		228*	228.028 716		1.913 yr
				229*	229.031 757		7 300 yr
		(Io)		230*	230.033 127		75.000 yr
		(UY)		231*	231.036 299		25.52 h
		(Th)		232*	232.038 051	100	1.40×10^{10} yr
		(UX_1)		234*	234.043 593		24.1 days
91	Protactinium	Pa		231*	231.035 880		32.760 yr
		(Uz)		234*	234.043 300		6.7 h
92	Uranium	U	238.028 9	232*	232.037 131		69 yr
				233*	233.039 630		1.59×10^5 yr
				234*	234.040 946	0.005 5	2.45×10^5 yr
		(Ac U)		235*	235.043 924	0.720	7.04×10^8 yr
				236*	236.045 562		2.34×10^7 yr
		(UI)		238*	238.050 784	99.274 5	4.47×10^9 yr
93	Neptunium	Np		235*	235.044 057		396 days
				236*	236.046 560		1.15×10^5 yr
				237*	237.048 168		2.14×10^6 yr
94	Plutonium	Pu		236*	236.046 033		2.87 yr
				238*	238.049 555		87.7 yr
				239*	239.052 157		2.412×10^4 yr
				240*	240.053 808		6 560 yr
				241*	241.056 846		14.4 yr
				242*	242.058 737		3.73×10^6 yr
				244*	244.064 200		8.1×10^7 yr

[a] The masses in the sixth column are atomic masses, which include the mass of Z electrons. Data are from the National Nuclear Data Center, Brookhaven National Laboratory, prepared by Jagdish K. Tuli, July 1990. The data are based on experimental results reported in *Nuclear Data Sheets* and *Nuclear Physics* and also from *Chart of the Nuclides,* 14th ed. Atomic masses are based on those by A. H. Wapstra, G. Audi, and R. Hoekstra. Isotopic abundances are based on those by N. E. Holden.

APPENDIX B • *Mathematics Review*

These appendices in mathematics are intended as a brief review of operations and methods. Early in this course, you should be totally familiar with basic algebraic techniques, analytic geometry, and trigonometry. The appendices on differential and integral calculus are more detailed and are intended for those students who have difficulty applying calculus concepts to physical situations.

B.1 SCIENTIFIC NOTATION

Many quantities that scientists deal with often have very large or very small values. For example, the speed of light is about 300 000 000 m/s, and the ink required to make the dot over an *i* in this textbook has a mass of about 0.000 000 001 kg. Obviously, it is very cumbersome to read, write, and keep track of numbers such as these. We avoid this problem by using a method dealing with powers of the number 10:

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 10 \times 10 = 100$$

$$10^3 = 10 \times 10 \times 10 = 1000$$

$$10^4 = 10 \times 10 \times 10 \times 10 = 10\,000$$

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\,000$$

and so on. The number of zeros corresponds to the power to which 10 is raised, called the **exponent** of 10. For example, the speed of light, 300 000 000 m/s, can be expressed as 3×10^8 m/s.

In this method, some representative numbers smaller than unity are

$$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-2} = \frac{1}{10 \times 10} = 0.01$$

$$10^{-3} = \frac{1}{10 \times 10 \times 10} = 0.001$$

$$10^{-4} = \frac{1}{10 \times 10 \times 10 \times 10} = 0.000\,1$$

$$10^{-5} = \frac{1}{10 \times 10 \times 10 \times 10 \times 10} = 0.000\,01$$

In these cases, the number of places the decimal point is to the left of the digit 1 equals the value of the (negative) exponent. Numbers expressed as some power of 10 multiplied by another number between 1 and 10 are said to be in **scientific notation.** For example, the scientific notation for 5 943 000 000 is 5.943×10^9 and that for 0.000 083 2 is 8.32×10^{-5}.

When numbers expressed in scientific notation are being multiplied, the following general rule is very useful:

$$10^n \times 10^m = 10^{n+m} \qquad \textbf{(B.1)}$$

where n and m can be *any* numbers (not necessarily integers). For example, $10^2 \times 10^5 = 10^7$. The rule also applies if one of the exponents is negative: $10^3 \times 10^{-8} = 10^{-5}$.

When dividing numbers expressed in scientific notation, note that

$$\frac{10^n}{10^m} = 10^n \times 10^{-m} = 10^{n-m} \qquad \textbf{(B.2)}$$

Exercises

With help from the above rules, verify the answers to the following:

1. $86\,400 = 8.64 \times 10^4$
2. $9\,816\,762.5 = 9.816\,762\,5 \times 10^6$
3. $0.000\,000\,039\,8 = 3.98 \times 10^{-8}$
4. $(4 \times 10^8)(9 \times 10^9) = 3.6 \times 10^{18}$
5. $(3 \times 10^7)(6 \times 10^{-12}) = 1.8 \times 10^{-4}$
6. $\dfrac{75 \times 10^{-11}}{5 \times 10^{-3}} = 1.5 \times 10^{-7}$
7. $\dfrac{(3 \times 10^6)(8 \times 10^{-2})}{(2 \times 10^{17})(6 \times 10^5)} = 2 \times 10^{-18}$

B.2 ALGEBRA

Some Basic Rules

When algebraic operations are performed, the laws of arithmetic apply. Symbols such as x, y, and z are usually used to represent quantities that are not specified, what are called the **unknowns.**

First, consider the equation

$$8x = 32$$

If we wish to solve for x, we can divide (or multiply) each side of the equation by the same factor without destroying the equality. In this case, if we divide both sides by 8, we have

$$\frac{8x}{8} = \frac{32}{8}$$

$$x = 4$$

Next consider the equation

$$x + 2 = 8$$

In this type of expression, we can add or subtract the same quantity from each side. If we subtract 2 from each side, we get

$$x + 2 - 2 = 8 - 2$$

$$x = 6$$

In general, if $x + a = b$, then $x = b - a$.

Now consider the equation

$$\frac{x}{5} = 9$$

If we multiply each side by 5, we are left with x on the left by itself and 45 on the right:

$$\left(\frac{x}{5}\right)(5) = 9 \times 5$$

$$x = 45$$

In all cases, *whatever operation is performed on the left side of the equality must also be performed on the right side.*

The following rules for multiplying, dividing, adding, and subtracting fractions should be recalled, where a, b, and c are three numbers:

	Rule	Example
Multiplying	$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$	$\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}$
Dividing	$\frac{(a/b)}{(c/d)} = \frac{ad}{bc}$	$\frac{2/3}{4/5} = \frac{(2)(5)}{(4)(3)} = \frac{10}{12}$
Adding	$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$	$\frac{2}{3} - \frac{4}{5} = \frac{(2)(5) - (4)(3)}{(3)(5)} = -\frac{2}{15}$

Exercises

In the following exercises, solve for x:

	Answers
1. $a = \dfrac{1}{1 + x}$	$x = \dfrac{1 - a}{a}$
2. $3x - 5 = 13$	$x = 6$
3. $ax - 5 = bx + 2$	$x = \dfrac{7}{a - b}$
4. $\dfrac{5}{2x + 6} = \dfrac{3}{4x + 8}$	$x = -\dfrac{11}{7}$

Powers

When powers of a given quantity x are multiplied, the following rule applies:

$$x^n x^m = x^{n+m} \qquad \textbf{(B.3)}$$

For example, $x^2x^4 = x^{2+4} = x^6$.

