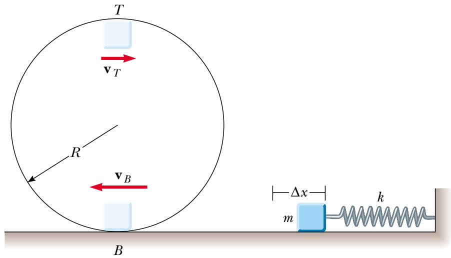


Figure P8.63

65. An object of mass m is suspended from a post on top of a cart by a string of length L as in Figure P8.65a. The cart and object are initially moving to the right at constant speed v_i. The cart comes to rest after colliding and sticking to a bumper as in Figure P8.65b, and the suspended object swings through an angle θ. (a) Show that the speed is $v_i = \sqrt{2gL(1 - \cos\theta)}$. (b) If $L = 1.20$ m and $\theta = 35.0°$, find the initial speed of the cart. (*Hint:* The force exerted by the string on the object does no work on the object.)

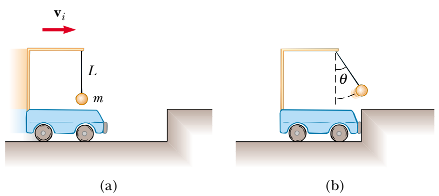


Figure P8.65

66. A child slides without friction from a height h along a curved water slide (Fig. P8.66). She is launched from a height $h/5$ into the pool. Determine her maximum airborne height y in terms of h and θ.

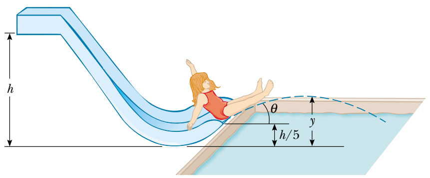


Figure P8.66

67. A ball having mass m is connected by a strong string of length L to a pivot point and held in place in a vertical position. A wind exerting constant force of magnitude F is blowing from left to right as in Figure P8.67a. (a) If the ball is released from rest, show that the maximum height H it reaches, as measured from its initial height, is

$$H = \frac{2L}{1 + (mg/F)^2}$$

Check that the above formula is valid both when $0 \le H \le L$ and when $L \le H \le 2L$. (*Hint:* First determine the potential energy associated with the *constant* wind force.) (b) Compute the value of H using the values $m = 2.00$ kg, $L = 2.00$ m, and $F = 14.7$ N. (c) Using these same values, determine the *equilibrium* height of the ball. (d) Could the equilibrium height ever be greater than L? Explain.

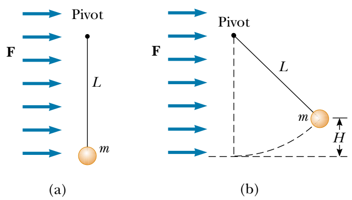


Figure P8.67

68. A ball is tied to one end of a string. The other end of the string is fixed. The ball is set in motion around a vertical circle without friction. At the top of the circle, the ball has a speed of $v_i = \sqrt{Rg}$, as shown in Figure P8.68. At what angle θ should the string be cut so that the ball will travel through the center of the circle?

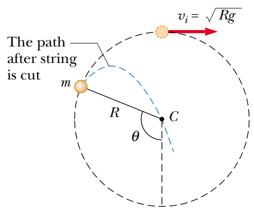


Figure P8.68

69. A ball at the end of a string whirls around in a vertical circle. If the ball's total energy remains constant, show that the tension in the string at the bottom is greater than the tension at the top by a value six times the weight of the ball.

70. A pendulum comprising a string of length L and a sphere swings in the vertical plane. The string hits a peg located a distance d below the point of suspension (Fig. P8.70). (a) Show that if the sphere is released from a height below that of the peg, it will return to this height after striking the peg. (b) Show that if the pendulum is released from the horizontal position ($\theta = 90°$) and is to swing in a complete circle centered on the peg, then the minimum value of d must be $3L/5$.

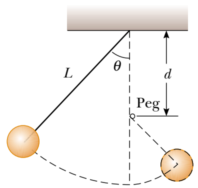


Figure P8.70

71. Jane, whose mass is 50.0 kg, needs to swing across a river (having width D) filled with man-eating crocodiles to save Tarzan from danger. However, she must swing into a wind exerting constant horizontal force $\mathbf{F}$ on a vine having length L and initially making an angle θ with the vertical (Fig. P8.71). Taking $D = 50.0$ m, $F = 110$ N, $L = 40.0$ m, and $\theta = 50.0°$, (a) with what minimum speed must Jane begin her swing to just make it to the other side? (*Hint:* First determine the potential energy associated with the wind force.) (b) Once the rescue is complete, Tarzan and Jane must swing back across the river. With what minimum speed must they begin their swing? Assume that Tarzan has a mass of 80.0 kg.

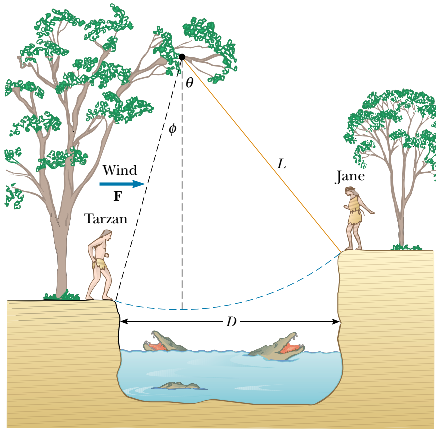


Figure P8.71

72. A child starts from rest and slides down the frictionless slide shown in Figure P8.72. In terms of R and H, at what height h will he lose contact with the section of radius R?

Figure P8.72

73. A 5.00-kg block free to move on a horizontal, frictionless surface is attached to one end of a light horizontal spring. The other end of the spring is fixed. The spring is compressed 0.100 m from equilibrium and is then released. The speed of the block is 1.20 m/s when it passes the equilibrium position of the spring. The same experiment is now repeated with the frictionless surface replaced by a surface for which $\mu_k = 0.300$. Determine the speed of the block at the equilibrium position of the spring.

74. A 50.0-kg block and a 100-kg block are connected by a string as in Figure P8.74. The pulley is frictionless and of negligible mass. The coefficient of kinetic friction between the 50.0-kg block and the incline is $\mu_k = 0.250$. Determine the change in the kinetic energy of the 50.0-kg block as it moves from Ⓐ to Ⓑ, a distance of 20.0 m.

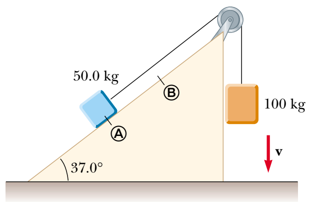


Figure P8.74

Answers to Quick Quizzes

8.1 Yes, because we are free to choose any point whatsoever as our origin of coordinates, which is the $U_g = 0$ point. If the object is below the origin of coordinates that we choose, then $U_g < 0$ for the object–Earth system.

8.2 Yes, the total mechanical energy of the system is conserved because the only forces acting are conservative: the force of gravity and the spring force. There are two forms of potential energy: (1) gravitational potential energy and (2) elastic potential energy stored in the spring.

8.3 The first and third balls speed up after they are thrown, while the second ball initially slows down but then speeds up after reaching its peak. The paths of all three balls are parabolas, and the balls take different times to reach the ground because they have different initial velocities. However, all three balls have the same speed at the moment they hit the ground because all start with the same kinetic energy and undergo the same change in gravitational potential energy. In other words, $E_{\text{total}} = \frac{1}{2}mv^2 + mgh$ is the same for all three balls at the start of the motion.

8.4 Designate one object as No. 1 and the other as No. 2. The external force does work W_{app} on the system. If $W_{\text{app}} > 0$, then the system energy increases. If $W_{\text{app}} < 0$, then the system energy decreases. The effect of friction is to decrease the total system energy. Equation 8.15 then becomes

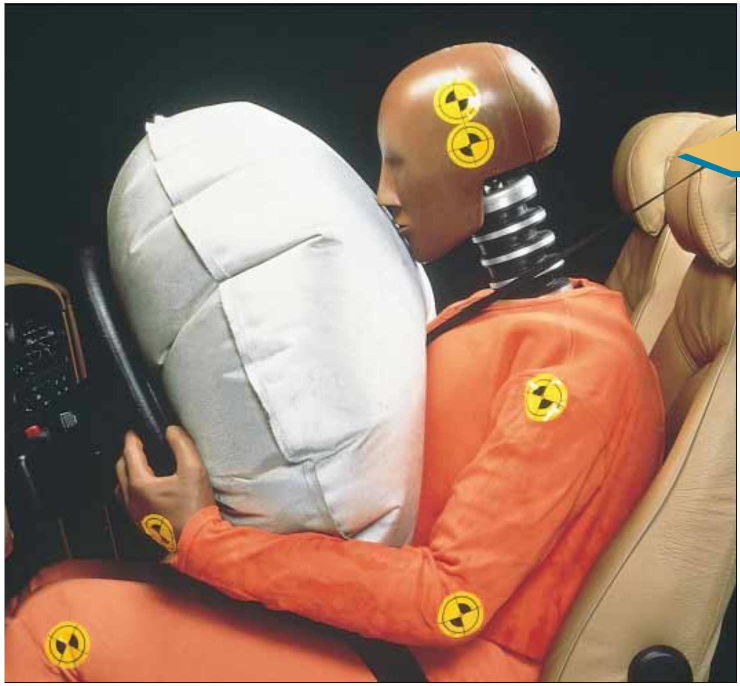
$$\begin{aligned}\Delta E &= W_{\text{app}} - \Delta E_{\text{friction}} \\ &= \Delta K + \Delta U \\ &= [K_{1f} + K_{2f}) - (K_{1i} + K_{2i})] \\ &\quad + [(U_{g1f} + U_{g2f} + U_{sf}) - (U_{g1i} + U_{g2i} + U_{si})]\end{aligned}$$

You may find it easier to think of this equation with its terms in a different order, saying

total initial energy + net change = total final energy

$$K_{1i} + K_{2i} + U_{g1i} + U_{g2i} + U_{si} + W_{\text{app}} - f_k d = K_{1f} + K_{2f} + U_{g1f} + U_{g2f} + U_{sf}$$

8.5 The slope of a $U(x)$-versus-x graph is by definition $dU(x)/dx$. From Equation 8.16, we see that this expression is equal to the negative of the x component of the conservative force acting on an object that is part of the system.



PUZZLER

Airbags have saved countless lives by reducing the forces exerted on vehicle occupants during collisions. How can airbags change the force needed to bring a person from a high speed to a complete stop? Why are they usually safer than seat belts alone? *(Courtesy of Saab)*

chapter 9

Linear Momentum and Collisions

Chapter Outline

Consider what happens when a golf ball is struck by a club. The ball is given a very large initial velocity as a result of the collision; consequently, it is able to travel more than 100 m through the air. The ball experiences a large acceleration. Furthermore, because the ball experiences this acceleration over a very short time interval, the average force exerted on it during the collision is very great. According to Newton's third law, the ball exerts on the club a reaction force that is equal in magnitude to and opposite in direction to the force exerted by the club on the ball. This reaction force causes the club to accelerate. Because the club is much more massive than the ball, however, the acceleration of the club is much less than the acceleration of the ball.

One of the main objectives of this chapter is to enable you to understand and analyze such events. As a first step, we introduce the concept of *momentum,* which is useful for describing objects in motion and as an alternate and more general means of applying Newton's laws. For example, a very massive football player is often said to have a great deal of momentum as he runs down the field. A much less massive player, such as a halfback, can have equal or greater momentum if his speed is greater than that of the more massive player. This follows from the fact that momentum is defined as the product of mass and velocity. The concept of momentum leads us to a second conservation law, that of conservation of momentum. This law is especially useful for treating problems that involve collisions between objects and for analyzing rocket propulsion. The concept of the center of mass of a system of particles also is introduced, and we shall see that the motion of a system of particles can be described by the motion of one representative particle located at the center of mass.

9.1 LINEAR MOMENTUM AND ITS CONSERVATION

In the preceding two chapters we studied situations too complex to analyze easily with Newton's laws. In fact, Newton himself used a form of his second law slightly different from $\Sigma \mathbf{F} = m\mathbf{a}$ (Eq. 5.2)—a form that is considerably easier to apply in complicated circumstances. Physicists use this form to study everything from subatomic particles to rocket propulsion. In studying situations such as these, it is often useful to know both something about the object and something about its motion. We start by defining a new term that incorporates this information:

Definition of linear momentum of a particle

The **linear momentum** of a particle of mass m moving with a velocity $\mathbf{v}$ is defined to be the product of the mass and velocity:

$$\mathbf{p} \equiv m\mathbf{v} \tag{9.1}$$

6.2

Linear momentum is a vector quantity because it equals the product of a scalar quantity m and a vector quantity $\mathbf{v}$. Its direction is along $\mathbf{v}$, it has dimensions ML/T, and its SI unit is kg · m/s.

If a particle is moving in an arbitrary direction, $\mathbf{p}$ must have three components, and Equation 9.1 is equivalent to the component equations

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \tag{9.2}$$

As you can see from its definition, the concept of momentum provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of a bowling ball moving at 10 m/s is much greater than that of a tennis ball moving at the same speed. Newton called the product $m\mathbf{v}$

quantity of motion; this is perhaps a more graphic description than our present-day word *momentum,* which comes from the Latin word for movement.

Quick Quiz 9.1

Two objects have equal kinetic energies. How do the magnitudes of their momenta compare? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) not enough information to tell.

Newton's second law for a particle

Using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle: **The time rate of change of the linear momentum of a particle is equal to the net force acting on the particle:**

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} \tag{9.3}$$

In addition to situations in which the velocity vector varies with time, we can use Equation 9.3 to study phenomena in which the mass changes. The real value of Equation 9.3 as a tool for analysis, however, stems from the fact that when the net force acting on a particle is zero, the time derivative of the momentum of the particle is zero, and therefore its linear momentum[1] is constant. Of course, if the particle is *isolated,* then by necessity $\sum \mathbf{F} = 0$ and $\mathbf{p}$ remains unchanged. This means that $\mathbf{p}$ is conserved. Just as the law of conservation of energy is useful in solving complex motion problems, the law of conservation of momentum can greatly simplify the analysis of other types of complicated motion.

Conservation of Momentum for a Two-Particle System

6.2 Consider two particles 1 and 2 that can interact with each other but are isolated from their surroundings (Fig. 9.1). That is, the particles may exert a force on each other, but no external forces are present. It is important to note the impact of Newton's third law on this analysis. If an internal force from particle 1 (for example, a gravitational force) acts on particle 2, then there must be a second internal force—equal in magnitude but opposite in direction—that particle 2 exerts on particle 1.

Suppose that at some instant, the momentum of particle 1 is $\mathbf{p}_1$ and that of particle 2 is $\mathbf{p}_2$. Applying Newton's second law to each particle, we can write

$$\mathbf{F}_{21} = \frac{d\mathbf{p}_1}{dt} \qquad \text{and} \qquad \mathbf{F}_{12} = \frac{d\mathbf{p}_2}{dt}$$

where $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1 and $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2. Newton's third law tells us that $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are equal in magnitude and opposite in direction. That is, they form an action–reaction pair $\mathbf{F}_{12} = -\mathbf{F}_{21}$. We can express this condition as

$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

or as

$$\frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

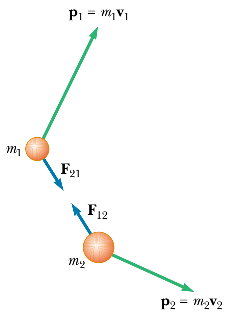


Figure 9.1 At some instant, the momentum of particle 1 is $\mathbf{p}_1 = m_1\mathbf{v}_1$ and the momentum of particle 2 is $\mathbf{p}_2 = m_2\mathbf{v}_2$. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. The total momentum of the system $\mathbf{p}_{\text{tot}}$ is equal to the vector sum $\mathbf{p}_1 + \mathbf{p}_2$.

[1]In this chapter, the terms *momentum* and *linear momentum* have the same meaning. Later, in Chapter 11, we shall use the term *angular momentum* when dealing with rotational motion.

Because the time derivative of the total momentum $\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2$ is *zero,* we conclude that the *total* momentum of the system must remain constant:

$$\mathbf{p}_{\text{tot}} = \sum_{\text{system}} \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \tag{9.4}$$

or, equivalently,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{9.5}$$

where $\mathbf{p}_{1i}$ and $\mathbf{p}_{2i}$ are the initial values and $\mathbf{p}_{1f}$ and $\mathbf{p}_{2f}$ the final values of the momentum during the time interval dt over which the reaction pair interacts. Equation 9.5 in component form demonstrates that the total momenta in the x, y, and z directions are all independently conserved:

$$\sum_{\text{system}} p_{ix} = \sum_{\text{system}} p_{fx} \qquad \sum_{\text{system}} p_{iy} = \sum_{\text{system}} p_{fy} \qquad \sum_{\text{system}} p_{iz} = \sum_{\text{system}} p_{fz} \tag{9.6}$$

This result, known as the **law of conservation of linear momentum,** can be extended to any number of particles in an isolated system. It is considered one of the most important laws of mechanics. We can state it as follows:

Conservation of momentum

> Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

This law tells us that **the total momentum of an isolated system at all times equals its initial momentum.**

Notice that we have made no statement concerning the nature of the forces acting on the particles of the system. The only requirement is that the forces must be *internal* to the system.

Quick Quiz 9.2

Your physical education teacher throws a baseball to you at a certain speed, and you catch it. The teacher is next going to throw you a medicine ball whose mass is ten times the mass of the baseball. You are given the following choices: You can have the medicine ball thrown with (a) the same speed as the baseball, (b) the same momentum, or (c) the same kinetic energy. Rank these choices from easiest to hardest to catch.

EXAMPLE 9.1 The Floating Astronaut

A SkyLab astronaut discovered that while concentrating on writing some notes, he had gradually floated to the middle of an open area in the spacecraft. Not wanting to wait until he floated to the opposite side, he asked his colleagues for a push. Laughing at his predicament, they decided not to help, and so he had to take off his uniform and throw it in one direction so that he would be propelled in the opposite direction. Estimate his resulting velocity.

Figure 9.2 A hapless astronaut has discarded his uniform to get somewhere.

Solution We begin by making some reasonable guesses of relevant data. Let us assume we have a 70-kg astronaut who threw his 1-kg uniform at a speed of 20 m/s. For convenience, we set the positive direction of the x axis to be the direction of the throw (Fig. 9.2). Let us also assume that the x axis is tangent to the circular path of the spacecraft.

We take the system to consist of the astronaut and the uniform. Because of the gravitational force (which keeps the astronaut, his uniform, and the entire spacecraft in orbit), the system is not really isolated. However, this force is directed perpendicular to the motion of the system. Therefore, momentum is constant in the x direction because there are no external forces in this direction.

The total momentum of the system before the throw is zero ($m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = 0$). Therefore, the total momentum after the throw must be zero; that is,

$$m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = 0$$

With $m_1 = 70$ kg, $\mathbf{v}_{2f} = 20\mathbf{i}$ m/s, and $m_2 = 1$ kg, solving for $\mathbf{v}_{1f}$, we find the recoil velocity of the astronaut to be

$$\mathbf{v}_{1f} = -\frac{m_2}{m_1}\mathbf{v}_{2f} = -\left(\frac{1\text{ kg}}{70\text{ kg}}\right)(20\mathbf{i}\text{ m/s}) = \boxed{-0.3\mathbf{i}\text{ m/s}}$$

The negative sign for $\mathbf{v}_{1f}$ indicates that the astronaut is moving to the left after the throw, in the direction opposite the direction of motion of the uniform, in accordance with Newton's third law. Because the astronaut is much more massive than his uniform, his acceleration and consequent velocity are much smaller than the acceleration and velocity of the uniform.

EXAMPLE 9.2 Breakup of a Kaon at Rest

One type of nuclear particle, called the *neutral kaon* (K^0), breaks up into a pair of other particles called *pions* (π^+ and π^-) that are oppositely charged but equal in mass, as illustrated in Figure 9.3. Assuming the kaon is initially at rest, prove that the two pions must have momenta that are equal in magnitude and opposite in direction.

Solution The breakup of the kaon can be written

$$K^0 \longrightarrow \pi^+ + \pi^-$$

If we let $\mathbf{p}^+$ be the momentum of the positive pion and $\mathbf{p}^-$ the momentum of the negative pion, the final momentum of the system consisting of the two pions can be written

$$\mathbf{p}_f = \mathbf{p}^+ + \mathbf{p}^-$$

Because the kaon is at rest before the breakup, we know that $\mathbf{p}_i = 0$. Because momentum is conserved, $\mathbf{p}_i = \mathbf{p}_f = 0$, so that $\mathbf{p}^+ + \mathbf{p}^- = 0$, or

$$\mathbf{p}^+ = -\mathbf{p}^-$$

The important point behind this problem is that even though it deals with objects that are very different from those in the preceding example, the physics is identical: Linear momentum is conserved in an isolated system.

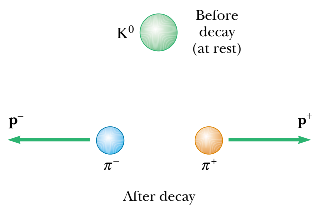


Figure 9.3 A kaon at rest breaks up spontaneously into a pair of oppositely charged pions. The pions move apart with momenta that are equal in magnitude but opposite in direction.

9.2 IMPULSE AND MOMENTUM

6.3 & 6.4

As we have seen, the momentum of a particle changes if a net force acts on the particle. Knowing the change in momentum caused by a force is useful in solving some types of problems. To begin building a better understanding of this important concept, let us assume that a single force $\mathbf{F}$ acts on a particle and that this force may vary with time. According to Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$, or

$$d\mathbf{p} = \mathbf{F}\,dt \tag{9.7}$$

We can integrate[2] this expression to find the change in the momentum of a particle when the force acts over some time interval. If the momentum of the particle

[2]Note that here we are integrating force with respect to time. Compare this with our efforts in Chapter 7, where we integrated force with respect to position to express the work done by the force.

changes from $\mathbf{p}_i$ at time t_i to $\mathbf{p}_f$ at time t_f, integrating Equation 9.7 gives

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f}\mathbf{F}\,dt \tag{9.8}$$

To evaluate the integral, we need to know how the force varies with time. The quantity on the right side of this equation is called the **impulse** of the force $\mathbf{F}$ acting on a particle over the time interval $\Delta t = t_f - t_i$. Impulse is a vector defined by

Impulse of a force

$$\mathbf{I} \equiv \int_{t_i}^{t_f}\mathbf{F}\,dt = \Delta\mathbf{p} \tag{9.9}$$

Impulse–momentum theorem

The impulse of the force $\mathbf{F}$ acting on a particle equals the change in the momentum of the particle caused by that force.

This statement, known as the **impulse–momentum theorem,**[3] is equivalent to Newton's second law. From this definition, we see that impulse is a vector quantity having a magnitude equal to the area under the force–time curve, as described in Figure 9.4a. In this figure, it is assumed that the force varies in time in the general manner shown and is nonzero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is the same as the direction of the change in momentum. Impulse has the dimensions of momentum—that is, ML/T. Note that impulse is *not* a property of a particle; rather, it is a measure of the degree to which an external force changes the momentum of the particle. Therefore, when we say that an impulse is given to a particle, we mean that momentum is transferred from an external agent to that particle.

Because the force imparting an impulse can generally vary in time, it is convenient to define a time-averaged force

$$\overline{\mathbf{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f}\mathbf{F}\,dt \tag{9.10}$$

where $\Delta t = t_f - t_i$. (This is an application of the mean value theorem of calculus.) Therefore, we can express Equation 9.9 as

$$\mathbf{I} \equiv \overline{\mathbf{F}}\,\Delta t \tag{9.11}$$

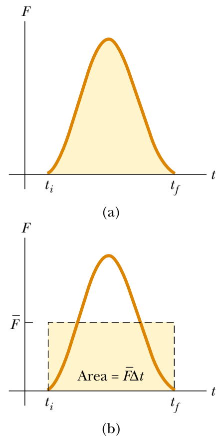


Figure 9.4 (a) A force acting on a particle may vary in time. The impulse imparted to the particle by the force is the area under the force versus time curve. (b) In the time interval Δt, the time-averaged force (horizontal dashed line) gives the same impulse to a particle as does the time-varying force described in part (a).

This time-averaged force, described in Figure 9.4b, can be thought of as the constant force that would give to the particle in the time interval Δt the same impulse that the time-varying force gives over this same interval.

In principle, if $\mathbf{F}$ is known as a function of time, the impulse can be calculated from Equation 9.9. The calculation becomes especially simple if the force acting on the particle is constant. In this case, $\overline{\mathbf{F}} = \mathbf{F}$ and Equation 9.11 becomes

$$\mathbf{I} = \mathbf{F}\,\Delta t \tag{9.12}$$

In many physical situations, we shall use what is called the **impulse approximation, in which we assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present.** This approximation is especially useful in treating collisions in which the duration of the

[3]Although we assumed that only a single force acts on the particle, the impulse–momentum theorem is valid when several forces act; in this case, we replace $\mathbf{F}$ in Equation 9.9 with $\Sigma\mathbf{F}$.

During the brief time the club is in contact with the ball, the ball gains momentum as a result of the collision, and the club loses the same amount of momentum.

collision is very short. When this approximation is made, we refer to the force as an *impulsive force*. For example, when a baseball is struck with a bat, the time of the collision is about 0.01 s and the average force that the bat exerts on the ball in this time is typically several thousand newtons. Because this is much greater than the magnitude of the gravitational force, the impulse approximation justifies our ignoring the weight of the ball and bat. When we use this approximation, it is important to remember that $\mathbf{p}_i$ and $\mathbf{p}_f$ represent the momenta *immediately* before and after the collision, respectively. Therefore, in any situation in which it is proper to use the impulse approximation, the particle moves very little during the collision.

QuickLab

If you can find someone willing, play catch with an egg. What is the best way to move your hands so that the egg does not break when you change its momentum to zero?

Quick Quiz 9.3

Two objects are at rest on a frictionless surface. Object 1 has a greater mass than object 2. When a force is applied to object 1, it accelerates through a distance d. The force is removed from object 1 and is applied to object 2. At the moment when object 2 has accelerated through the same distance d, which statements are true? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) $K_1 < K_2$, (e) $K_1 = K_2$, (f) $K_1 > K_2$.

EXAMPLE 9.3 Teeing Off

A golf ball of mass 50 g is struck with a club (Fig. 9.5). The force exerted on the ball by the club varies from zero, at the instant before contact, up to some maximum value (at which the ball is deformed) and then back to zero when the ball leaves the club. Thus, the force–time curve is qualitatively described by Figure 9.4. Assuming that the ball travels 200 m, estimate the magnitude of the impulse caused by the collision.

Solution Let us use Ⓐ to denote the moment when the club first contacts the ball, Ⓑ to denote the moment when the club loses contact with the ball as the ball starts on its trajectory, and Ⓒ to denote its landing. Neglecting air resistance, we can use Equation 4.14 for the range of a projectile:

$$R = x_{\text{C}} = \frac{{v_{\text{B}}}^2}{g} \sin 2\theta_{\text{B}}$$

Let us assume that the launch angle θ_{B} is 45°, the angle that provides the maximum range for any given launch velocity. This assumption gives $\sin 2\theta_{\text{B}} = 1$, and the launch velocity of

the ball is

$$v_{\text{B}} = \sqrt{x_{\text{C}} g} = \sqrt{(200 \text{ m})(9.80 \text{ m/s}^2)} = 44 \text{ m/s}$$

Considering the time interval for the collision, $v_i = v_{\text{A}} = 0$ and $v_f = v_{\text{B}}$ for the ball. Hence, the magnitude of the impulse imparted to the ball is

$$I = \Delta p = mv_{\text{B}} - mv_{\text{A}} = (50 \times 10^{-3} \text{ kg})(44 \text{ m/s}) - 0$$
$$= 2.2 \text{ kg}\cdot\text{m/s}$$

Exercise If the club is in contact with the ball for a time of 4.5×10^{-4} s, estimate the magnitude of the average force exerted by the club on the ball.

Answer 4.9×10^3 N, a value that is extremely large when compared with the weight of the ball, 0.49 N.

Figure 9.5 A golf ball being struck by a club. *(© Harold E. Edgerton/ Courtesy of Palm Press, Inc.)*

EXAMPLE 9.4 How Good Are the Bumpers?

In a particular crash test, an automobile of mass 1 500 kg collides with a wall, as shown in Figure 9.6. The initial and final velocities of the automobile are $\mathbf{v}_i = -15.0\mathbf{i}$ m/s and $\mathbf{v}_f = 2.60\mathbf{i}$ m/s, respectively. If the collision lasts for 0.150 s, find the impulse caused by the collision and the average force exerted on the automobile.

Solution Let us assume that the force exerted on the car by the wall is large compared with other forces on the car so that we can apply the impulse approximation. Furthermore, we note that the force of gravity and the normal force exerted by the road on the car are perpendicular to the motion and therefore do not affect the horizontal momentum.

The initial and final momenta of the automobile are

$$\mathbf{p}_i = m\mathbf{v}_i = (1\,500 \text{ kg})(-15.0\mathbf{i} \text{ m/s}) = -2.25 \times 10^4\mathbf{i} \text{ kg}\cdot\text{m/s}$$

$$\mathbf{p}_f = m\mathbf{v}_f = (1\,500 \text{ kg})(2.60\mathbf{i} \text{ m/s}) = 0.39 \times 10^4\mathbf{i} \text{ kg}\cdot\text{m/s}$$

Hence, the impulse is

$$\mathbf{I} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = 0.39 \times 10^4\mathbf{i} \text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\mathbf{i} \text{ kg}\cdot\text{m/s})$$

$$\mathbf{I} = 2.64 \times 10^4\mathbf{i} \text{ kg}\cdot\text{m/s}$$

The average force exerted on the automobile is

$$\overline{\mathbf{F}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{2.64 \times 10^4\mathbf{i} \text{ kg}\cdot\text{m/s}}{0.150 \text{ s}} = 1.76 \times 10^5\mathbf{i} \text{ N}$$

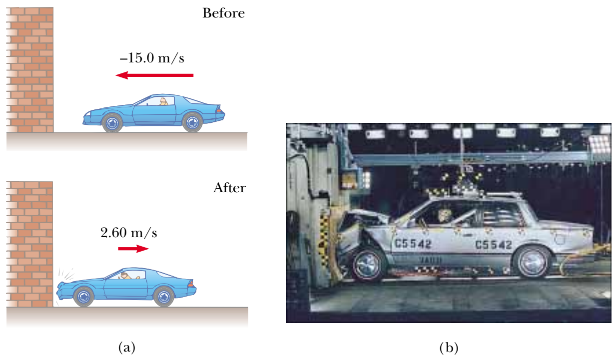


Figure 9.6 (a) This car's momentum changes as a result of its collision with the wall. (b) In a crash test, much of the car's initial kinetic energy is transformed into energy used to damage the car.