When dividing the powers of a given quantity, the rule is

$$\frac{x^n}{x^m} = x^{n-m} \quad \textbf{(B.4)}$$

For example, $x^8/x^2 = x^{8-2} = x^6$.

A power that is a fraction, such as $\frac{1}{3}$, corresponds to a root as follows:

$$x^{1/n} = \sqrt[n]{x} \quad \textbf{(B.5)}$$

TABLE B.1
Rules of Exponents

$$x^0 = 1$$
$$x^1 = x$$
$$x^n x^m = x^{n+m}$$
$$x^n/x^m = x^{n-m}$$
$$x^{1/n} = \sqrt[n]{x}$$
$$(x^n)^m = x^{nm}$$

For example, $4^{1/3} = \sqrt[3]{4} = 1.5874$. (A scientific calculator is useful for such calculations.)

Finally, any quantity x^n raised to the mth power is

$$(x^n)^m = x^{nm} \quad \textbf{(B.6)}$$

Table B.1 summarizes the rules of exponents.

Exercises

Verify the following:

1. $3^2 \times 3^3 = 243$
2. $x^5x^{-8} = x^{-3}$
3. $x^{10}/x^{-5} = x^{15}$
4. $5^{1/3} = 1.709\ 975$ (Use your calculator.)
5. $60^{1/4} = 2.783\ 158$ (Use your calculator.)
6. $(x^4)^3 = x^{12}$

Factoring

Some useful formulas for factoring an equation are

$ax + ay + az = a(x + y + x)$	common factor
$a^2 + 2ab + b^2 = (a + b)^2$	perfect square
$a^2 - b^2 = (a + b)(a - b)$	differences of squares

Quadratic Equations

The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \quad \textbf{(B.7)}$$

where x is the unknown quantity and a, b, and c are numerical factors referred to as **coefficients** of the equation. This equation has two roots, given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \textbf{(B.8)}$$

If $b^2 \geq 4ac$, the roots are real.

Example 1

The equation $x^2 + 5x + 4 = 0$ has the following roots corresponding to the two signs of the square-root term:

$$x = \frac{-5 \pm \sqrt{5^2 - (4)(1)(4)}}{2(1)} = \frac{-5 \pm \sqrt{9}}{2} = \frac{-5 \pm 3}{2}$$

$$x_+ = \frac{-5 + 3}{2} = \boxed{-1} \qquad x_- = \frac{-5 - 3}{2} = \boxed{-4}$$

where x_+ refers to the root corresponding to the positive sign and x_- refers to the root corresponding to the negative sign.

Exercises

Solve the following quadratic equations:

	Answers	
1. $x^2 + 2x - 3 = 0$	$x_+ = 1$	$x_- = -3$
2. $2x^2 - 5x + 2 = 0$	$x_+ = 2$	$x_- = \frac{1}{2}$
3. $2x^2 - 4x - 9 = 0$	$x_+ = 1 + \sqrt{22}/2$	$x_- = 1 - \sqrt{22}/2$

Linear Equations

A linear equation has the general form

$$y = mx + b \quad \textbf{(B.9)}$$

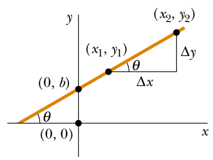


Figure B.1

where m and b are constants. This equation is referred to as being linear because the graph of y versus x is a straight line, as shown in Figure B.1. The constant b, called the ***y*-intercept,** represents the value of y at which the straight line intersects the y axis. The constant m is equal to the **slope** of the straight line and is also equal to the tangent of the angle that the line makes with the x axis. If any two points on the straight line are specified by the coordinates (x_1, y_1) and (x_2, y_2), as in Figure B.1, then the slope of the straight line can be expressed as

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \tan\theta \quad \textbf{(B.10)}$$

Note that m and b can have either positive or negative values. If $m > 0$, the straight line has a *positive* slope, as in Figure B1. If $m < 0$, the straight line has a *negative* slope. In Figure B.1, both m and b are positive. Three other possible situations are shown in Figure B.2.

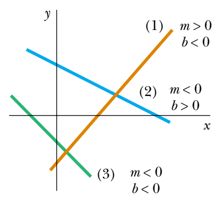


Figure B.2

Exercises

1. Draw graphs of the following straight lines:
   (a) $y = 5x + 3$ (b) $y = -2x + 4$ (c) $y = -3x - 6$
2. Find the slopes of the straight lines described in Exercise 1.

Answers (a) 5 (b) -2 (c) -3

3. Find the slopes of the straight lines that pass through the following sets of points:
(a) $(0, -4)$ and $(4, 2)$, (b) $(0, 0)$ and $(2, -5)$, and (c) $(-5, 2)$ and $(4, -2)$

Answers (a) $3/2$ (b) $-5/2$ (c) $-4/9$

Solving Simultaneous Linear Equations

Consider the equation $3x + 5y = 15$, which has two unknowns, x and y. Such an equation does not have a unique solution. For example, note that $(x = 0, y = 3)$, $(x = 5, y = 0)$, and $(x = 2, y = 9/5)$ are all solutions to this equation.

If a problem has two unknowns, a unique solution is possible only if we have *two* equations. In general, if a problem has n unknowns, its solution requires n equations. In order to solve two simultaneous equations involving two unknowns, x and y, we solve one of the equations for x in terms of y and substitute this expression into the other equation.

EXAMPLE 2

Solve the following two simultaneous equations:

$$(1)\quad 5x + y = -8$$

$$(2)\quad 2x - 2y = 4$$

Solution From (2), $x = y + 2$. Substitution of this into (1) gives

$$5(y + 2) + y = -8$$

$$6y = -18$$

$$y = -3$$

$$x = y + 2 = -1$$

Alternate Solution Multiply each term in (1) by the factor 2 and add the result to (2):

$$10x + 2y = -16$$

$$\underline{2x - 2y = 4}$$

$$12x = -12$$

$$x = -1$$

$$y = x - 2 = -3$$

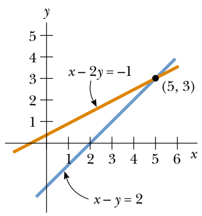


Figure B.3

Two linear equations containing two unknowns can also be solved by a graphical method. If the straight lines corresponding to the two equations are plotted in a conventional coordinate system, the intersection of the two lines represents the solution. For example, consider the two equations

$$x - y = 2$$

$$x - 2y = -1$$

These are plotted in Figure B.3. The intersection of the two lines has the coordinates $x = 5$, $y = 3$. This represents the solution to the equations. You should check this solution by the analytical technique discussed above.

EXERCISES

Solve the following pairs of simultaneous equations involving two unknowns:

	Answers
1. $x + y = 8$ $x - y = 2$	$x = 5, y = 3$
2. $98 - T = 10a$ $T - 49 = 5a$	$T = 65, a = 3.27$
3. $6x + 2y = 6$ $8x - 4y = 28$	$x = 2, y = -3$

Logarithms

Suppose that a quantity x is expressed as a power of some quantity a:

$$x = a^y \tag{B.11}$$

The number a is called the **base** number. The **logarithm** of x with respect to the base a is equal to the exponent to which the base must be raised in order to satisfy the expression $x = a^y$:

$$y = \log_a x \tag{B.12}$$

Conversely, the **antilogarithm** of y is the number x:

$$x = \text{antilog}_a\, y \tag{B.13}$$

In practice, the two bases most often used are base 10, called the *common* logarithm base, and base $e = 2.718 \ldots$, called Euler's constant or the *natural* logarithm base. When common logarithms are used,

$$y = \log_{10} x \qquad (\text{or } x = 10^y) \tag{B.14}$$

When natural logarithms are used,

$$y = \ln_e x \qquad (\text{or } x = e^y) \tag{B.15}$$

For example, $\log_{10} 52 = 1.716$, so that $\text{antilog}_{10}\, 1.716 = 10^{1.716} = 52$. Likewise, $\ln_e 52 = 3.951$, so $\text{antiln}_e\, 3.951 = e^{3.951} = 52$.