Note that the magnitude of this force is large compared with the weight of the car ($mg = 1.47 \times 10^4$ N), which justifies our initial assumption. Of note in this problem is how the signs of the velocities indicated the reversal of directions. What would the mathematics be describing if both the initial and final velocities had the same sign?

Quick Quiz 9.4

Rank an automobile dashboard, seatbelt, and airbag in terms of (a) the impulse and (b) the average force they deliver to a front-seat passenger during a collision.

9.3 COLLISIONS

6.5 & 6.6

In this section we use the law of conservation of linear momentum to describe what happens when two particles collide. We use the term **collision** to represent the event of two particles' coming together for a short time and thereby producing impulsive forces on each other. **These forces are assumed to be much greater than any external forces present.**

A collision may entail physical contact between two macroscopic objects, as described in Figure 9.7a, but the notion of what we mean by collision must be generalized because "physical contact" on a submicroscopic scale is ill-defined and hence meaningless. To understand this, consider a collision on an atomic scale (Fig. 9.7b), such as the collision of a proton with an alpha particle (the nucleus of a helium atom). Because the particles are both positively charged, they never come into physical contact with each other; instead, they repel each other because of the strong electrostatic force between them at close separations. When two particles 1 and 2 of masses m_1 and m_2 collide as shown in Figure 9.7, the impulsive forces may vary in time in complicated ways, one of which is described in Figure 9.8. If $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1, and if we assume that no external forces act on the particles, then the change in momentum of particle 1 due to the collision is given by Equation 9.8:

$$\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_{21}\, dt$$

Likewise, if $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2, then the change in momentum of particle 2 is

$$\Delta \mathbf{p}_2 = \int_{t_i}^{t_f} \mathbf{F}_{12}\, dt$$

From Newton's third law, we conclude that

$$\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2$$

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0$$

Because the total momentum of the system is $\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2$, we conclude that the *change* in the momentum of the system due to the collision is zero:

$$\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

This is precisely what we expect because no external forces are acting on the system (see Section 9.2). Because the impulsive forces are internal, they do not change the total momentum of the system (only external forces can do that).

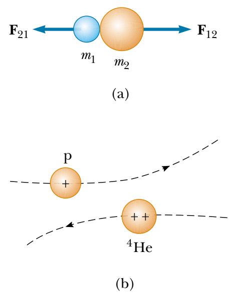


Figure 9.7 (a) The collision between two objects as the result of direct contact. (b) The "collision" between two charged particles.

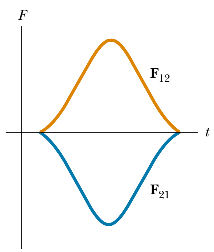


Figure 9.8 The impulse force as a function of time for the two colliding particles described in Figure 9.7a. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Momentum is conserved for any collision

Therefore, we conclude that **the total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.**

EXAMPLE 9.5 Carry Collision Insurance!

A car of mass 1800 kg stopped at a traffic light is struck from the rear by a 900-kg car, and the two become entangled. If the smaller car was moving at 20.0 m/s before the collision, what is the velocity of the entangled cars after the collision?

Solution We can guess that the final speed is less than 20.0 m/s, the initial speed of the smaller car. The total momentum of the system (the two cars) before the collision must equal the total momentum immediately after the collision because momentum is conserved in any type of collision. The magnitude of the total momentum before the collision is equal to that of the smaller car because the larger car is initially at rest:

$$p_i = m_1 v_{1i} = (900 \text{ kg})(20.0 \text{ m/s}) = 1.80 \times 10^4 \text{ kg}\cdot\text{m/s}$$

After the collision, the magnitude of the momentum of the entangled cars is

$$p_f = (m_1 + m_2)v_f = (2\,700 \text{ kg})v_f$$

Equating the momentum before to the momentum after and solving for v_f, the final velocity of the entangled cars, we have

$$v_f = \frac{p_i}{m_1 + m_2} = \frac{1.80 \times 10^4 \text{ kg}\cdot\text{m/s}}{2\,700 \text{ kg}} = \boxed{6.67 \text{ m/s}}$$

The direction of the final velocity is the same as the velocity of the initially moving car.

Exercise What would be the final speed if the two cars each had a mass of 900 kg?

Answer 10.0 m/s.

When the bowling ball and pin collide, part of the ball's momentum is transferred to the pin. Consequently, the pin acquires momentum and kinetic energy, and the ball loses momentum and kinetic energy. However, the total momentum of the system (ball and pin) remains constant.

Quick Quiz 9.5

As a ball falls toward the Earth, the ball's momentum increases because its speed increases. Does this mean that momentum is not conserved in this situation?

Quick Quiz 9.6

A skater is using very low-friction rollerblades. A friend throws a Frisbee straight at her. In which case does the Frisbee impart the greatest impulse to the skater: (a) she catches the Frisbee and holds it, (b) she catches it momentarily but drops it, (c) she catches it and at once throws it back to her friend?

9.4 ELASTIC AND INELASTIC COLLISIONS IN ONE DIMENSION

As we have seen, momentum is conserved in any collision in which external forces are negligible. In contrast, kinetic energy may or may not be constant, depending on the type of collision. In fact, whether or not kinetic energy is the same before and after the collision is used to classify collisions as being either elastic or inelastic.

Elastic collision

An **elastic collision** between two objects is one in which *total kinetic energy (as well as total momentum) is the same before and after the collision.* Billiard-ball collisions and the collisions of air molecules with the walls of a container at ordinary temperatures are approximately elastic. Truly elastic collisions do occur, however, between atomic and subatomic particles. Collisions between certain objects in the macroscopic world, such as billiard-ball collisions, are only approximately elastic because some deformation and loss of kinetic energy take place.

An **inelastic collision** is one in which *total kinetic energy is not the same before and after the collision (even though momentum is constant).* Inelastic collisions are of two types. When the colliding objects stick together after the collision, as happens when a meteorite collides with the Earth, the collision is called **perfectly inelastic.** When the colliding objects do not stick together, but some kinetic energy is lost, as in the case of a rubber ball colliding with a hard surface, the collision is called **inelastic** (with no modifying adverb). For example, when a rubber ball collides with a hard surface, the collision is inelastic because some of the kinetic energy of the ball is lost when the ball is deformed while it is in contact with the surface.

Inelastic collision

In most collisions, kinetic energy is *not* the same before and after the collision because some of it is converted to internal energy, to elastic potential energy when the objects are deformed, and to rotational energy. Elastic and perfectly inelastic collisions are limiting cases; most collisions fall somewhere between them.

In the remainder of this section, we treat collisions in one dimension and consider the two extreme cases—perfectly inelastic and elastic collisions. The important distinction between these two types of collisions is that **momentum is constant in all collisions, but kinetic energy is constant only in elastic collisions.**

QuickLab

Hold a Ping-Pong ball or tennis ball on top of a basketball. Drop them both at the same time so that the basketball hits the floor, bounces up, and hits the smaller falling ball. What happens and why?

Perfectly Inelastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ along a straight line, as shown in Figure 9.9. The two particles collide head-on, stick together, and then move with some common velocity $\mathbf{v}_f$ after the collision. Because momentum is conserved in any collision, we can say that the total momentum before the collision equals the total momentum of the composite system after the collision:

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f \qquad \textbf{(9.13)}$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \qquad \textbf{(9.14)}$$

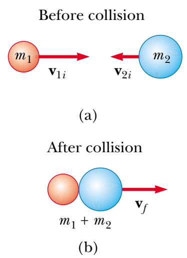


Figure 9.9 Schematic representation of a perfectly inelastic head-on collision between two particles: (a) before collision and (b) after collision.

Quick Quiz 9.7

Which is worse, crashing into a brick wall at 40 mi/h or crashing head-on into an oncoming car that is identical to yours and also moving at 40 mi/h?

Elastic Collisions

6.6 Now consider two particles that undergo an elastic head-on collision (Fig. 9.10). In this case, both momentum and kinetic energy are conserved; therefore, we have

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \qquad \textbf{(9.15)}$$

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 \qquad \textbf{(9.16)}$$

Because all velocities in Figure 9.10 are either to the left or the right, they can be represented by the corresponding speeds along with algebraic signs indicating direction. We shall indicate v as positive if a particle moves to the right and negative

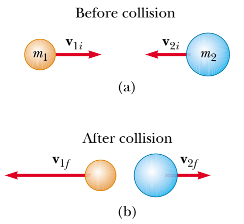


Figure 9.10 Schematic representation of an elastic head-on collision between two particles: (a) before collision and (b) after collision.

if it moves to the left. As has been seen in earlier chapters, it is common practice to call these values "speed" even though this term technically refers to the magnitude of the velocity vector, which does not have an algebraic sign.

In a typical problem involving elastic collisions, there are two unknown quantities, and Equations 9.15 and 9.16 can be solved simultaneously to find these. An alternative approach, however—one that involves a little mathematical manipulation of Equation 9.16—often simplifies this process. To see how, let us cancel the factor $\frac{1}{2}$ in Equation 9.16 and rewrite it as

$$m_1({v_{1i}}^2 - {v_{1f}}^2) = m_2({v_{2f}}^2 - {v_{2i}}^2)$$

and then factor both sides:

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \qquad \textbf{(9.17)}$$

Next, let us separate the terms containing m_1 and m_2 in Equation 9.15 to get

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \qquad \textbf{(9.18)}$$

To obtain our final result, we divide Equation 9.17 by Equation 9.18 and get

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \qquad \textbf{(9.19)}$$

This equation, in combination with Equation 9.15, can be used to solve problems dealing with elastic collisions. According to Equation 9.19, the relative speed of the two particles before the collision $v_{1i} - v_{2i}$ equals the negative of their relative speed after the collision, $-(v_{1f} - v_{2f})$.

Suppose that the masses and initial velocities of both particles are known. Equations 9.15 and 9.19 can be solved for the final speeds in terms of the initial speeds because there are two equations and two unknowns:

Elastic collision: relationships between final and initial velocities

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \qquad \textbf{(9.20)}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} \qquad \textbf{(9.21)}$$

It is important to remember that the appropriate signs for v_{1i} and v_{2i} must be included in Equations 9.20 and 9.21. For example, if particle 2 is moving to the left initially, then v_{2i} is negative.

Let us consider some special cases: If $m_1 = m_2$, then $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, the particles exchange speeds if they have equal masses. This is approximately what one observes in head-on billiard ball collisions—the cue ball stops, and the struck ball moves away from the collision with the same speed that the cue ball had.

If particle 2 is initially at rest, then $v_{2i} = 0$ and Equations 9.20 and 9.21 become

Elastic collision: particle 2 initially at rest

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} \qquad \textbf{(9.22)}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} \qquad \textbf{(9.23)}$$

If m_1 is much greater than m_2 and $v_{2i} = 0$, we see from Equations 9.22 and 9.23 that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. That is, when a very heavy particle collides head-on with a very light one that is initially at rest, the heavy particle continues its motion unaltered after the collision, and the light particle rebounds with a speed equal to about twice the initial speed of the heavy particle. An example of such a collision would be that of a moving heavy atom, such as uranium, with a light atom, such as hydrogen.

If m_2 is much greater than m_1 and particle 2 is initially at rest, then $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx v_{2i} = 0$. That is, when a very light particle collides head-on with a very heavy particle that is initially at rest, the light particle has its velocity reversed and the heavy one remains approximately at rest.

EXAMPLE 9.6 The Ballistic Pendulum

The ballistic pendulum (Fig. 9.11) is a system used to measure the speed of a fast-moving projectile, such as a bullet. The bullet is fired into a large block of wood suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h. The collision is perfectly inelastic, and because momentum is conserved, Equation 9.14 gives the speed of the system right after the collision, when we assume the impulse approximation. If we call the bullet particle 1 and the block particle 2, the total kinetic energy right after the collision is

$$(1) \qquad K_f = \tfrac{1}{2}(m_1 + m_2)v_f^2$$

With $v_{2i} = 0$, Equation 9.14 becomes

$$(2) \qquad v_f = \frac{m_1 v_{1i}}{m_1 + m_2}$$

Substituting this value of v_f into (1) gives

$$K_f = \frac{m_1^2 v_{1i}^2}{2(m_1 + m_2)}$$

Note that this kinetic energy immediately after the collision is less than the initial kinetic energy of the bullet. In all the energy changes that take place *after* the collision, however, the total amount of mechanical energy remains constant; thus, we can say that after the collision, the kinetic energy of the block and bullet at the bottom is transformed to potential energy at the height h:

$$\frac{m_1^2 v_{1i}^2}{2(m_1 + m_2)} = (m_1 + m_2)gh$$

Solving for v_{1i}, we obtain

$$v_{1i} = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{2gh}$$

This expression tells us that it is possible to obtain the initial speed of the bullet by measuring h and the two masses.

Because the collision is perfectly inelastic, some mechanical energy is converted to internal energy and it would be *incorrect* to equate the initial kinetic energy of the incoming bullet to the final gravitational potential energy of the bullet–block combination.

Exercise In a ballistic pendulum experiment, suppose that $h = 5.00$ cm, $m_1 = 5.00$ g, and $m_2 = 1.00$ kg. Find (a) the initial speed of the bullet and (b) the loss in mechanical energy due to the collision.

Answer 199 m/s; 98.5 J.

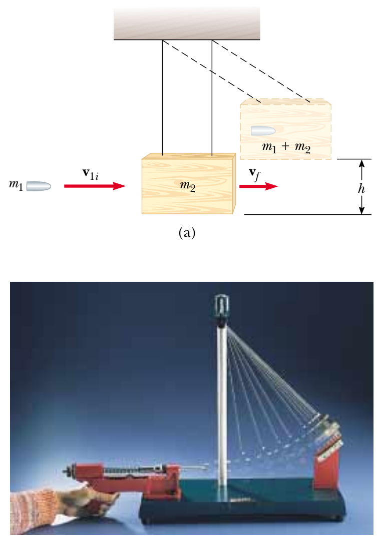


Figure 9.11 (a) Diagram of a ballistic pendulum. Note that $\mathbf{v}_{1i}$ is the velocity of the bullet just before the collision and $\mathbf{v}_f = \mathbf{v}_{1f} = \mathbf{v}_{2f}$ is the velocity of the bullet + block system just after the perfectly inelastic collision. (b) Multiflash photograph of a ballistic pendulum used in the laboratory.

EXAMPLE 9.7 A Two-Body Collision with a Spring

A block of mass $m_1 = 1.60$ kg initially moving to the right with a speed of 4.00 m/s on a frictionless horizontal track collides with a spring attached to a second block of mass $m_2 = 2.10$ kg initially moving to the left with a speed of 2.50 m/s, as shown in Figure 9.12a. The spring constant is 600 N/m. (a) At the instant block 1 is moving to the right with a speed of 3.00 m/s, as in Figure 9.12b, determine the velocity of block 2.

Solution First, note that the initial velocity of block 2 is -2.50 m/s because its direction is to the left. Because momentum is conserved for the system of two blocks, we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(1.60 \text{ kg})(4.00 \text{ m/s}) + (2.10 \text{ kg})(-2.50 \text{ m/s}) = (1.60 \text{ kg})(3.00 \text{ m/s}) + (2.10 \text{ kg})v_{2f}$$

$$v_{2f} = \boxed{-1.74 \text{ m/s}}$$

The negative value for v_{2f} means that block 2 is still moving to the left at the instant we are considering.

(b) Determine the distance the spring is compressed at that instant.

Solution To determine the distance that the spring is compressed, shown as x in Figure 9.12b, we can use the concept of conservation of mechanical energy because no friction or other nonconservative forces are acting on the system. Thus, we have

$$\tfrac{1}{2}m_1 v_{1i}^2 + \tfrac{1}{2}m_2 v_{2i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 + \tfrac{1}{2}kx^2$$

Substituting the given values and the result to part (a) into this expression gives

$$x = \boxed{0.173 \text{ m}}$$

It is important to note that we needed to use the principles of both conservation of momentum and conservation of mechanical energy to solve the two parts of this problem.

Exercise Find the velocity of block 1 and the compression in the spring at the instant that block 2 is at rest.

Answer 0.719 m/s to the right; 0.251 m.

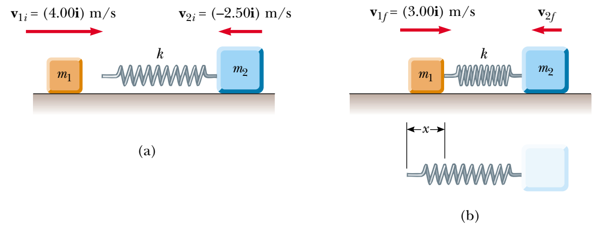


Figure 9.12

EXAMPLE 9.8 Slowing Down Neutrons by Collisions

In a nuclear reactor, neutrons are produced when a ${}^{235}_{92}U$ atom splits in a process called *fission*. These neutrons are moving at about 10^7 m/s and must be slowed down to about 10^3 m/s before they take part in another fission event. They are slowed down by being passed through a solid or liquid material called a *moderator*. The slowing-down process involves elastic collisions. Let us show that a neutron can lose most of its kinetic energy if it collides elastically with a moderator containing light nuclei, such as deuterium (in "heavy water," D_2O) or carbon (in graphite).

Solution Let us assume that the moderator nucleus of mass m_m is at rest initially and that a neutron of mass m_n and initial speed v_{ni} collides with it head-on.

Because these are elastic collisions, the first thing we do is recognize that both momentum and kinetic energy are constant. Therefore, Equations 9.22 and 9.23 can be applied to the head-on collision of a neutron with a moderator nucleus. We can represent this process by a drawing such as Figure 9.10.

The initial kinetic energy of the neutron is

$$K_{ni} = \frac{1}{2}m_n v_{ni}^2$$

After the collision, the neutron has kinetic energy $\frac{1}{2}m_n v_{nf}^2$, and we can substitute into this the value for v_{nf} given by Equation 9.22:

$$K_{nf} = \frac{1}{2}m_n v_{nf}^2 = \frac{m_n}{2}\left(\frac{m_n - m_m}{m_n + m_m}\right)^2 v_{ni}^2$$

Therefore, the fraction f_n of the initial kinetic energy possessed by the neutron after the collision is

$$(1) \qquad f_n = \frac{K_{nf}}{K_{ni}} = \left(\frac{m_n - m_m}{m_n + m_m}\right)^2$$

From this result, we see that the final kinetic energy of the neutron is small when m_m is close to m_n and zero when $m_n = m_m$.

We can use Equation 9.23, which gives the final speed of the particle that was initially at rest, to calculate the kinetic energy of the moderator nucleus after the collision:

$$K_{mf} = \frac{1}{2}m_m v_{mf}^2 = \frac{2m_n^2 m_m}{(m_n + m_m)^2} v_{ni}^2$$

Hence, the fraction f_m of the initial kinetic energy transferred to the moderator nucleus is

$$(2) \qquad f_m = \frac{K_{mf}}{K_{ni}} = \frac{4m_n m_m}{(m_n + m_m)^2}$$

Because the total kinetic energy of the system is conserved, (2) can also be obtained from (1) with the condition that $f_n + f_m = 1$, so that $f_m = 1 - f_n$.

Suppose that heavy water is used for the moderator. For collisions of the neutrons with deuterium nuclei in D_2O ($m_m = 2m_n$), $f_n = 1/9$ and $f_m = 8/9$. That is, 89% of the neutron's kinetic energy is transferred to the deuterium nucleus. In practice, the moderator efficiency is reduced because head-on collisions are very unlikely.

How do the results differ when graphite (^{12}C, as found in pencil lead) is used as the moderator?

Quick Quiz 9.8

An ingenious device that illustrates conservation of momentum and kinetic energy is shown in Figure 9.13a. It consists of five identical hard balls supported by strings of equal lengths. When ball 1 is pulled out and released, after the almost-elastic collision between it and ball 2, ball 5 moves out, as shown in Figure 9.13b. If balls 1 and 2 are pulled out and released, balls 4 and 5 swing out, and so forth. Is it ever possible that, when ball 1 is released, balls 4 and 5 will swing out on the opposite side and travel with half the speed of ball 1, as in Figure 9.13c?

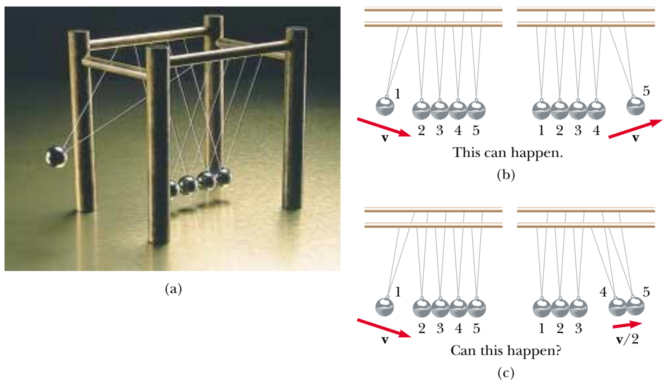


Figure 9.13 An executive stress reliever.

9.5 TWO-DIMENSIONAL COLLISIONS

In Sections 9.1 and 9.3, we showed that the momentum of a system of two particles is constant when the system is isolated. For any collision of two particles, this result implies that the momentum in each of the directions *x*, *y*, and *z* is constant. However, an important subset of collisions takes place in a plane. The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for conservation of momentum:

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Let us consider a two-dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2, where particle 2 is initially at rest, as shown in Figure 9.14. After the collision, particle 1 moves at an angle θ with respect to the horizontal and particle 2 moves at an angle ϕ with respect to the horizontal. This is called a *glancing* collision. Applying the law of conservation of momentum in component form, and noting that the initial *y* component of the momentum of the two-particle system is zero, we obtain

$$m_1 v_{1i} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi \qquad \textbf{(9.24)}$$

$$0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi \qquad \textbf{(9.25)}$$

where the minus sign in Equation 9.25 comes from the fact that after the collision, particle 2 has a *y* component of velocity that is downward. We now have two independent equations. As long as no more than two of the seven quantities in Equations 9.24 and 9.25 are unknown, we can solve the problem.

If the collision is elastic, we can also use Equation 9.16 (conservation of kinetic energy), with $v_{2i} = 0$, to give

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \qquad \textbf{(9.26)}$$

Knowing the initial speed of particle 1 and both masses, we are left with four unknowns (v_{1f}, v_{2f}, θ, ϕ). Because we have only three equations, one of the four remaining quantities must be given if we are to determine the motion after the collision from conservation principles alone.

If the collision is inelastic, kinetic energy is *not* conserved and Equation 9.26 does *not* apply.

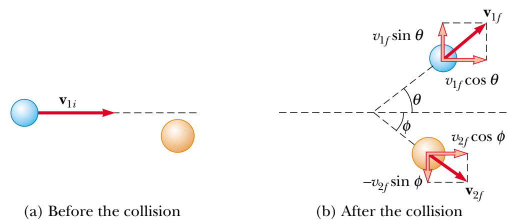


Figure 9.14 An elastic glancing collision between two particles.

Problem-Solving Hints

Collisions

The following procedure is recommended when dealing with problems involving collisions between two objects:

- Set up a coordinate system and define your velocities with respect to that system. It is usually convenient to have the *x* axis coincide with one of the initial velocities.
- In your sketch of the coordinate system, draw and label all velocity vectors and include all the given information.
- Write expressions for the *x* and *y* components of the momentum of each object before and after the collision. Remember to include the appropriate signs for the components of the velocity vectors.
- Write expressions for the total momentum in the *x* direction before and after the collision and equate the two. Repeat this procedure for the total momentum in the *y* direction. These steps follow from the fact that, because the momentum of the *system* is conserved in any collision, the total momentum along any direction must also be constant. Remember, it is the momentum of the *system* that is constant, not the momenta of the individual objects.
- If the collision is inelastic, kinetic energy is not conserved, and additional information is probably required. If the collision is perfectly inelastic, the final velocities of the two objects are equal. Solve the momentum equations for the unknown quantities.
- If the collision is elastic, kinetic energy is conserved, and you can equate the total kinetic energy before the collision to the total kinetic energy after the collision to get an additional relationship between the velocities.

Example 9.9 Collision at an Intersection

A 1 500-kg car traveling east with a speed of 25.0 m/s collides at an intersection with a 2 500-kg van traveling north at a speed of 20.0 m/s, as shown in Figure 9.15. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision (that is, they stick together).

Solution Let us choose east to be along the positive *x* direction and north to be along the positive *y* direction. Before the collision, the only object having momentum in the *x* direction is the car. Thus, the magnitude of the total initial momentum of the system (car plus van) in the *x* direction is

$$\sum p_{xi} = (1\,500\text{ kg})(25.0\text{ m/s}) = 3.75 \times 10^4\text{ kg}\cdot\text{m/s}$$

Let us assume that the wreckage moves at an angle θ and speed v_f after the collision. The magnitude of the total momentum in the *x* direction after the collision is

$$\sum p_{xf} = (4\,000\text{ kg})v_f\cos\theta$$

Because the total momentum in the *x* direction is constant, we can equate these two equations to obtain

$$(1)\qquad 3.75 \times 10^4\text{ kg}\cdot\text{m/s} = (4\,000\text{ kg})v_f\cos\theta$$

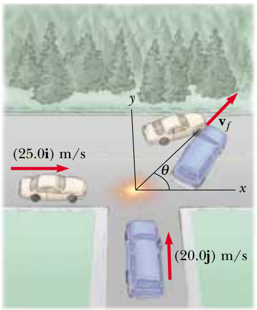


Figure 9.15 An eastbound car colliding with a northbound van.

Similarly, the total initial momentum of the system in the *y* direction is that of the van, and the magnitude of this momentum is (2 500 kg)(20.0 m/s). Applying conservation of

momentum to the *y* direction, we have

$$\sum p_{yi} = \sum p_{yf}$$

$$(2\,500\text{ kg})(20.0\text{ m/s}) = (4\,000\text{ kg})v_f\sin\theta$$

$$(2)\qquad 5.00 \times 10^4\text{ kg}\cdot\text{m/s} = (4\,000\text{ kg})v_f\sin\theta$$

If we divide (2) by (1), we get

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{5.00 \times 10^4}{3.75 \times 10^4} = 1.33$$

$$\theta = 53.1°$$

When this angle is substituted into (2), the value of v_f is

$$v_f = \frac{5.00 \times 10^4\text{ kg}\cdot\text{m/s}}{(4\,000\text{ kg})\sin 53.1°} = 15.6\text{ m/s}$$

It might be instructive for you to draw the momentum vectors of each vehicle before the collision and the two vehicles together after the collision.

Example 9.10 Proton–Proton Collision

Proton 1 collides elastically with proton 2 that is initially at rest. Proton 1 has an initial speed of 3.50×10^5 m/s and makes a glancing collision with proton 2, as was shown in Figure 9.14. After the collision, proton 1 moves at an angle of 37.0° to the horizontal axis, and proton 2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.

Solution Because both particles are protons, we know that $m_1 = m_2$. We also know that $\theta = 37.0°$ and $v_{1i} = 3.50 \times 10^5$ m/s. Equations 9.24, 9.25, and 9.26 become

$$v_{1f}\cos 37.0° + v_{2f}\cos\phi = 3.50 \times 10^5\text{ m/s}$$

$$v_{1f}\sin 37.0° - v_{2f}\sin\phi = 0$$

$$v_{1f}^2 + v_{2f}^2 = (3.50 \times 10^5\text{ m/s})^2$$

Solving these three equations with three unknowns simultaneously gives

$$v_{1f} = 2.80 \times 10^5\text{ m/s}$$

$$v_{2f} = 2.11 \times 10^5\text{ m/s}$$

$$\phi = 53.0°$$

Note that $\theta + \phi = 90°$. This result is not accidental. **Whenever two equal masses collide elastically in a glancing collision and one of them is initially at rest, their final velocities are always at right angles to each other.** The next example illustrates this point in more detail.

Example 9.11 Billiard Ball Collision

In a game of billiards, a player wishes to sink a target ball 2 in the corner pocket, as shown in Figure 9.16. If the angle to the corner pocket is 35°, at what angle θ is the cue ball 1 deflected? Assume that friction and rotational motion are unimportant and that the collision is elastic.

Solution Because the target ball is initially at rest, conservation of energy (Eq. 9.16) gives

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2$$

But $m_1 = m_2$, so that

$$(1)\qquad v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

Applying conservation of momentum to the two-dimensional collision gives

$$(2)\qquad \mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f}$$

Note that because $m_1 = m_2$, the masses also cancel in (2). If we square both sides of (2) and use the definition of the dot

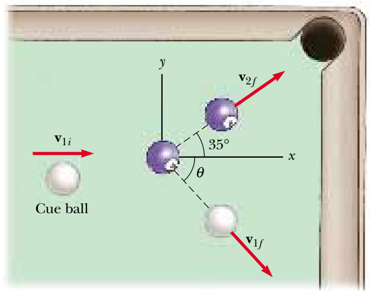


Figure 9.16

product of two vectors from Section 7.2, we get

$$v_{1i}^2 = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f}) = v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

Because the angle between $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$ is $\theta + 35°$, $\mathbf{v}_{1f} \cdot \mathbf{v}_{2f} = v_{1f}v_{2f}\cos(\theta + 35°)$, and so

$$(3) \qquad v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\cos(\theta + 35°)$$

Subtracting (1) from (3) gives

$$0 = 2v_{1f}v_{2f}\cos(\theta + 35°)$$

$$0 = \cos(\theta + 35°)$$

$$\theta + 35° = 90° \qquad \text{or} \qquad \theta = 55°$$

This result shows that whenever two equal masses undergo a glancing elastic collision and one of them is initially at rest, they move at right angles to each other after the collision. The same physics describes two very different situations, protons in Example 9.10 and billiard balls in this example.