In general, note that you can convert between base 10 and base e with the equality

$$\ln_e x = (2.302\,585) \log_{10} x \tag{B.16}$$

Finally, some useful properties of logarithms are

$$\log(ab) = \log a + \log b$$
$$\log(a/b) = \log a - \log b$$
$$\log(a^n) = n \log a$$
$$\ln e = 1$$
$$\ln e^a = a$$
$$\ln\left(\frac{1}{a}\right) = -\ln a$$

B.3 GEOMETRY

The **distance** d between two points having coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.17}$$

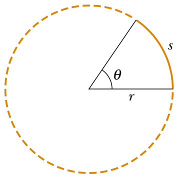


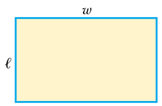
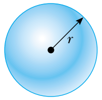
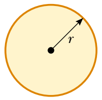
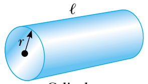
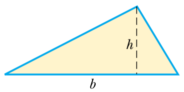
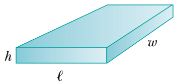
Figure B.4

Radian measure: The arc length s of a circular arc (Fig. B.4) is proportional to the radius r for a fixed value of θ (in radians):

$$s = r\theta \qquad \theta = \frac{s}{r} \tag{B.18}$$

Table B.2 gives the areas and volumes for several geometric shapes used throughout this text:

TABLE B.2 Useful Information for Geometry

Shape	Area or Volume	Shape	Area or Volume
 Rectangle	Area = ℓw	 Sphere	Surface area = $4\pi r^2$ Volume = $\frac{4\pi r^3}{3}$
 Circle	Area = πr^2 (Circumference = $2\pi r$)	 Cylinder	Lateral surface area = $2\pi r\ell$ Volume = $\pi r^2\ell$
 Triangle	Area = $\frac{1}{2}bh$	 Rectangular box	Area = $2(\ell h + \ell w + hw)$ Volume = ℓwh

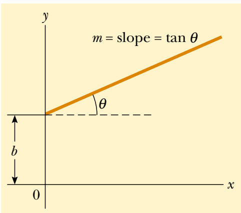


Figure B.5

The equation of a **straight line** (Fig. B.5) is

$$y = mx + b \tag{B.19}$$

where b is the y-intercept and m is the slope of the line.

The equation of a **circle** of radius R centered at the origin is

$$x^2 + y^2 = R^2 \tag{B.20}$$

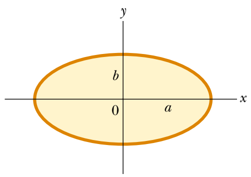


Figure B.6

The equation of an **ellipse** having the origin at its center (Fig. B.6) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{B.21}$$

where a is the length of the semi-major axis (the longer one) and b is the length of the semi-minor axis (the shorter one).

The equation of a **parabola** the vertex of which is at $y = b$ (Fig. B.7) is

$$y = ax^2 + b \tag{B.22}$$

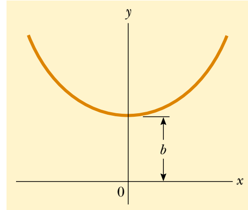


Figure B.7

The equation of a **rectangular hyperbola** (Fig. B.8) is

$$xy = \text{constant} \tag{B.23}$$

B.4 TRIGONOMETRY

That portion of mathematics based on the special properties of the right triangle is called trigonometry. By definition, a right triangle is one containing a 90° angle. Consider the right triangle shown in Figure B.9, where side a is opposite the angle θ, side b is adjacent to the angle θ, and side c is the hypotenuse of the triangle. The three basic trigonometric functions defined by such a triangle are the sine (sin), cosine (cos), and tangent (tan) functions. In terms of the angle θ, these functions are defined by

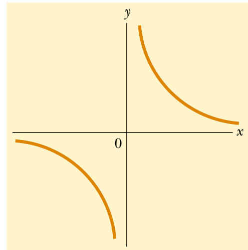


Figure B.8

$$\sin\theta \equiv \frac{\text{side opposite }\theta}{\text{hypotenuse}} = \frac{a}{c} \tag{B.24}$$

$$\cos\theta \equiv \frac{\text{side adjacent to }\theta}{\text{hypotenuse}} = \frac{b}{c} \tag{B.25}$$

$$\tan\theta \equiv \frac{\text{side opposite }\theta}{\text{side adjacent to }\theta} = \frac{a}{b} \tag{B.26}$$

The Pythagorean theorem provides the following relationship between the sides of a right triangle:

$$c^2 = a^2 + b^2 \tag{B.27}$$

From the above definitions and the Pythagorean theorem, it follows that

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The cosecant, secant, and cotangent functions are defined by

$$\csc\theta \equiv \frac{1}{\sin\theta} \qquad \sec\theta \equiv \frac{1}{\cos\theta} \qquad \cot\theta \equiv \frac{1}{\tan\theta}$$

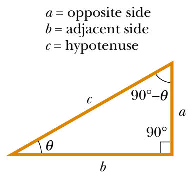


Figure B.9

The relationships below follow directly from the right triangle shown in Figure B.9:

$$\sin\theta = \cos(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta)$$

$$\cot\theta = \tan(90° - \theta)$$

Some properties of trigonometric functions are

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

The following relationships apply to *any* triangle, as shown in Figure B.10:

$$\alpha + \beta + \gamma = 180°$$

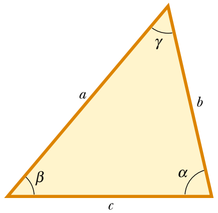


Figure B.10

Law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Table B.3 lists a number of useful trigonometric identities.

TABLE B.3 Some Trigonometric Identities

$\sin^2\theta + \cos^2\theta = 1$	$\csc^2\theta = 1 + \cot^2\theta$
$\sec^2\theta = 1 + \tan^2\theta$	$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$
$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\tan\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$
$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	

EXAMPLE 3

Consider the right triangle in Figure B.11, in which $a = 2$, $b = 5$, and c is unknown. From the Pythagorean theorem, we have

$$c^2 = a^2 + b^2 = 2^2 + 5^2 = 4 + 25 = 29$$

$$c = \sqrt{29} = 5.39$$

To find the angle θ, note that

$$\tan\theta = \frac{a}{b} = \frac{2}{5} = 0.400$$

From a table of functions or from a calculator, we have

$$\theta = \tan^{-1}(0.400) = 21.8°$$

where $\tan^{-1}$ (0.400) is the notation for "angle whose tangent is 0.400," sometimes written as arctan (0.400).