9.6 THE CENTER OF MASS

6.7 In this section we describe the overall motion of a mechanical system in terms of a special point called the **center of mass** of the system. The mechanical system can be either a system of particles, such as a collection of atoms in a container, or an extended object, such as a gymnast leaping through the air. We shall see that the center of mass of the system moves as if all the mass of the system were concentrated at that point. Furthermore, if the resultant external force on the system is $\Sigma\mathbf{F}_{\text{ext}}$ and the total mass of the system is M, the center of mass moves with an acceleration given by $\mathbf{a} = \Sigma\mathbf{F}_{\text{ext}}/M$. That is, the system moves as if the resultant external force were applied to a single particle of mass M located at the center of mass. This behavior is independent of other motion, such as rotation or vibration of the system. This result was implicitly assumed in earlier chapters because many examples referred to the motion of extended objects that were treated as particles.

Consider a mechanical system consisting of a pair of particles that have different masses and are connected by a light, rigid rod (Fig. 9.17). One can describe the position of the center of mass of a system as being the *average position* of the system's mass. The center of mass of the system is located somewhere on the line joining the particles and is closer to the particle having the larger mass. If a single force is applied at some point on the rod somewhere between the center of mass and the less massive particle, the system rotates clockwise (see Fig. 9.17a). If the force is applied at a point on the rod somewhere between the center of mass and the more massive particle, the system rotates counterclockwise (see Fig. 9.17b). If the force is applied at the center of mass, the system moves in the direction of $\mathbf{F}$ without rotating (see Fig. 9.17c). Thus, the center of mass can be easily located.

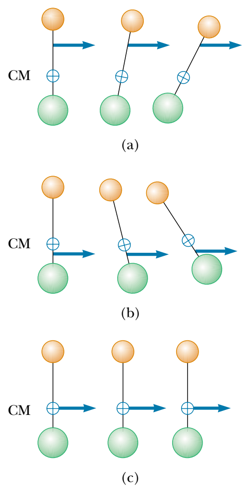
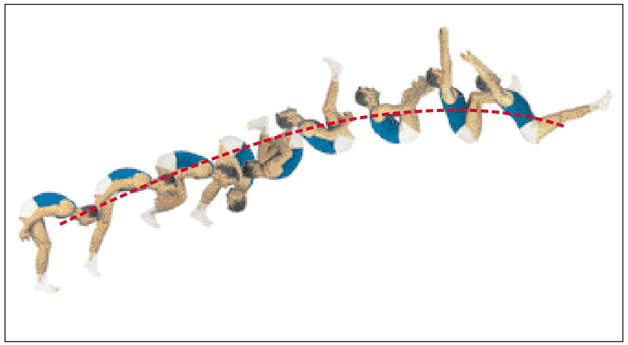


Figure 9.17 Two particles of unequal mass are connected by a light, rigid rod. (a) The system rotates clockwise when a force is applied between the less massive particle and the center of mass. (b) The system rotates counterclockwise when a force is applied between the more massive particle and the center of mass. (c) The system moves in the direction of the force without rotating when a force is applied at the center of mass.



This multiflash photograph shows that as the acrobat executes a somersault, his center of mass follows a parabolic path, the same path that a particle would follow.

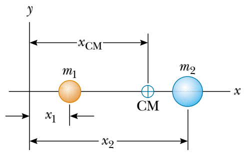


Figure 9.18 The center of mass of two particles of unequal mass on the x axis is located at x_{CM}, a point between the particles, closer to the one having the larger mass.

The center of mass of the pair of particles described in Figure 9.18 is located on the x axis and lies somewhere between the particles. Its x coordinate is

$$x_{\text{CM}} \equiv \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \qquad \textbf{(9.27)}$$

For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{\text{CM}} = \frac{2}{3}d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.

We can extend this concept to a system of many particles in three dimensions. The x coordinate of the center of mass of n particles is defined to be

$$x_{\text{CM}} \equiv \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_ix_i}{\sum_i m_i} \qquad \textbf{(9.28)}$$

where x_i is the x coordinate of the ith particle. For convenience, we express the total mass as $M \equiv \sum_i m_i$, where the sum runs over all n particles. The y and z coordinates of the center of mass are similarly defined by the equations

$$y_{\text{CM}} \equiv \frac{\sum_i m_iy_i}{M} \qquad \text{and} \qquad z_{\text{CM}} \equiv \frac{\sum_i m_iz_i}{M} \qquad \textbf{(9.29)}$$

The center of mass can also be located by its position vector, $\mathbf{r}_{\text{CM}}$. The cartesian coordinates of this vector are x_{CM}, y_{CM}, and z_{CM}, defined in Equations 9.28 and 9.29. Therefore,

Vector position of the center of mass for a system of particles

$$\mathbf{r}_{\text{CM}} = x_{\text{CM}}\mathbf{i} + y_{\text{CM}}\mathbf{j} + z_{\text{CM}}\mathbf{k}$$

$$= \frac{\sum_i m_ix_i\mathbf{i} + \sum_i m_iy_i\mathbf{j} + \sum_i m_iz_i\mathbf{k}}{M}$$

$$\mathbf{r}_{\text{CM}} \equiv \frac{\sum_i m_i\mathbf{r}_i}{M} \qquad \textbf{(9.30)}$$

where $\mathbf{r}_i$ is the position vector of the ith particle, defined by

$$\mathbf{r}_i \equiv x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$$

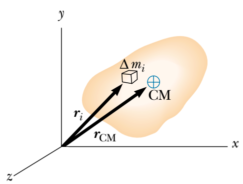


Figure 9.19 An extended object can be considered a distribution of small elements of mass Δm_i. The center of mass is located at the vector position $\mathbf{r}_{\text{CM}}$, which has coordinates x_{CM}, y_{CM}, and z_{CM}.

Although locating the center of mass for an extended object is somewhat more cumbersome than locating the center of mass of a system of particles, the basic ideas we have discussed still apply. We can think of an extended object as a system containing a large number of particles (Fig. 9.19). The particle separation is very small, and so the object can be considered to have a continuous mass distribution. By dividing the object into elements of mass Δm_i, with coordinates x_i, y_i, z_i, we see that the x coordinate of the center of mass is approximately

$$x_{\text{CM}} \approx \frac{\sum_i x_i\,\Delta m_i}{M}$$

with similar expressions for y_{CM} and z_{CM}. If we let the number of elements n approach infinity, then x_{CM} is given precisely. In this limit, we replace the sum by an integral and Δm_i by the differential element dm:

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M}\int x\,dm \qquad \textbf{(9.31)}$$

Likewise, for y_{CM} and z_{CM} we obtain

$$y_{CM} = \frac{1}{M}\int y\,dm \quad \text{and} \quad z_{CM} = \frac{1}{M}\int z\,dm \qquad \textbf{(9.32)}$$

We can express the vector position of the center of mass of an extended object in the form

$$\mathbf{r}_{CM} = \frac{1}{M}\int \mathbf{r}\,dm \qquad \textbf{(9.33)}$$

which is equivalent to the three expressions given by Equations 9.31 and 9.32.

The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.[4] For example, the center of mass of a rod lies in the rod, midway between its ends. The center of mass of a sphere or a cube lies at its geometric center.

One can determine the center of mass of an irregularly shaped object by suspending the object first from one point and then from another. In Figure 9.20, a wrench is hung from point A, and a vertical line AB (which can be established with a plumb bob) is drawn when the wrench has stopped swinging. The wrench is then hung from point C, and a second vertical line CD is drawn. The center of mass is halfway through the thickness of the wrench, under the intersection of these two lines. In general, if the wrench is hung freely from any point, the vertical line through this point must pass through the center of mass.

Because an extended object is a continuous distribution of mass, each small mass element is acted upon by the force of gravity. The net effect of all these forces is equivalent to the effect of a single force, $M\mathbf{g}$, acting through a special point, called the **center of gravity.** If $\mathbf{g}$ is constant over the mass distribution, then the center of gravity coincides with the center of mass. If an extended object is pivoted at its center of gravity, it balances in any orientation.

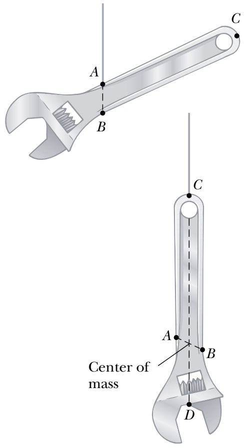


Figure 9.20 An experimental technique for determining the center of mass of a wrench. The wrench is hung freely first from point A and then from point C. The intersection of the two lines AB and CD locates the center of mass.

Quick Quiz 9.9

If a baseball bat is cut at the location of its center of mass as shown in Figure 9.21, do the two pieces have the same mass?

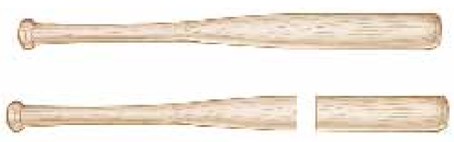


Figure 9.21 A baseball bat cut at the location of its center of mass.

QuickLab

Cut a triangle from a piece of cardboard and draw a set of adjacent strips inside it, parallel to one of the sides. Put a dot at the approximate location of the center of mass of each strip and then draw a straight line through the dots and into the angle opposite your starting side. The center of mass for the triangle must lie on this bisector of the angle. Repeat these steps for the other two sides. The three angle bisectors you have drawn will intersect at the center of mass of the triangle. If you poke a hole anywhere in the triangle and hang the cardboard from a string attached at that hole, the center of mass will be vertically aligned with the hole.

[4]This statement is valid only for objects that have a uniform mass per unit volume.

EXAMPLE 9.12 The Center of Mass of Three Particles

A system consists of three particles located as shown in Figure 9.22a. Find the center of mass of the system.

Solution We set up the problem by labeling the masses of the particles as shown in the figure, with $m_1 = m_2 = 1.0$ kg and $m_3 = 2.0$ kg. Using the basic defining equations for the coordinates of the center of mass and noting that $z_{CM} = 0$, we obtain

$$x_{CM} = \frac{\sum_i m_i x_i}{M} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(1.0\text{ m}) + (1.0\text{ kg})(2.0\text{ m}) + (2.0\text{ kg})(0\text{ m})}{1.0\text{ kg} + 1.0\text{ kg} + 2.0\text{ kg}}$$

$$= \frac{3.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 0.75\text{ m}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{M} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

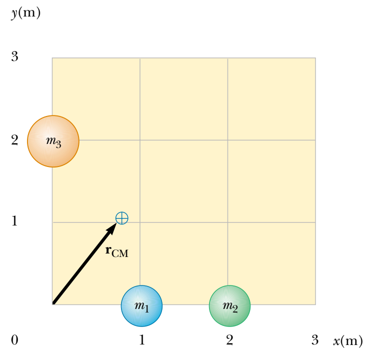
$$= \frac{(1.0\text{ kg})(0) + (1.0\text{ kg})(0) + (2.0\text{ kg})(2.0\text{ m})}{4.0\text{ kg}}$$

$$= \frac{4.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 1.0\text{ m}$$

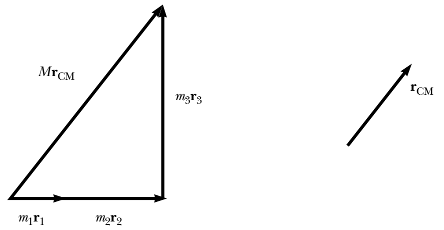
The position vector to the center of mass measured from the origin is therefore

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} = 0.75\mathbf{i}\text{ m} + 1.0\,\mathbf{j}\text{ m}$$

We can verify this result graphically by adding together $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$ and dividing the vector sum by M, the total mass. This is shown in Figure 9.22b.



(a)



(b)

Figure 9.22 (a) Two 1-kg masses and a single 2-kg mass are located as shown. The vector indicates the location of the system's center of mass. (b) The vector sum of $m_i\mathbf{r}_i$.

EXAMPLE 9.13 The Center of Mass of a Rod

(a) Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

Solution The rod is shown aligned along the x axis in Figure 9.23, so that $y_{CM} = z_{CM} = 0$. Furthermore, if we call the mass per unit length λ (this quantity is called the *linear mass density*), then $\lambda = M/L$ for the uniform rod we assume here. If we divide the rod into elements of length dx, then the mass of each element is $dm = \lambda\,dx$. For an arbitrary element located a distance x from the origin, Equation 9.31 gives

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{\lambda}{M}\left.\frac{x^2}{2}\right|_0^L = \frac{\lambda L^2}{2M}$$

Because $\lambda = M/L$, this reduces to

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \frac{L}{2}$$

One can also use symmetry arguments to obtain the same result.

(b) Suppose a rod is *nonuniform* such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

Solution In this case, we replace dm by λdx where λ is not constant. Therefore, x_{CM} is

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx$$
$$= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

We can eliminate α by noting that the total mass of the rod is related to α through the relationship

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

Substituting this into the expression for x_{CM} gives

$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\frac{2}{3}L}$$

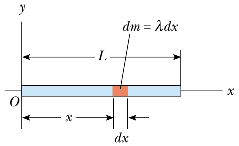


Figure 9.23 The center of mass of a uniform rod of length L is located at $x_{CM} = L/2$.

EXAMPLE 9.14 The Center of Mass of a Right Triangle

An object of mass M is in the shape of a right triangle whose dimensions are shown in Figure 9.24. Locate the coordinates of the center of mass, assuming the object has a uniform mass per unit area.

Solution By inspection we can estimate that the x coordinate of the center of mass must be past the center of the base, that is, greater than $a/2$, because the largest part of the triangle lies beyond that point. A similar argument indicates that its y coordinate must be less than $b/2$. To evaluate the x coordinate, we divide the triangle into narrow strips of width dx and height y as in Figure 9.24. The mass dm of each strip is

$$dm = \frac{\text{total mass of object}}{\text{total area of object}} \times \text{area of strip}$$
$$= \frac{M}{1/2\,ab}(y\,dx) = \left(\frac{2M}{ab}\right) y\,dx$$

Therefore, the x coordinate of the center of mass is

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^a x\left(\frac{2M}{ab}\right) y\,dx = \frac{2}{ab}\int_0^a xy\,dx$$

To evaluate this integral, we must express y in terms of x. From similar triangles in Figure 9.24, we see that

$$\frac{y}{x} = \frac{b}{a} \quad \text{or} \quad y = \frac{b}{a}x$$

With this substitution, x_{CM} becomes

$$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}x\right)dx = \frac{2}{a^2}\int_0^a x^2\,dx = \frac{2}{a^2}\left[\frac{x^3}{3}\right]_0^a$$
$$= \boxed{\frac{2}{3}a}$$

By a similar calculation, we get for the y coordinate of the center of mass

$$y_{CM} = \boxed{\frac{1}{3}b}$$

These values fit our original estimates.

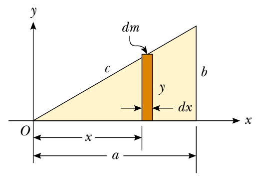


Figure 9.24

9.7 MOTION OF A SYSTEM OF PARTICLES

6.8

We can begin to understand the physical significance and utility of the center of mass concept by taking the time derivative of the position vector given by Equation 9.30. From Section 4.1 we know that the time derivative of a position vector is by definition a velocity. Assuming M remains constant for a system of particles, that is, no particles enter or leave the system, we get the following expression for the **velocity of the center of mass** of the system:

Velocity of the center of mass

$$\mathbf{v}_{CM} = \frac{d\mathbf{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum_i m_i\mathbf{v}_i}{M} \qquad \textbf{(9.34)}$$

where $\mathbf{v}_i$ is the velocity of the ith particle. Rearranging Equation 9.34 gives

Total momentum of a system of particles

$$M\mathbf{v}_{CM} = \sum_i m_i\mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{tot} \qquad \textbf{(9.35)}$$

Therefore, we conclude that the **total linear momentum of the system** equals the total mass multiplied by the velocity of the center of mass. In other words, the total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity $\mathbf{v}_{CM}$.

If we now differentiate Equation 9.34 with respect to time, we get the **acceleration of the center of mass** of the system:

Acceleration of the center of mass

$$\mathbf{a}_{CM} = \frac{d\mathbf{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i\frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum_i m_i\mathbf{a}_i \qquad \textbf{(9.36)}$$

Rearranging this expression and using Newton's second law, we obtain

$$M\mathbf{a}_{CM} = \sum_i m_i\mathbf{a}_i = \sum_i \mathbf{F}_i \qquad \textbf{(9.37)}$$

where $\mathbf{F}_i$ is the net force on particle i.

The forces on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). However, by Newton's third law, the internal force exerted by particle 1 on particle 2, for example, is equal in magnitude and opposite in direction to the internal force exerted by particle 2 on particle 1. Thus, when we sum over all internal forces in Equation 9.37, they cancel in pairs and the net force on the system is caused *only* by external forces. Thus, we can write Equation 9.37 in the form

Newton's second law for a system of particles

$$\sum \mathbf{F}_{ext} = M\mathbf{a}_{CM} = \frac{d\mathbf{p}_{tot}}{dt} \qquad \textbf{(9.38)}$$

That is, the resultant external force on a system of particles equals the total mass of the system multiplied by the acceleration of the center of mass. If we compare this with Newton's second law for a single particle, we see that

> The center of mass of a system of particles of combined mass M moves like an equivalent particle of mass M would move under the influence of the resultant external force on the system.

Finally, we see that if the resultant external force is zero, then from Equation 9.38 it follows that

$$\frac{d\mathbf{p}_{tot}}{dt} = M\mathbf{a}_{CM} = 0$$

Figure 9.25 Multiflash photograph showing an overhead view of a wrench moving on a horizontal surface. The center of mass of the wrench moves in a straight line as the wrench rotates about this point, shown by the white dots.

so that

$$\mathbf{p}_{\text{tot}} = M\mathbf{v}_{\text{CM}} = \text{constant} \qquad (\text{when } \Sigma \mathbf{F}_{\text{ext}} = 0) \tag{9.39}$$

That is, the total linear momentum of a system of particles is conserved if no net external force is acting on the system. It follows that for an isolated system of particles, both the total momentum and the velocity of the center of mass are constant in time, as shown in Figure 9.25. This is a generalization to a many-particle system of the law of conservation of momentum discussed in Section 9.1 for a two-particle system.

Suppose an isolated system consisting of two or more members is at rest. The center of mass of such a system remains at rest unless acted upon by an external force. For example, consider a system made up of a swimmer standing on a raft, with the system initially at rest. When the swimmer dives horizontally off the raft, the center of mass of the system remains at rest (if we neglect friction between raft and water). Furthermore, the linear momentum of the diver is equal in magnitude to that of the raft but opposite in direction.

As another example, suppose an unstable atom initially at rest suddenly breaks up into two fragments of masses M_A and M_B, with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. Because the total momentum of the system before the breakup is zero, the total momentum of the system after the breakup must also be zero. Therefore, $M_A\mathbf{v}_A + M_B\mathbf{v}_B = 0$. If the velocity of one of the fragments is known, the recoil velocity of the other fragment can be calculated.

EXAMPLE 9.15 The Sliding Bear

Suppose you tranquilize a polar bear on a smooth glacier as part of a research effort. How might you estimate the bear's mass using a measuring tape, a rope, and knowledge of your own mass?

Solution Tie one end of the rope around the bear, and then lay out the tape measure on the ice with one end at the bear's original position, as shown in Figure 9.26. Grab hold of the free end of the rope and position yourself as shown, noting your location. Take off your spiked shoes and pull on the rope hand over hand. Both you and the bear will slide over the ice until you meet. From the tape, observe how far you have slid, x_p, and how far the bear has slid, x_b. The point where you meet the bear is the constant location of the center of mass of the system (bear plus you), and so you can determine the mass of the bear from $m_b x_b = m_p x_p$. (Unfortunately, you cannot get back to your spiked shoes and so are in big trouble if the bear wakes up!)

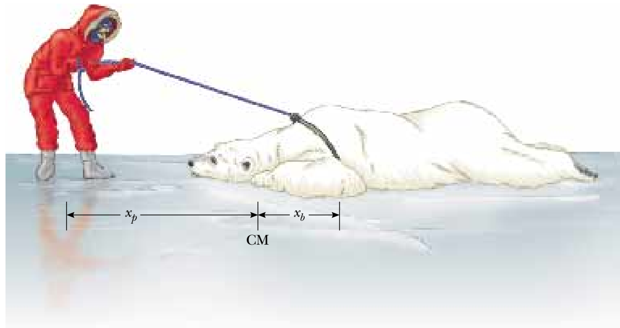
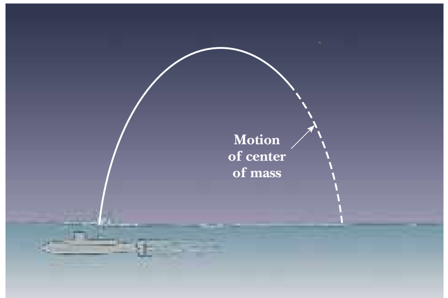


Figure 9.26 The center of mass of an isolated system remains at rest unless acted on by an external force. How can you determine the mass of the polar bear?

CONCEPTUAL EXAMPLE 9.16 Exploding Projectile

A projectile fired into the air suddenly explodes into several fragments (Fig. 9.27). What can be said about the motion of the center of mass of the system made up of all the fragments after the explosion?



Solution Neglecting air resistance, the only external force on the projectile is the gravitational force. Thus, if the projectile did not explode, it would continue to move along the parabolic path indicated by the broken line in Figure 9.27. Because the forces caused by the explosion are internal, they do not affect the motion of the center of mass. Thus, after the explosion the center of mass of the system (the fragments) follows the same parabolic path the projectile would have followed if there had been no explosion.

Figure 9.27 When a projectile explodes into several fragments, the center of mass of the system made up of all the fragments follows the same parabolic path the projectile would have taken had there been no explosion.

EXAMPLE 9.17 The Exploding Rocket

A rocket is fired vertically upward. At the instant it reaches an altitude of 1 000 m and a speed of 300 m/s, it explodes into three equal fragments. One fragment continues to move upward with a speed of 450 m/s following the explosion. The second fragment has a speed of 240 m/s and is moving east right after the explosion. What is the velocity of the third fragment right after the explosion?

Solution Let us call the total mass of the rocket M; hence, the mass of each fragment is $M/3$. Because the forces of the explosion are internal to the system and cannot affect its total momentum, the total momentum $\mathbf{p}_i$ of the rocket just before the explosion must equal the total momentum $\mathbf{p}_f$ of the fragments right after the explosion.

Before the explosion:

$$\mathbf{p}_i = M\mathbf{v}_i = M(300\mathbf{j})\ \text{m/s}$$

After the explosion:

$$\mathbf{p}_f = \frac{M}{3}(240\mathbf{i})\ \text{m/s} + \frac{M}{3}(450\mathbf{j})\ \text{m/s} + \frac{M}{3}\mathbf{v}_f$$

where $\mathbf{v}_f$ is the unknown velocity of the third fragment. Equating these two expressions (because $\mathbf{p}_i = \mathbf{p}_f$) gives

$$\frac{M}{3}\mathbf{v}_f + M(80\mathbf{i})\ \text{m/s} + M(150\mathbf{j})\ \text{m/s} = M(300\,\mathbf{j})\ \text{m/s}$$

$$\mathbf{v}_f = (-240\mathbf{i} + 450\mathbf{j})\ \text{m/s}$$

What does the sum of the momentum vectors for all the fragments look like?

Exercise Find the position of the center of mass of the system of fragments relative to the ground 3.00 s after the explosion. Assume the rocket engine is nonoperative after the explosion.

Answer The x coordinate does not change; $y_{\text{CM}} = 1.86$ km.

Optional Section

9.8 ROCKET PROPULSION

When ordinary vehicles, such as automobiles and locomotives, are propelled, the driving force for the motion is friction. In the case of the automobile, the driving force is the force exerted by the road on the car. A locomotive "pushes" against the tracks; hence, the driving force is the force exerted by the tracks on the locomotive. However, a rocket moving in space has no road or tracks to push against. Therefore, the source of the propulsion of a rocket must be something other than friction. Figure 9.28 is a dramatic photograph of a spacecraft at liftoff. **The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel.**

Rocket propulsion can be understood by first considering the mechanical system consisting of a machine gun mounted on a cart on wheels. As the gun is fired,

Figure 9.28 Liftoff of the space shuttle *Columbia.* Enormous thrust is generated by the shuttle's liquid-fuel engines, aided by the two solid-fuel boosters. Many physical principles from mechanics, thermodynamics, and electricity and magnetism are involved in such a launch.

The force from a nitrogen-propelled, hand-controlled device allows an astronaut to move about freely in space without restrictive tethers.

each bullet receives a momentum $m\mathbf{v}$ in some direction, where $\mathbf{v}$ is measured with respect to a stationary Earth frame. The momentum of the system made up of cart, gun, and bullets must be conserved. Hence, for each bullet fired, the gun and cart must receive a compensating momentum in the opposite direction. That is, the reaction force exerted by the bullet on the gun accelerates the cart and gun, and the cart moves in the direction opposite that of the bullets. If n is the number of bullets fired each second, then the average force exerted on the gun is $\mathbf{F}_{\text{av}} = nm\mathbf{v}$.

In a similar manner, as a rocket moves in free space, its linear momentum changes when some of its mass is released in the form of ejected gases. **Because the gases are given momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction.** Therefore, the rocket is accelerated as a result of the "push," or thrust, from the exhaust gases. In free space, the center of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process.[5]

Suppose that at some time t, the magnitude of the momentum of a rocket plus its fuel is $(M + \Delta m)v$, where v is the speed of the rocket relative to the Earth (Fig. 9.29a). Over a short time interval Δt, the rocket ejects fuel of mass Δm, and so at the end of this interval the rocket's speed is $v + \Delta v$, where Δv is the change in speed of the rocket (Fig. 9.29b). If the fuel is ejected with a speed v_e relative to the rocket (the subscript "e" stands for *exhaust,* and v_e is usually called the *exhaust speed*), the velocity of the fuel relative to a stationary frame of reference is $v - v_e$. Thus, if we equate the total initial momentum of the system to the total final momentum, we obtain

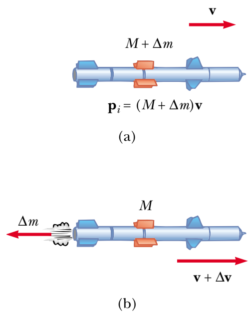


Figure 9.29 Rocket propulsion. (a) The initial mass of the rocket plus all its fuel is $M + \Delta m$ at a time t, and its speed is v. (b) At a time $t + \Delta t$, the rocket's mass has been reduced to M and an amount of fuel Δm has been ejected. The rocket's speed increases by an amount Δv.

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

where M represents the mass of the rocket and its remaining fuel after an amount of fuel having mass Δm has been ejected. Simplifying this expression gives

$$M\,\Delta v = v_e\,\Delta m$$

We also could have arrived at this result by considering the system in the center-of-mass frame of reference, which is a frame having the same velocity as the center of mass of the system. In this frame, the total momentum of the system is zero; therefore, if the rocket gains a momentum $M\,\Delta v$ by ejecting some fuel, the exhausted fuel obtains a momentum $v_e\,\Delta m$ in the *opposite* direction, so that $M\,\Delta v - v_e\,\Delta m = 0$. If we now take the limit as Δt goes to zero, we get $\Delta v \rightarrow dv$ and $\Delta m \rightarrow dm$. Futhermore, the increase in the exhaust mass dm corresponds to an equal decrease in the rocket mass, so that $dm = -dM$. Note that dM is given a negative sign because it represents a decrease in mass. Using this fact, we obtain

$$M\,dv = v_e\,dm = -v_e\,dM \tag{9.40}$$

Integrating this equation and taking the initial mass of the rocket plus fuel to be M_i and the final mass of the rocket plus its remaining fuel to be M_f, we obtain

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

Expression for rocket propulsion

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right) \tag{9.41}$$

[5]It is interesting to note that the rocket and machine gun represent cases of the reverse of a perfectly inelastic collision: Momentum is conserved, but the kinetic energy of the system increases (at the expense of chemical potential energy in the fuel).

This is the basic expression of rocket propulsion. First, it tells us that the increase in rocket speed is proportional to the exhaust speed of the ejected gases, v_e. Therefore, the exhaust speed should be very high. Second, the increase in rocket speed is proportional to the natural logarithm of the ratio M_i/M_f. Therefore, this ratio should be as large as possible, which means that the mass of the rocket without its fuel should be as small as possible and the rocket should carry as much fuel as possible.

The **thrust** on the rocket is the force exerted on it by the ejected exhaust gases. We can obtain an expression for the thrust from Equation 9.40:

$$\text{Thrust} = M\frac{dv}{dt} = \left| v_e \frac{dM}{dt} \right| \tag{9.42}$$

This expression shows us that the thrust increases as the exhaust speed increases and as the rate of change of mass (called the *burn rate*) increases.

EXAMPLE 9.18 A Rocket in Space

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to one-half its mass before ignition?

Solution We can guess that the speed we are looking for must be greater than the original speed because the rocket is accelerating. Applying Equation 9.41, we obtain

$$v_f = v_i + v_e \ln\left(\frac{M_i}{M_f}\right)$$

$$= 3.0 \times 10^3 \text{ m/s} + (5.0 \times 10^3 \text{ m/s})\ln\left(\frac{M_i}{0.5\,M_i}\right)$$

$$= \boxed{6.5 \times 10^3 \text{ m/s}}$$

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

Solution

$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right| = (5.0 \times 10^3 \text{ m/s})(50 \text{ kg/s})$$

$$= \boxed{2.5 \times 10^5 \text{ N}}$$

EXAMPLE 9.19 Fighting a Fire

Two firefighters must apply a total force of 600 N to steady a hose that is discharging water at 3 600 L/min. Estimate the speed of the water as it exits the nozzle.

Solution The water is exiting at 3 600 L/min, which is 60 L/s. Knowing that 1 L of water has a mass of 1 kg, we can say that about 60 kg of water leaves the nozzle every second. As the water leaves the hose, it exerts on the hose a thrust that must be counteracted by the 600-N force exerted on the hose by the firefighters. So, applying Equation 9.42 gives

$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right|$$

$$600 \text{ N} = |v_e(60 \text{ kg/s})|$$

$$v_e = \boxed{10 \text{ m/s}}$$

Firefighting is dangerous work. If the nozzle should slip from their hands, the movement of the hose due to the thrust it receives from the rapidly exiting water could injure the firefighters.

Firefighters attack a burning house with a hose line.