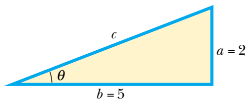


Figure B.11

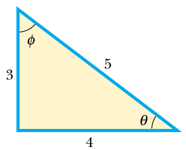


Figure B.12

EXERCISES

1. In Figure B.12, identify (a) the side opposite θ and (b) the side adjacent to ϕ and then find (c) $\cos\theta$, (d) $\sin\phi$, and (e) $\tan\phi$.

Answers (a) 3, (b) 3, (c) $\frac{4}{5}$, (d) $\frac{4}{5}$, and (e) $\frac{4}{3}$

2. In a certain right triangle, the two sides that are perpendicular to each other are 5 m and 7 m long. What is the length of the third side?

Answer 8.60 m

3. A right triangle has a hypotenuse of length 3 m, and one of its angles is 30°. What is the length of (a) the side opposite the 30° angle and (b) the side adjacent to the 30° angle?

Answers (a) 1.5 m, (b) 2.60 m

B.5 SERIES EXPANSIONS

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1 \pm x) = \pm x - \tfrac{1}{2}x^2 \pm \tfrac{1}{3}x^3 - \cdots$$

$$\left.\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad |x| < \pi/2 \end{aligned}\right\} x \text{ in radians}$$

For $x \ll 1$, the following approximations can be used[1]:

$(1 + x)^n \approx 1 + nx$	$\sin x \approx x$
$e^x \approx 1 + x$	$\cos x \approx 1$
$\ln(1 \pm x) \approx \pm x$	$\tan x \approx x$

B.6 DIFFERENTIAL CALCULUS

In various branches of science, it is sometimes necessary to use the basic tools of calculus, invented by Newton, to describe physical phenomena. The use of calculus is fundamental in the treatment of various problems in Newtonian mechanics, electricity, and magnetism. In this section, we simply state some basic properties and "rules of thumb" that should be a useful review to the student.

First, a **function** must be specified that relates one variable to another (such as a coordinate as a function of time). Suppose one of the variables is called y (the dependent variable), the other x (the independent variable). We might have a function relationship such as

$$y(x) = ax^3 + bx^2 + cx + d$$

If a, b, c, and d are specified constants, then y can be calculated for any value of x. We usually deal with continuous functions, that is, those for which y varies "smoothly" with x.

[1] The approximations for the functions $\sin x$, $\cos x$, and $\tan x$ are for $x \leq 0.1$ rad.

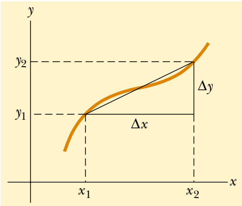


Figure B.13

The **derivative** of y with respect to x is defined as the limit, as Δx approaches zero, of the slopes of chords drawn between two points on the y versus x curve. Mathematically, we write this definition as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \qquad \textbf{(B.28)}$$

where Δy and Δx are defined as $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ (Fig. B.13). It is important to note that dy/dx *does not* mean dy divided by dx, but is simply a notation of the limiting process of the derivative as defined by Equation B.28.

A useful expression to remember when $y(x) = ax^n$, where a is a *constant* and n is *any* positive or negative number (integer or fraction), is

$$\frac{dy}{dx} = nax^{n-1} \qquad \textbf{(B.29)}$$

If $y(x)$ is a polynomial or algebraic function of x, we apply Equation B.29 to *each* term in the polynomial and take $d[\text{constant}]/dx = 0$. In Examples 4 through 7, we evaluate the derivatives of several functions.

EXAMPLE 4

Suppose $y(x)$ (that is, y as a function of x) is given by

$$y(x) = ax^3 + bx + c$$

where a and b are constants. Then it follows that

$$y(x + \Delta x) = a(x + \Delta x)^3 + b(x + \Delta x) + c$$

$$y(x + \Delta x) = a(x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) + b(x + \Delta x) + c$$

so

$$\Delta y = y(x + \Delta x) - y(x) = a(3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) + b\Delta x$$

Substituting this into Equation B.28 gives

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [3ax^2 + 3x\Delta x + \Delta x^2] + b$$

$$\frac{dy}{dx} = 3ax^2 + b$$

EXAMPLE 5

$$y(x) = 8x^5 + 4x^3 + 2x + 7$$

Solution Applying Equation B.29 to each term independently, and remembering that d/dx (constant) = 0, we have

$$\frac{dy}{dx} = 8(5)x^4 + 4(3)x^2 + 2(1)x^0 + 0$$

$$\frac{dy}{dx} = 40x^4 + 12x^2 + 2$$

Special Properties of the Derivative

A. Derivative of the product of two functions If a function $f(x)$ is given by the product of two functions, say, $g(x)$ and $h(x)$, then the derivative of $f(x)$ is defined as

$$\frac{d}{dx} f(x) = \frac{d}{dx}[g(x)h(x)] = g\frac{dh}{dx} + h\frac{dg}{dx} \qquad \textbf{(B.30)}$$

B. Derivative of the sum of two functions If a function $f(x)$ is equal to the sum of two functions, then the derivative of the sum is equal to the sum of the derivatives:

$$\frac{d}{dx} f(x) = \frac{d}{dx}[g(x) + h(x)] = \frac{dg}{dx} + \frac{dh}{dx} \qquad \textbf{(B.31)}$$

C. Chain rule of differential calculus If $y = f(x)$ and $x = g(z)$, then dy/dz can be written as the product of two derivatives:

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} \qquad \textbf{(B.32)}$$

D. The second derivative The second derivative of y with respect to x is defined as the derivative of the function dy/dx (the derivative of the derivative). It is usually written

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad \textbf{(B.33)}$$

EXAMPLE 6

Find the derivative of $y(x) = x^3/(x + 1)^2$ with respect to x.

Solution We can rewrite this function as $y(x) = x^3(x + 1)^{-2}$ and apply Equation B.30:

$$\frac{dy}{dx} = (x + 1)^{-2}\frac{d}{dx}(x^3) + x^3\frac{d}{dx}(x + 1)^{-2}$$

$$= (x + 1)^{-2}3x^2 + x^3(-2)(x + 1)^{-3}$$

$$\frac{dy}{dx} = \frac{3x^2}{(x + 1)^2} - \frac{2x^3}{(x + 1)^3}$$

EXAMPLE 7

A useful formula that follows from Equation B.30 is the derivative of the quotient of two functions. Show that

$$\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \frac{h\frac{dg}{dx} - g\frac{dh}{dx}}{h^2}$$

Solution We can write the quotient as gh^{-1} and then apply Equations B.29 and B.30:

$$\frac{d}{dx}\left(\frac{g}{h}\right) = \frac{d}{dx}(gh^{-1}) = g\frac{d}{dx}(h^{-1}) + h^{-1}\frac{d}{dx}(g)$$

$$= -gh^{-2}\frac{dh}{dx} + h^{-1}\frac{dg}{dx}$$

$$= \frac{h\frac{dg}{dx} - g\frac{dh}{dx}}{h^2}$$

Some of the more commonly used derivatives of functions are listed in Table B.4.

B.7 INTEGRAL CALCULUS

We think of integration as the inverse of differentiation. As an example, consider the expression

$$f(x) = \frac{dy}{dx} = 3ax^2 + b \qquad \textbf{(B.34)}$$

which was the result of differentiating the function

$$y(x) = ax^3 + bx + c$$

TABLE B.4
Derivatives for Several Functions

$\frac{d}{dx}(a) = 0$

$\frac{d}{dx}(ax^n) = nax^{n-1}$

$\frac{d}{dx}(e^{ax}) = ae^{ax}$

$\frac{d}{dx}(\sin ax) = a\cos ax$

$\frac{d}{dx}(\cos ax) = -a\sin ax$

$\frac{d}{dx}(\tan ax) = a\sec^2 ax$

$\frac{d}{dx}(\cot ax) = -a\csc^2 ax$

$\frac{d}{dx}(\sec x) = \tan x\sec x$

$\frac{d}{dx}(\csc x) = -\cot x\csc x$

$\frac{d}{dx}(\ln ax) = \frac{1}{x}$

Note: The letters a and n are constants.

in Example 4. We can write Equation B.34 as $dy = f(x)\,dx = (3ax^2 + b)\,dx$ and obtain $y(x)$ by "summing" over all values of x. Mathematically, we write this inverse operation

$$y(x) = \int f(x)\,dx$$

For the function $f(x)$ given by Equation B.34, we have

$$y(x) = \int (3ax^2 + b)\,dx = ax^3 + bx + c$$

where c is a constant of the integration. This type of integral is called an *indefinite integral* because its value depends on the choice of c.

A general **indefinite integral** $I(x)$ is defined as

$$I(x) = \int f(x)\,dx \qquad \textbf{(B.35)}$$

where $f(x)$ is called the *integrand* and $f(x) = \frac{dI(x)}{dx}$.

For a *general continuous* function $f(x)$, the integral can be described as the area under the curve bounded by $f(x)$ and the x axis, between two specified values of x, say, x_1 and x_2, as in Figure B.14.

The area of the blue element is approximately $f(x_i)\Delta x_i$. If we sum all these area elements from x_1 and x_2 and take the limit of this sum as $\Delta x_i \rightarrow 0$, we obtain the *true* area under the curve bounded by $f(x)$ and x, between the limits x_1 and x_2:

$$\text{Area} = \lim_{\Delta x_i \to 0} \sum_i f(x_i)\,\Delta x_i = \int_{x_1}^{x_2} f(x)\,dx \qquad \textbf{(B.36)}$$

Integrals of the type defined by Equation B.36 are called **definite integrals.**

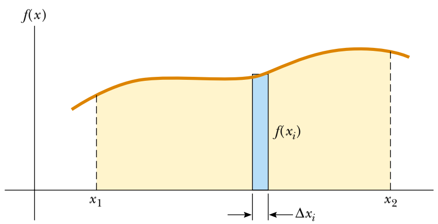


Figure B.14

One common integral that arises in practical situations has the form

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1) \qquad \textbf{(B.37)}$$

This result is obvious, being that differentiation of the right-hand side with respect to x gives $f(x) = x^n$ directly. If the limits of the integration are known, this integral becomes a *definite integral* and is written

$$\int_{x_1}^{x_2} x^n\,dx = \frac{x_2^{\,n+1} - x_1^{\,n+1}}{n+1} \qquad (n \neq -1) \qquad \textbf{(B.38)}$$

EXAMPLES

1. $\int_0^a x^2\,dx = \frac{x^3}{3}\Big]_0^a = \frac{a^3}{3}$
2. $\int_0^b x^{3/2}\,dx = \frac{x^{5/2}}{5/2}\Big]_0^b = \frac{2}{5}b^{5/2}$
3. $\int_3^5 x\,dx = \frac{x^2}{2}\Big]_3^5 = \frac{5^2 - 3^2}{2} = 8$

Partial Integration

Sometimes it is useful to apply the method of *partial integration* (also called "integrating by parts") to evaluate certain integrals. The method uses the property that

$$\int u\,dv = uv - \int v\,du \qquad \textbf{(B.39)}$$

where u and v are *carefully* chosen so as to reduce a complex integral to a simpler one. In many cases, several reductions have to be made. Consider the function

$$I(x) = \int x^2 e^x\,dx$$

This can be evaluated by integrating by parts twice. First, if we choose $u = x^2$, $v = e^x$, we get

$$\int x^2 e^x\,dx = \int x^2\,d(e^x) = x^2 e^x - 2\int e^x x\,dx + c_1$$

Now, in the second term, choose $u = x$, $v = e^x$, which gives

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2\int e^x\,dx + c_1$$

or

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + c_2$$

The Perfect Differential

Another useful method to remember is the use of the *perfect differential,* in which we look for a change of variable such that the differential of the function is the differential of the independent variable appearing in the integrand. For example, consider the integral

$$I(x) = \int \cos^2 x\sin x\,dx$$

This becomes easy to evaluate if we rewrite the differential as $d(\cos x) = -\sin x\,dx$. The integral then becomes

$$\int \cos^2 x\sin x\,dx = -\int \cos^2 x\,d(\cos x)$$

If we now change variables, letting $y = \cos x$, we obtain

$$\int \cos^2 x\sin x\,dx = -\int y^2 dy = -\frac{y^3}{3} + c = -\frac{\cos^3 x}{3} + c$$

Table B.5 lists some useful indefinite integrals. Table B.6 gives Gauss's probability integral and other definite integrals. A more complete list can be found in various handbooks, such as *The Handbook of Chemistry and Physics,* CRC Press.

TABLE B.5 Some Indefinite Integrals (An arbitrary constant should be added to each of these integrals.)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (\text{provided } n \neq -1)$$

$$\int \frac{dx}{x} = \int x^{-1}\,dx = \ln x$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$$

$$\int \frac{x\,dx}{a+bx} = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$$

$$\int \frac{dx}{x(x+a)} = -\frac{1}{a}\ln\frac{x+a}{x}$$

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} \qquad (a^2-x^2>0)$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} \qquad (x^2-a^2>0)$$

$$\int \frac{x\,dx}{a^2\pm x^2} = \pm\tfrac{1}{2}\ln(a^2\pm x^2)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a} \qquad (a^2-x^2>0)$$

$$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln(x+\sqrt{x^2\pm a^2})$$

$$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$$

$$\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2}$$

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right)$$

$$\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}(a^2-x^2)^{3/2}$$

$$\int \sqrt{x^2\pm a^2}\,dx = \tfrac{1}{2}[x\sqrt{x^2\pm a^2} \pm a^2\ln(x+\sqrt{x^2\pm a^2})]$$

$$\int x(\sqrt{x^2\pm a^2})\,dx = \tfrac{1}{3}(x^2\pm a^2)^{3/2}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int \ln ax\,dx = (x\ln ax) - x$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int \frac{dx}{a+be^{cx}} = \frac{x}{a} - \frac{1}{ac}\ln(a+be^{cx})$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \tan ax\,dx = \frac{1}{a}\ln(\cos ax) = \frac{1}{a}\ln(\sec ax)$$

$$\int \cot ax\,dx = \frac{1}{a}\ln(\sin ax)$$

$$\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\left[\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right]$$

$$\int \csc ax\,dx = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\left(\tan\frac{ax}{2}\right)$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

$$\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax$$

$$\int \tan^2 ax\,dx = \frac{1}{a}(\tan ax) - x$$

$$\int \cot^2 ax\,dx = -\frac{1}{a}(\cot ax) - x$$

$$\int \sin^{-1} ax\,dx = x(\sin^{-1} ax) + \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \cos^{-1} ax\,dx = x(\cos^{-1} ax) - \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