SUMMARY

The **linear momentum p** of a particle of mass m moving with a velocity **v** is

$$\mathbf{p} \equiv m\mathbf{v} \tag{9.1}$$

The law of **conservation of linear momentum** indicates that the total momentum of an isolated system is conserved. If two particles form an isolated system, their total momentum is conserved regardless of the nature of the force between them. Therefore, the total momentum of the system at all times equals its initial total momentum, or

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{9.5}$$

The **impulse** imparted to a particle by a force **F** is equal to the change in the momentum of the particle:

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\,dt = \Delta\mathbf{p} \tag{9.9}$$

This is known as the **impulse–momentum theorem.**

Impulsive forces are often very strong compared with other forces on the system and usually act for a very short time, as in the case of collisions.

When two particles collide, the total momentum of the system before the collision always equals the total momentum after the collision, regardless of the nature of the collision. An **inelastic collision** is one for which the total kinetic energy is not conserved. A **perfectly inelastic collision** is one in which the colliding bodies stick together after the collision. An **elastic collision** is one in which kinetic energy is constant.

In a two- or three-dimensional collision, the components of momentum in each of the three directions (x, y, and z) are conserved independently.

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} \equiv \frac{\sum_i m_i\mathbf{r}_i}{M} \tag{9.30}$$

where $M = \sum_i m_i$ is the total mass of the system and $\mathbf{r}_i$ is the position vector of the ith particle.

The position vector of the center of mass of a rigid body can be obtained from the integral expression

$$\mathbf{r}_{\text{CM}} = \frac{1}{M}\int \mathbf{r}\,dm \tag{9.33}$$

The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{\sum_i m_i\mathbf{v}_i}{M} \tag{9.34}$$

The total momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass.

Newton's second law applied to a system of particles is

$$\sum \mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}} = \frac{d\mathbf{p}_{\text{tot}}}{dt} \tag{9.38}$$

where $\mathbf{a}_{\text{CM}}$ is the acceleration of the center of mass and the sum is over all external forces. The center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system. It follows from Equation 9.38 that the total momentum of the system is conserved if there are no external forces acting on it.

QUESTIONS

1. If the kinetic energy of a particle is zero, what is its linear momentum?
2. If the speed of a particle is doubled, by what factor is its momentum changed? By what factor is its kinetic energy changed?
3. If two particles have equal kinetic energies, are their momenta necessarily equal? Explain.
4. If two particles have equal momenta, are their kinetic energies necessarily equal? Explain.
5. An isolated system is initially at rest. Is it possible for parts of the system to be in motion at some later time? If so, explain how this might occur.
6. If two objects collide and one is initially at rest, is it possible for both to be at rest after the collision? Is it possible for one to be at rest after the collision? Explain.
7. Explain how linear momentum is conserved when a ball bounces from a floor.
8. Is it possible to have a collision in which all of the kinetic energy is lost? If so, cite an example.
9. In a perfectly elastic collision between two particles, does the kinetic energy of each particle change as a result of the collision?
10. When a ball rolls down an incline, its linear momentum increases. Does this imply that momentum is not conserved? Explain.
11. Consider a perfectly inelastic collision between a car and a large truck. Which vehicle loses more kinetic energy as a result of the collision?
12. Can the center of mass of a body lie outside the body? If so, give examples.
13. Three balls are thrown into the air simultaneously. What is the acceleration of their center of mass while they are in motion?
14. A meter stick is balanced in a horizontal position with the index fingers of the right and left hands. If the two fingers are slowly brought together, the stick remains balanced and the two fingers always meet at the 50-cm mark regardless of their original positions (try it!). Explain.
15. A sharpshooter fires a rifle while standing with the butt of the gun against his shoulder. If the forward momentum of a bullet is the same as the backward momentum of the gun, why is it not as dangerous to be hit by the gun as by the bullet?
16. A piece of mud is thrown against a brick wall and sticks to the wall. What happens to the momentum of the mud? Is momentum conserved? Explain.
17. Early in this century, Robert Goddard proposed sending a rocket to the Moon. Critics took the position that in a vacuum, such as exists between the Earth and the Moon, the gases emitted by the rocket would have nothing to push against to propel the rocket. According to *Scientific American* (January 1975), Goddard placed a gun in a vacuum and fired a blank cartridge from it. (A blank cartridge fires only the wadding and hot gases of the burning gunpowder.) What happened when the gun was fired?
18. A pole-vaulter falls from a height of 6.0 m onto a foam rubber pad. Can you calculate his speed just before he reaches the pad? Can you estimate the force exerted on him due to the collision? Explain.
19. Explain how you would use a balloon to demonstrate the mechanism responsible for rocket propulsion.
20. Does the center of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.
21. A ball is dropped from a tall building. Identify the system for which linear momentum is conserved.
22. A bomb, initially at rest, explodes into several pieces. (a) Is linear momentum conserved? (b) Is kinetic energy conserved? Explain.
23. NASA often uses the gravity of a planet to "slingshot" a probe on its way to a more distant planet. This is actually a collision where the two objects do not touch. How can the probe have its speed increased in this manner?
24. The Moon revolves around the Earth. Is the Moon's linear momentum conserved? Is its kinetic energy conserved? Assume that the Moon's orbit is circular.
25. A raw egg dropped to the floor breaks apart upon impact. However, a raw egg dropped onto a thick foam rubber cushion from a height of about 1 m rebounds without breaking. Why is this possible? (If you try this experiment, be sure to catch the egg after the first bounce.)
26. On the subject of the following positions, state your own view and argue to support it: (a) The best theory of motion is that force causes acceleration. (b) The true measure of a force's effectiveness is the work it does, and the best theory of motion is that work on an object changes its energy. (c) The true measure of a force's effect is impulse, and the best theory of motion is that impulse injected into an object changes its momentum.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 9.1 Linear Momentum and Its Conservation

1. A 3.00-kg particle has a velocity of $(3.00\mathbf{i} - 4.00\mathbf{j})$ m/s. (a) Find its x and y components of momentum. (b) Find the magnitude and direction of its momentum.
2. A 0.100-kg ball is thrown straight up into the air with an initial speed of 15.0 m/s. Find the momentum of the ball (a) at its maximum height and (b) halfway up to its maximum height.
3. A 40.0-kg child standing on a frozen pond throws a 0.500-kg stone to the east with a speed of 5.00 m/s. Neglecting friction between child and ice, find the recoil velocity of the child.
4. A pitcher claims he can throw a baseball with as much momentum as a 3.00-g bullet moving with a speed of 1 500 m/s. A baseball has a mass of 0.145 kg. What must be its speed if the pitcher's claim is valid?
5. How fast can you set the Earth moving? In particular, when you jump straight up as high as you can, you give the Earth a maximum recoil speed of what order of magnitude? Model the Earth as a perfectly solid object. In your solution, state the physical quantities you take as data and the values you measure or estimate for them.
6. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one of them, and the blocks are pushed together with the spring between them (Fig. P9.6). A cord initially holding the blocks together is burned; after this, the block of mass $3M$ moves to the right with a speed of 2.00 m/s. (a) What is the speed of the block of mass M? (b) Find the original elastic energy in the spring if $M =$ 0.350 kg.

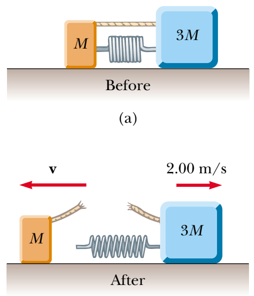


Figure P9.6

7. (a) A particle of mass m moves with momentum p. Show that the kinetic energy of the particle is given by $K = p^2/2m$. (b) Express the magnitude of the particle's momentum in terms of its kinetic energy and mass.

Section 9.2 Impulse and Momentum

8. A car is stopped for a traffic signal. When the light turns green, the car accelerates, increasing its speed from zero to 5.20 m/s in 0.832 s. What linear impulse and average force does a 70.0-kg passenger in the car experience?
9. An estimated force–time curve for a baseball struck by a bat is shown in Figure P9.9. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

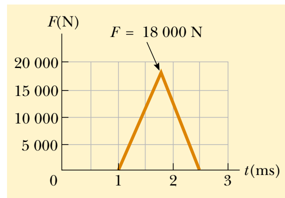


Figure P9.9

10. A tennis player receives a shot with the ball (0.060 0 kg) traveling horizontally at 50.0 m/s and returns the shot with the ball traveling horizontally at 40.0 m/s in the opposite direction. (a) What is the impulse delivered to the ball by the racket? (b) What work does the racket do on the ball?
WEB 11. A 3.00-kg steel ball strikes a wall with a speed of 10.0 m/s at an angle of 60.0° with the surface. It bounces off with the same speed and angle (Fig. P9.11). If the ball is in contact with the wall for 0.200 s, what is the average force exerted on the ball by the wall?
12. In a slow-pitch softball game, a 0.200-kg softball crossed the plate at 15.0 m/s at an angle of 45.0° below the horizontal. The ball was hit at 40.0 m/s, 30.0° above the horizontal. (a) Determine the impulse delivered to the ball. (b) If the force on the ball increased linearly for 4.00 ms, held constant for 20.0 ms, and then decreased to zero linearly in another 4.00 ms, what was the maximum force on the ball?

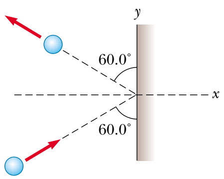


Figure P9.11

13. A garden hose is held in the manner shown in Figure P9.13. The hose is initially full of motionless water. What additional force is necessary to hold the nozzle stationary after the water is turned on if the discharge rate is 0.600 kg/s with a speed of 25.0 m/s?

Figure P9.13

14. A professional diver performs a dive from a platform 10 m above the water surface. Estimate the order of magnitude of the average impact force she experiences in her collision with the water. State the quantities you take as data and their values.

Section 9.3 Collisions

Section 9.4 Elastic and Inelastic Collisions in One Dimension

15. High-speed stroboscopic photographs show that the head of a golf club of mass 200 g is traveling at 55.0 m/s just before it strikes a 46.0-g golf ball at rest on a tee. After the collision, the club head travels (in the same direction) at 40.0 m/s. Find the speed of the golf ball just after impact.

16. A 75.0-kg ice skater, moving at 10.0 m/s, crashes into a stationary skater of equal mass. After the collision, the two skaters move as a unit at 5.00 m/s. Suppose the average force a skater can experience without breaking a bone is 4 500 N. If the impact time is 0.100 s, does a bone break?

17. A 10.0-g bullet is fired into a stationary block of wood (m = 5.00 kg). The relative motion of the bullet stops inside the block. The speed of the bullet-plus-wood combination immediately after the collision is measured as 0.600 m/s. What was the original speed of the bullet?

18. As shown in Figure P9.18, a bullet of mass m and speed v passes completely through a pendulum bob of mass M. The bullet emerges with a speed of $v/2$. The pendulum bob is suspended by a stiff rod of length ℓ and negligible mass. What is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle?

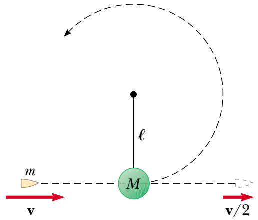


Figure P9.18

19. A 45.0-kg girl is standing on a plank that has a mass of 150 kg. The plank, originally at rest, is free to slide on a frozen lake, which is a flat, frictionless supporting surface. The girl begins to walk along the plank at a constant speed of 1.50 m/s relative to the plank. (a) What is her speed relative to the ice surface? (b) What is the speed of the plank relative to the ice surface?

20. Gayle runs at a speed of 4.00 m/s and dives on a sled, which is initially at rest on the top of a frictionless snow-covered hill. After she has descended a vertical distance of 5.00 m, her brother, who is initially at rest, hops on her back and together they continue down the hill. What is their speed at the bottom of the hill if the total vertical drop is 15.0 m? Gayle's mass is 50.0 kg, the sled has a mass of 5.00 kg and her brother has a mass of 30.0 kg.

21. A 1 200-kg car traveling initially with a speed of 25.0 m/s in an easterly direction crashes into the rear end of a 9 000-kg truck moving in the same direction at 20.0 m/s (Fig. P9.21). The velocity of the car right after the collision is 18.0 m/s to the east. (a) What is the velocity of the truck right after the collision? (b) How much mechanical energy is lost in the collision? Account for this loss in energy.

22. A railroad car of mass 2.50×10^4 kg is moving with a speed of 4.00 m/s. It collides and couples with three other coupled railroad cars, each of the same mass as the single car and moving in the same direction with an initial speed of 2.00 m/s. (a) What is the speed of the four cars after the collision? (b) How much energy is lost in the collision?

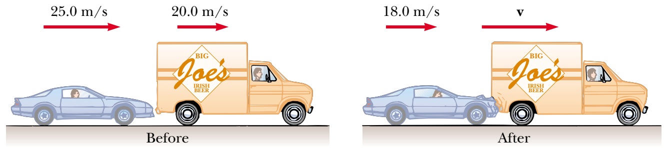


Figure P9.21

23. Four railroad cars, each of mass 2.50×10^4 kg, are coupled together and coasting along horizontal tracks at a speed of v_i toward the south. A very strong but foolish movie actor, riding on the second car, uncouples the front car and gives it a big push, increasing its speed to 4.00 m/s southward. The remaining three cars continue moving toward the south, now at 2.00 m/s. (a) Find the initial speed of the cars. (b) How much work did the actor do? (c) State the relationship between the process described here and the process in Problem 22.

24. A 7.00-kg bowling ball collides head-on with a 2.00-kg bowling pin. The pin flies forward with a speed of 3.00 m/s. If the ball continues forward with a speed of 1.80 m/s, what was the initial speed of the ball? Ignore rotation of the ball.

WEB **25.** A neutron in a reactor makes an elastic head-on collision with the nucleus of a carbon atom initially at rest. (a) What fraction of the neutron's kinetic energy is transferred to the carbon nucleus? (b) If the initial kinetic energy of the neutron is 1.60×10^{-13} J, find its final kinetic energy and the kinetic energy of the carbon nucleus after the collision. (The mass of the carbon nucleus is about 12.0 times greater than the mass of the neutron.)

26. Consider a frictionless track ABC as shown in Figure P9.26. A block of mass m_1 = 5.00 kg is released from A. It makes a head-on elastic collision at B with a block of mass m_2 = 10.0 kg that is initially at rest. Calculate the maximum height to which m_1 rises after the collision.

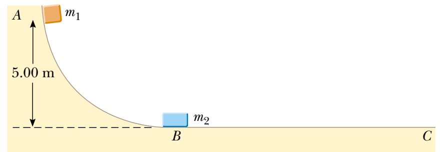


Figure P9.26

WEB **27.** A 12.0-g bullet is fired into a 100-g wooden block initially at rest on a horizontal surface. After impact, the block slides 7.50 m before coming to rest. If the coefficient of friction between the block and the surface is 0.650, what was the speed of the bullet immediately before impact?

28. A 7.00-g bullet, when fired from a gun into a 1.00-kg block of wood held in a vise, would penetrate the block to a depth of 8.00 cm. This block of wood is placed on a frictionless horizontal surface, and a 7.00-g bullet is fired from the gun into the block. To what depth will the bullet penetrate the block in this case?

Section 9.5 Two-Dimensional Collisions

29. A 90.0-kg fullback running east with a speed of 5.00 m/s is tackled by a 95.0-kg opponent running north with a speed of 3.00 m/s. If the collision is perfectly inelastic, (a) calculate the speed and direction of the players just after the tackle and (b) determine the energy lost as a result of the collision. Account for the missing energy.