TABLE B.6 Gauss's Probability Integral and Other Definite Integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

$$I_0 = \int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad (\text{Gauss's probability integral})$$

$$I_1 = \int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a}$$

$$I_2 = \int_0^\infty x^2e^{-ax^2}\,dx = -\frac{dI_0}{da} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$I_3 = \int_0^\infty x^3e^{-ax^2}\,dx = -\frac{dI_1}{da} = \frac{1}{2a^2}$$

$$I_4 = \int_0^\infty x^4e^{-ax^2}\,dx = \frac{d^2I_0}{da^2} = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$

$$I_5 = \int_0^\infty x^5e^{-ax^2}\,dx = \frac{d^2I_1}{da^2} = \frac{1}{a^3}$$

$$\vdots$$

$$I_{2n} = (-1)^n\frac{d^n}{da^n}I_0$$

$$I_{2n+1} = (-1)^n\frac{d^n}{da^n}I_1$$

APPENDIX C • Periodic Table of the Elements

Symbol	Ca 20	Atomic number
Atomic mass †	40.08	
	$4s^2$	Electron configuration

Group I	Group II	Transition elements										Group III	Group IV	Group V	Group VI	Group VII	Group 0
H 1 1.008 0 $1s^1$																H 1 1.008 0 $1s^1$	He 2 4.002 6 $1s^2$
Li 3 6.94 $2s^1$	Be 4 9.012 $2s^2$											B 5 10.81 $2p^1$	C 6 12.011 $2p^2$	N 7 14.007 $2p^3$	O 8 15.999 $2p^4$	F 9 18.998 $2p^5$	Ne 10 20.18 $2p^6$
Na 11 22.99 $3s^1$	Mg 12 24.31 $3s^2$											Al 13 26.98 $3p^1$	Si 14 28.09 $3p^2$	P 15 30.97 $3p^3$	S 16 32.06 $3p^4$	Cl 17 35.453 $3p^5$	Ar 18 39.948 $3p^6$
K 19 39.102 $4s^1$	Ca 20 40.08 $4s^2$	Sc 21 44.96 $3d^14s^2$	Ti 22 47.90 $3d^24s^2$	V 23 50.94 $3d^34s^2$	Cr 24 51.996 $3d^54s^1$	Mn 25 54.94 $3d^54s^2$	Fe 26 55.85 $3d^64s^2$	Co 27 58.93 $3d^74s^2$	Ni 28 58.71 $3d^84s^2$	Cu 29 63.54 $3d^{10}4s^1$	Zn 30 65.37 $3d^{10}4s^2$	Ga 31 69.72 $4p^1$	Ge 32 72.59 $4p^2$	As 33 74.92 $4p^3$	Se 34 78.96 $4p^4$	Br 35 79.91 $4p^5$	Kr 36 83.80 $4p^6$
Rb 37 85.47 $5s^1$	Sr 38 87.62 $5s^2$	Y 39 88.906 $4d^15s^2$	Zr 40 91.22 $4d^25s^2$	Nb 41 92.91 $4d^45s^1$	Mo 42 95.94 $4d^55s^1$	Tc 43 (99) $4d^55s^2$	Ru 44 101.1 $4d^75s^1$	Rh 45 102.91 $4d^85s^1$	Pd 46 106.4 $4d^{10}$	Ag 47 107.87 $4d^{10}5s^1$	Cd 48 112.40 $4d^{10}5s^2$	In 49 114.82 $5p^1$	Sn 50 118.69 $5p^2$	Sb 51 121.75 $5p^3$	Te 52 127.60 $5p^4$	I 53 126.90 $5p^5$	Xe 54 131.30 $5p^6$
Cs 55 132.91 $6s^1$	Ba 56 137.34 $6s^2$	57-71*	Hf 72 178.49 $5d^26s^2$	Ta 73 180.95 $5d^36s^2$	W 74 183.85 $5d^46s^2$	Re 75 186.2 $5d^56s^2$	Os 76 190.2 $5d^66s^2$	Ir 77 192.2 $5d^76s^2$	Pt 78 195.09 $5d^96s^1$	Au 79 196.97 $5d^{10}6s^1$	Hg 80 200.59 $5d^{10}6s^2$	Tl 81 204.37 $6p^1$	Pb 82 207.2 $6p^2$	Bi 83 208.98 $6p^3$	Po 84 (210) $6p^4$	At 85 (218) $6p^5$	Rn 86 (222) $6p^6$
Fr 87 (223) $7s^1$	Ra 88 (226) $7s^2$	89-103**	Rf 104 (261) $6d^27s^2$	Db 105 (262) $6d^37s^2$	Sg 106 (263)	Bh 107 (262)	Hs 108 (265)	Mt 109 (266)	110†† (269)	111†† (272)	112†† (277)		114†† (289)				

Series															
*Lanthanide series	La 57 138.91 $5d^16s^2$	Ce 58 140.12 $5d^14f^16s^2$	Pr 59 140.91 $4f^36s^2$	Nd 60 144.24 $4f^46s^2$	Pm 61 (147) $4f^56s^2$	Sm 62 150.4 $4f^66s^2$	Eu 63 152.0 $4f^76s^2$	Gd 64 157.25 $5d^14f^76s^2$	Tb 65 158.92 $5d^14f^86s^2$	Dy 66 162.50 $4f^{10}6s^2$	Ho 67 164.93 $4f^{11}6s^2$	Er 68 167.26 $4f^{12}6s^2$	Tm 69 168.93 $4f^{13}6s^2$	Yb 70 173.04 $4f^{14}6s^2$	Lu 71 174.97 $5d^14f^{14}6s^2$
**Actinide series	Ac 89 (227) $6d^17s^2$	Th 90 (232) $6d^27s^2$	Pa 91 (231) $5f^26d^17s^2$	U 92 (238) $5f^36d^17s^2$	Np 93 (239) $5f^46d^17s^2$	Pu 94 (239) $5f^66d^07s^2$	Am 95 (243) $5f^76d^07s^2$	Cm 96 (245) $5f^76d^17s^2$	Bk 97 (247) $5f^86d^17s^2$	Cf 98 (249) $5f^{10}6d^07s^2$	Es 99 (254) $5f^{11}6d^07s^2$	Fm 100 (253) $5f^{12}6d^07s^2$	Md 101 (255) $5f^{13}6d^07s^2$	No 102 (255) $6d^07s^2$	Lr 103 (257) $6d^17s^2$

Atomic mass values given are averaged over isotopes in the percentages in which they exist in nature.
† For an unstable element, mass number of the most stable known isotope is given in parentheses.
†† Elements 110, 111, 112, and 114 have not yet been named.
††† For a description of the atomic data, visit **physics.nist.gov/atomic**

APPENDIX D • SI Units

TABLE D.1 SI Units

Base Quantity	SI Base Unit: Name	SI Base Unit: Symbol
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

TABLE D.2 Some Derived SI Units

Quantity	Name	Symbol	Expression in Terms of Base Units	Expression in Terms of Other SI Units
Plane angle	radian	rad	m/m	
Frequency	hertz	Hz	s^{-1}	
Force	newton	N	$kg \cdot m/s^2$	J/m
Pressure	pascal	Pa	$kg/m \cdot s^2$	N/m^2
Energy; work	joule	J	$kg \cdot m^2/s^2$	$N \cdot m$
Power	watt	W	$kg \cdot m^2/s^3$	J/s
Electric charge	coulomb	C	$A \cdot s$	
Electric potential	volt	V	$kg \cdot m^2/A \cdot s^3$	W/A
Capacitance	farad	F	$A^2 \cdot s^4/kg \cdot m^2$	C/V
Electric resistance	ohm	Ω	$kg \cdot m^2/A^2 \cdot s^3$	V/A
Magnetic flux	weber	Wb	$kg \cdot m^2/A \cdot s^2$	$V \cdot s$
Magnetic field intensity	tesla	T	$kg/A \cdot s^2$	
Inductance	henry	H	$kg \cdot m^2/A^2 \cdot s^2$	$T \cdot m^2/A$

APPENDIX E • *Nobel Prizes*

All Nobel Prizes in physics are listed (and marked with a P), as well as relevant Nobel Prizes in Chemistry (C). The key dates for some of the scientific work are supplied; they often antedate the prize considerably.

1901 (P) *Wilhelm Roentgen* for discovering x-rays (1895).

1902 (P) *Hendrik A. Lorentz* for predicting the Zeeman effect and *Pieter Zeeman* for discovering the Zeeman effect, the splitting of spectral lines in magnetic fields.

1903 (P) *Antoine-Henri Becquerel* for discovering radioactivity (1896) and *Pierre* and *Marie Curie* for studying radioactivity.

1904 (P) *Lord Rayleigh* for studying the density of gases and discovering argon.
(C) *William Ramsay* for discovering the inert gas elements helium, neon, xenon, and krypton, and placing them in the periodic table.

1905 (P) *Philipp Lenard* for studying cathode rays, electrons (1898–1899).

1906 (P) *J. J. Thomson* for studying electrical discharge through gases and discovering the electron (1897).

1907 (P) *Albert A. Michelson* for inventing optical instruments and measuring the speed of light (1880s).

1908 (P) *Gabriel Lippmann* for making the first color photographic plate, using interference methods (1891).
(C) *Ernest Rutherford* for discovering that atoms can be broken apart by alpha rays and for studying radioactivity.

1909 (P) *Guglielmo Marconi* and *Carl Ferdinand Braun* for developing wireless telegraphy.

1910 (P) *Johannes D. van der Waals* for studying the equation of state for gases and liquids (1881).

1911 (P) *Wilhelm Wien* for discovering Wien's law giving the peak of a blackbody spectrum (1893).
(C) *Marie Curie* for discovering radium and polonium (1898) and isolating radium.

1912 (P) *Nils Dalén* for inventing automatic gas regulators for lighthouses.

1913 (P) *Heike Kamerlingh Onnes* for the discovery of superconductivity and liquefying helium (1908).

1914 (P) *Max T. F. von Laue* for studying x-rays from their diffraction by crystals, showing that x-rays are electromagnetic waves (1912).
(C) *Theodore W. Richards* for determining the atomic weights of sixty elements, indicating the existence of isotopes.

1915 (P) *William Henry Bragg* and *William Lawrence Bragg*, his son, for studying the diffraction of x-rays in crystals.

1917 (P) *Charles Barkla* for studying atoms by x-ray scattering (1906).

1918 (P) *Max Planck* for discovering energy quanta (1900).

1919 (P) *Johannes Stark*, for discovering the Stark effect, the splitting of spectral lines in electric fields (1913).

1920 (P) *Charles-Édouard Guillaume* for discovering invar, a nickel-steel alloy with low coefficient of expansion.
(C) *Walther Nernst* for studying heat changes in chemical reactions and formulating the third law of thermodynamics (1918).
1921 (P) *Albert Einstein* for explaining the photoelectric effect and for his services to theoretical physics (1905).
(C) *Frederick Soddy* for studying the chemistry of radioactive substances and discovering isotopes (1912).
1922 (P) *Niels Bohr* for his model of the atom and its radiation (1913).
(C) *Francis W. Aston* for using the mass spectrograph to study atomic weights, thus discovering 212 of the 287 naturally occurring isotopes.
1923 (P) *Robert A. Millikan* for measuring the charge on an electron (1911) and for studying the photoelectric effect experimentally (1914).
1924 (P) *Karl M. G. Siegbahn* for his work in x-ray spectroscopy.
1925 (P) *James Franck* and *Gustav Hertz* for discovering the Franck-Hertz effect in electron-atom collisions.
1926 (P) *Jean-Baptiste Perrin* for studying Brownian motion to validate the discontinuous structure of matter and measure the size of atoms.
1927 (P) *Arthur Holly Compton* for discovering the Compton effect on x-rays, their change in wavelength when they collide with matter (1922), and *Charles T. R. Wilson* for inventing the cloud chamber, used to study charged particles (1906).
1928 (P) *Owen W. Richardson* for studying the thermionic effect and electrons emitted by hot metals (1911).
1929 (P) *Louis Victor de Broglie* for discovering the wave nature of electrons (1923).
1930 (P) *Chandrasekhara Venkata Raman* for studying Raman scattering, the scattering of light by atoms and molecules with a change in wavelength (1928).
1932 (P) *Werner Heisenberg* for creating quantum mechanics (1925).
1933 (P) *Erwin Schrödinger* and *Paul A. M. Dirac* for developing wave mechanics (1925) and relativistic quantum mechanics (1927).
(C) *Harold Urey* for discovering heavy hydrogen, deuterium (1931).
1935 (P) *James Chadwick* for discovering the neutron (1932).
(C) *Irène* and *Frédéric Joliot-Curie* for synthesizing new radioactive elements.
1936 (P) *Carl D. Anderson* for discovering the positron in particular and antimatter in general (1932) and *Victor F. Hess* for discovering cosmic rays.
(C) *Peter J. W. Debye* for studying dipole moments and diffraction of x-rays and electrons in gases.
1937 (P) *Clinton Davisson* and *George Thomson* for discovering the diffraction of electrons by crystals, confirming de Broglie's hypothesis (1927).
1938 (P) *Enrico Fermi* for producing the transuranic radioactive elements by neutron irradiation (1934–1937).
1939 (P) *Ernest O. Lawrence* for inventing the cyclotron.
1943 (P) *Otto Stern* for developing molecular-beam studies (1923), and using them to discover the magnetic moment of the proton (1933).
1944 (P) *Isidor I. Rabi* for discovering nuclear magnetic resonance in atomic and molecular beams.
(C) *Otto Hahn* for discovering nuclear fission (1938).
1945 (P) *Wolfgang Pauli* for discovering the exclusion principle (1924).
1946 (P) *Percy W. Bridgman* for studying physics at high pressures.
1947 (P) *Edward V. Appleton* for studying the ionosphere.

1948 (P) *Patrick M. S. Blackett* for studying nuclear physics with cloud-chamber photographs of cosmic-ray interactions.
1949 (P) *Hideki Yukawa* for predicting the existence of mesons (1935).
1950 (P) *Cecil F. Powell* for developing the method of studying cosmic rays with photographic emulsions and discovering new mesons.
1951 (P) *John D. Cockcroft* and *Ernest T. S. Walton* for transmuting nuclei in an accelerator (1932).
(C) *Edwin M. McMillan* for producing neptunium (1940) and *Glenn T. Seaborg* for producing plutonium (1941) and further transuranic elements.
1952 (P) *Felix Bloch* and *Edward Mills Purcell* for discovering nuclear magnetic resonance in liquids and gases (1946).
1953 (P) *Frits Zernike* for inventing the phase-contrast microscope, which uses interference to provide high contrast.
1954 (P) *Max Born* for interpreting the wave function as a probability (1926) and other quantum-mechanical discoveries and *Walther Bothe* for developing the coincidence method to study subatomic particles (1930–1931), producing, in particular, the particle interpreted by Chadwick as the neutron.
1955 (P) *Willis E. Lamb, Jr.,* for discovering the Lamb shift in the hydrogen spectrum (1947) and *Polykarp Kusch* for determining the magnetic moment of the electron (1947).
1956 (P) *John Bardeen, Walter H. Brattain,* and *William Shockley* for inventing the transistor (1956).
1957 (P) *T.-D. Lee* and *C.-N. Yang* for predicting that parity is not conserved in beta decay (1956).
1958 (P) *Pavel A. Čerenkov* for discovering Čerenkov radiation (1935) and *Ilya M. Frank* and *Igor Tamm* for interpreting it (1937).
1959 (P) *Emilio G. Segrè* and *Owen Chamberlain* for discovering the antiproton (1955).
1960 (P) *Donald A. Glaser* for inventing the bubble chamber to study elementary particles (1952).
(C) *Willard Libby* for developing radiocarbon dating (1947).
1961 (P) *Robert Hofstadter* for discovering internal structure in protons and neutrons and *Rudolf L. Mössbauer* for discovering the Mössbauer effect of recoilless gamma-ray emission (1957).
1962 (P) *Lev Davidovich Landau* for studying liquid helium and other condensed matter theoretically.
1963 (P) *Eugene P. Wigner* for applying symmetry principles to elementary-particle theory and *Maria Goeppert Mayer* and *J. Hans D. Jensen* for studying the shell model of nuclei (1947).
1964 (P) *Charles H. Townes, Nikolai G. Basov,* and *Alexandr M. Prokhorov* for developing masers (1951–1952) and lasers.
1965 (P) *Sin-itiro Tomonaga, Julian S. Schwinger,* and *Richard P. Feynman* for developing quantum electrodynamics (1948).
1966 (P) *Alfred Kastler* for his optical methods of studying atomic energy levels.
1967 (P) *Hans Albrecht Bethe* for discovering the routes of energy production in stars (1939).
1968 (P) *Luis W. Alvarez* for discovering resonance states of elementary particles.
1969 (P) *Murray Gell-Mann* for classifying elementary particles (1963).
1970 (P) *Hannes Alfvén* for developing magnetohydrodynamic theory and *Louis Eugène Félix Néel* for discovering antiferromagnetism and ferrimagnetism (1930s).

1971 (P) *Dennis Gabor* for developing holography (1947).
(C) *Gerhard Herzberg* for studying the structure of molecules spectroscopically.

1972 (P) *John Bardeen, Leon N. Cooper,* and *John Robert Schrieffer* for explaining superconductivity (1957).

1973 (P) *Leo Esaki* for discovering tunneling in semiconductors, *Ivar Giaever* for discovering tunneling in superconductors, and *Brian D. Josephson* for predicting the Josephson effect, which involves tunneling of paired electrons (1958–1962).

1974 (P) *Anthony Hewish* for discovering pulsars and *Martin Ryle* for developing radio interferometry.

1975 (P) *Aage N. Bohr, Ben R. Mottelson,* and *James Rainwater* for discovering why some nuclei take asymmetric shapes.

1976 (P) *Burton Richter* and *Samuel C. C. Ting* for discovering the J/psi particle, the first charmed particle (1974).

1977 (P) *John H. Van Vleck, Nevill F. Mott,* and *Philip W. Anderson* for studying solids quantum-mechanically.
(C) *Ilya Prigogine* for extending thermodynamics to show how life could arise in the face of the second law.

1978 (P) *Arno A. Penzias* and *Robert W. Wilson* for discovering the cosmic background radiation (1965) and *Pyotr Kapitsa* for his studies of liquid helium.

1979 (P) *Sheldon L. Glashow, Abdus Salam,* and *Steven Weinberg* for developing the theory that unified the weak and electromagnetic forces (1958–1971).

1980 (P) *Val Fitch* and *James W. Cronin* for discovering CP (charge-parity) violation (1964), which possibly explains the cosmological dominance of matter over antimatter.

1981 (P) *Nicolaas Bloembergen* and *Arthur L. Schawlow* for developing laser spectroscopy and *Kai M. Siegbahn* for developing high-resolution electron spectroscopy (1958).

1982 (P) *Kenneth G. Wilson* for developing a method of constructing theories of phase transitions to analyze critical phenomena.

1983 (P) *William A. Fowler* for theoretical studies of astrophysical nucleosynthesis and *Subramanyan Chandrasekhar* for studying physical processes of importance to stellar structure and evolution, including the prediction of white dwarf stars (1930).

1984 (P) *Carlo Rubbia* for discovering the W and Z particles, verifying the electroweak unification, and *Simon van der Meer,* for developing the method of stochastic cooling of the CERN beam that allowed the discovery (1982–1983).

1985 (P) *Klaus von Klitzing* for the quantized Hall effect, relating to conductivity in the presence of a magnetic field (1980).

1986 (P) *Ernst Ruska* for inventing the electron microscope (1931), and *Gerd Binnig* and *Heinrich Rohrer* for inventing the scanning-tunneling electron microscope (1981).

1987 (P) *J. Georg Bednorz* and *Karl Alex Müller* for the discovery of high temperature superconductivity (1986).

1988 (P) *Leon M. Lederman, Melvin Schwartz,* and *Jack Steinberger* for a collaborative experiment that led to the development of a new tool for studying the weak nuclear force, which affects the radioactive decay of atoms.

1989 (P) *Norman Ramsay* (U.S.) for various techniques in atomic physics; and *Hans Dehmelt* (U.S.) and *Wolfgang Paul* (Germany) for the development of techniques for trapping single charge particles.

1990 (P) *Jerome Friedman, Henry Kendall* (both U.S.), and *Richard Taylor* (Canada) for experiments important to the development of the quark model.

1991 (P) *Pierre-Gilles de Gennes* for discovering that methods developed for studying order phenomena in simple systems can be generalized to more complex forms of matter, in particular to liquid crystals and polymers.

1992 (P) *George Charpak* for developing detectors that trace the paths of evanescent subatomic particles produced in particle accelerators.

1993 (P) *Russell Hulse* and *Joseph Taylor* for discovering evidence of gravitational waves.

1994 (P) *Bertram N. Brockhouse* and *Clifford G. Shull* for pioneering work in neutron scattering.

1995 (P) *Martin L. Perl* and *Frederick Reines* for discovering the tau particle and the neutrino, respectively.

1996 (P) *David M. Lee, Douglas C. Osheroff,* and *Robert C. Richardson* for developing a superfluid using helium-3.

1997 (P) *Steven Chu, Claude Cohen-Tannoudji,* and *William D. Phillips* for developing methods to cool and trap atoms with laser light.

1998 (P) *Robert B. Laughlin, Horst L. Störmer,* and *Daniel C. Tsui* for discovering a new form of quantum fluid with fractionally charged excitations.