30. The mass of the blue puck in Figure P9.30 is 20.0% greater than the mass of the green one. Before colliding, the pucks approach each other with equal and opposite momenta, and the green puck has an initial speed of 10.0 m/s. Find the speeds of the pucks after the collision if half the kinetic energy is lost during the collision.

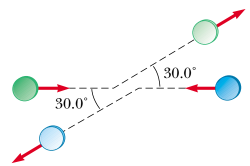


Figure P9.30

31. Two automobiles of equal mass approach an intersection. One vehicle is traveling with velocity 13.0 m/s toward the east and the other is traveling north with a speed of v_{2i}. Neither driver sees the other. The vehicles collide in the intersection and stick together, leaving parallel skid marks at an angle of 55.0° north of east. The speed limit for both roads is 35 mi/h, and the driver of the northward-moving vehicle claims he was within the speed limit when the collision occurred. Is he telling the truth?

32. A proton, moving with a velocity of $v_i\mathbf{i}$, collides elastically with another proton that is initially at rest. If the two protons have equal speeds after the collision, find (a) the speed of each proton after the collision in terms of v_i and (b) the direction of the velocity vectors after the collision.

33. A billiard ball moving at 5.00 m/s strikes a stationary ball of the same mass. After the collision, the first ball moves at 4.33 m/s and at an angle of 30.0° with respect to the original line of motion. Assuming an elastic collision (and ignoring friction and rotational motion), find the struck ball's velocity.

34. A 0.300-kg puck, initially at rest on a horizontal, frictionless surface, is struck by a 0.200-kg puck moving initially along the x axis with a speed of 2.00 m/s. After the collision, the 0.200-kg puck has a speed of 1.00 m/s at an angle of $\theta = 53.0°$ to the positive x axis (see Fig. 9.14). (a) Determine the velocity of the 0.300-kg puck after the collision. (b) Find the fraction of kinetic energy lost in the collision.

35. A 3.00-kg mass with an initial velocity of $5.00\mathbf{i}$ m/s collides with and sticks to a 2.00-kg mass with an initial velocity of $-3.00\mathbf{j}$ m/s. Find the final velocity of the composite mass.

36. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed of 5.00 m/s. After the collision, the orange disk moves along a direction that makes an angle of 37.0° with its initial direction of motion, and the velocity of the yellow disk is perpendicular to that of the orange disk (after the collision). Determine the final speed of each disk.

37. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed v_i. After the collision, the orange disk moves along a direction that makes an angle θ with its initial direction of motion, and the velocity of the yellow disk is perpendicular to that of the orange disk (after the collision). Determine the final speed of each disk.

38. During the battle of Gettysburg, the gunfire was so intense that several bullets collided in midair and fused together. Assume a 5.00-g Union musket ball was moving to the right at a speed of 250 m/s, 20.0° above the horizontal, and that a 3.00-g Confederate ball was moving to the left at a speed of 280 m/s, 15.0° above the horizontal. Immediately after they fuse together, what is their velocity?

WEB **39.** An unstable nucleus of mass 17.0×10^{-27} kg initially at rest disintegrates into three particles. One of the particles, of mass 5.00×10^{-27} kg, moves along the y axis with a velocity of 6.00×10^{6} m/s. Another particle, of mass 8.40×10^{-27} kg, moves along the x axis with a speed of 4.00×10^{6} m/s. Find (a) the velocity of the third particle and (b) the total kinetic energy increase in the process.

Section 9.6 The Center of Mass

40. Four objects are situated along the y axis as follows: A 2.00-kg object is at +3.00 m, a 3.00-kg object is at +2.50 m, a 2.50-kg object is at the origin, and a 4.00-kg object is at −0.500 m. Where is the center of mass of these objects?

41. A uniform piece of sheet steel is shaped as shown in Figure P9.41. Compute the x and y coordinates of the center of mass of the piece.

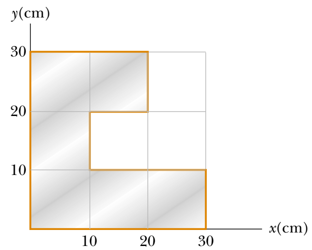


Figure P9.41

42. The mass of the Earth is 5.98×10^{24} kg, and the mass of the Moon is 7.36×10^{22} kg. The distance of separation, measured between their centers, is 3.84×10^{8} m. Locate the center of mass of the Earth–Moon system as measured from the center of the Earth.

43. A water molecule consists of an oxygen atom with two hydrogen atoms bound to it (Fig. P9.43). The angle between the two bonds is 106°. If the bonds are 0.100 nm long, where is the center of mass of the molecule?

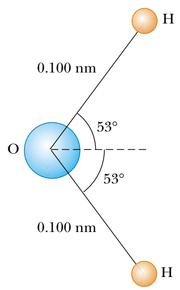


Figure P9.43

44. A 0.400-kg mass m_1 has position $\mathbf{r}_1 = 12.0\mathbf{j}$ cm. A 0.800-kg mass m_2 has position $\mathbf{r}_2 = -12.0\mathbf{i}$ cm. Another 0.800-kg mass m_3 has position $\mathbf{r}_3 = (12.0\mathbf{i} - 12.0\mathbf{j})$ cm. Make a drawing of the masses. Start from the origin and, to the scale 1 cm = 1 kg·cm, construct the vector $m_1\mathbf{r}_1$, then the vector $m_1\mathbf{r}_1 + m_2\mathbf{r}_2$, then the vector $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$, and at last $\mathbf{r}_{CM} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)/(m_1 + m_2 + m_3)$. Observe that the head of the vector $\mathbf{r}_{CM}$ indicates the position of the center of mass.

45. A rod of length 30.0 cm has linear density (mass-per-length) given by

$$\lambda = 50.0 \text{ g/m} + 20.0x \text{ g/m}^2$$

where x is the distance from one end, measured in meters. (a) What is the mass of the rod? (b) How far from the $x = 0$ end is its center of mass?

Section 9.7 Motion of a System of Particles

46. Consider a system of two particles in the xy plane: $m_1 = 2.00$ kg is at $\mathbf{r}_1 = (1.00\mathbf{i} + 2.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} + 0.500\mathbf{j})$ m/s; $m_2 = 3.00$ kg is at $\mathbf{r}_2 = (-4.00\mathbf{i} - 3.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} - 2.00\mathbf{j})$ m/s. (a) Plot these particles on a grid or graph paper. Draw their position vectors and show their velocities. (b) Find the position of the center of mass of the system and mark it on the grid. (c) Determine the velocity of the center of mass and also show it on the diagram. (d) What is the total linear momentum of the system?

47. Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat at rest in still water, 2.70 m away from Juliet who is in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0-kg boat move toward the shore it is facing?

48. Two masses, 0.600 kg and 0.300 kg, begin uniform motion at the same speed, 0.800 m/s, from the origin at $t = 0$ and travel in the directions shown in Figure P9.48. (a) Find the velocity of the center of mass in unit–vector notation. (b) Find the magnitude and direction of the velocity of the center of mass. (c) Write the position vector of the center of mass as a function of time.

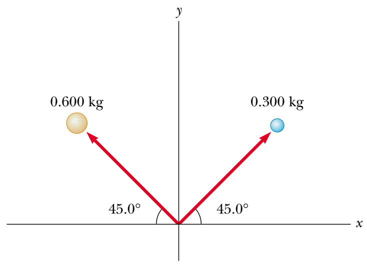


Figure P9.48

49. A 2.00-kg particle has a velocity of $(2.00\mathbf{i} - 3.00\mathbf{j})$ m/s, and a 3.00-kg particle has a velocity of $(1.00\mathbf{i} + 6.00\mathbf{j})$ m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system.

50. A ball of mass 0.200 kg has a velocity of $1.50\mathbf{i}$ m/s; a ball of mass 0.300 kg has a velocity of $-0.400\mathbf{i}$ m/s. They meet in a head-on elastic collision. (a) Find their velocities after the collision. (b) Find the velocity of their center of mass before and after the collision.

(Optional)

Section 9.8 Rocket Propulsion

WEB **51.** The first stage of a Saturn V space vehicle consumes fuel and oxidizer at the rate of 1.50×10^{4} kg/s, with an exhaust speed of 2.60×10^{3} m/s. (a) Calculate the thrust produced by these engines. (b) Find the initial acceleration of the vehicle on the launch pad if its initial mass is 3.00×10^{6} kg. [*Hint:* You must include the force of gravity to solve part (b).]

52. A large rocket with an exhaust speed of $v_e = 3\,000$ m/s develops a thrust of 24.0 million newtons. (a) How much mass is being blasted out of the rocket exhaust per second? (b) What is the maximum speed the rocket can attain if it starts from rest in a force-free environment with $v_e = 3.00$ km/s and if 90.0% of its initial mass is fuel and oxidizer?

53. A rocket for use in deep space is to have the capability of boosting a total load (payload plus rocket frame and engine) of 3.00 metric tons to a speed of 10 000 m/s. (a) It has an engine and fuel designed to produce an exhaust speed of 2 000 m/s. How much fuel plus oxidizer is required? (b) If a different fuel and engine design could give an exhaust speed of 5 000 m/s, what amount of fuel and oxidizer would be required for the same task?

54. A rocket car has a mass of 2 000 kg unfueled and a mass of 5 000 kg when completely fueled. The exhaust velocity is 2 500 m/s. (a) Calculate the amount of fuel used to accelerate the completely fueled car from rest to 225 m/s (about 500 mi/h). (b) If the burn rate is constant at 30.0 kg/s, calculate the time it takes the car to reach this speed. Neglect friction and air resistance.

ADDITIONAL PROBLEMS

55. Review Problem. A 60.0-kg person running at an initial speed of 4.00 m/s jumps onto a 120-kg cart initially at rest (Fig. P9.55). The person slides on the cart's top surface and finally comes to rest relative to the cart. The coefficient of kinetic friction between the person and the cart is 0.400. Friction between the cart and ground can be neglected. (a) Find the final velocity of the person and cart relative to the ground. (b) Find the frictional force acting on the person while he is sliding across the top surface of the cart. (c) How long does the frictional force act on the person? (d) Find the change in momentum of the person and the change in momentum of the cart. (e) Determine the displacement of the person relative to the ground while he is sliding on the cart. (f) Determine the displacement of the cart relative to the ground while the person is sliding. (g) Find the change in kinetic energy of the person. (h) Find the change in kinetic energy of the cart. (i) Explain why the answers to parts (g) and (h) differ. (What kind of collision is this, and what accounts for the loss of mechanical energy?)

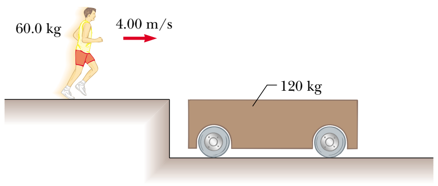


Figure P9.55

56. A golf ball (m = 46.0 g) is struck a blow that makes an angle of 45.0° with the horizontal. The ball lands 200 m away on a flat fairway. If the golf club and ball are in contact for 7.00 ms, what is the average force of impact? (Neglect air resistance.)

57. An 8.00-g bullet is fired into a 2.50-kg block that is initially at rest at the edge of a frictionless table of height 1.00 m (Fig. P9.57). The bullet remains in the block, and after impact the block lands 2.00 m from the bottom of the table. Determine the initial speed of the bullet.

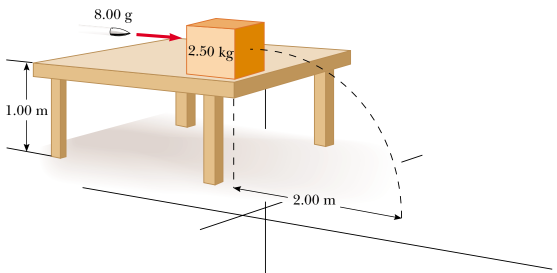


Figure P9.57 Problems 57 and 58.

58. A bullet of mass m is fired into a block of mass M that is initially at rest at the edge of a frictionless table of height h (see Fig. P9.57). The bullet remains in the block, and after impact the block lands a distance d from the bottom of the table. Determine the initial speed of the bullet.

59. An 80.0-kg astronaut is working on the engines of his ship, which is drifting through space with a constant velocity. The astronaut, wishing to get a better view of the Universe, pushes against the ship and much later finds himself 30.0 m behind the ship and at rest with respect to it. Without a thruster, the only way to return to the ship is to throw his 0.500-kg wrench directly away from the ship. If he throws the wrench with a speed of 20.0 m/s relative to the ship, how long does it take the astronaut to reach the ship?

60. A small block of mass m_1 = 0.500 kg is released from rest at the top of a curve-shaped frictionless wedge of mass m_2 = 3.00 kg, which sits on a frictionless horizontal surface, as shown in Figure P9.60a. When the block leaves the wedge, its velocity is measured to be 4.00 m/s to the right, as in Figure P9.60b. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height h of the wedge?

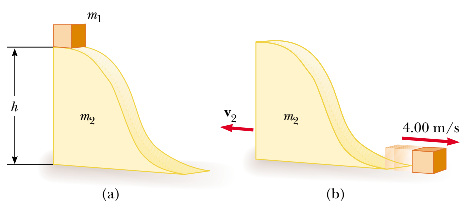


Figure P9.60

61. Tarzan, whose mass is 80.0 kg, swings from a 3.00-m vine that is horizontal when he starts. At the bottom of his arc, he picks up 60.0-kg Jane in a perfectly inelastic collision. What is the height of the highest tree limb they can reach on their upward swing?

62. A jet aircraft is traveling at 500 mi/h (223 m/s) in horizontal flight. The engine takes in air at a rate of 80.0 kg/s and burns fuel at a rate of 3.00 kg/s. If the exhaust gases are ejected at 600 m/s relative to the aircraft, find the thrust of the jet engine and the delivered horsepower.

63. A 75.0-kg firefighter slides down a pole while a constant frictional force of 300 N retards her motion. A horizontal 20.0-kg platform is supported by a spring at the bottom of the pole to cushion the fall. The firefighter starts from rest 4.00 m above the platform, and the spring constant is 4 000 N/m. Find (a) the firefighter's speed just before she collides with the platform and (b) the maximum distance the spring is compressed. (Assume the frictional force acts during the entire motion.)

64. A cannon is rigidly attached to a carriage, which can move along horizontal rails but is connected to a post by a large spring, initially unstretched and with force constant $k = 2.00 \times 10^4$ N/m, as shown in Figure P9.64. The cannon fires a 200-kg projectile at a velocity of 125 m/s directed 45.0° above the horizontal. (a) If the mass of the cannon and its carriage is 5 000 kg, find the recoil speed of the cannon. (b) Determine the maximum extension of the spring. (c) Find the maximum force the spring exerts on the carriage. (d) Consider the system consisting of the cannon, carriage, and shell. Is the momentum of this system conserved during the firing? Why or why not?

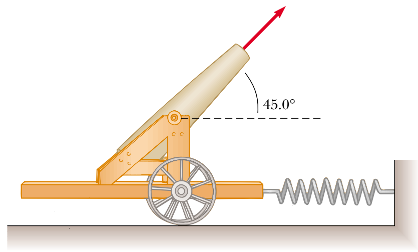


Figure P9.64

65. A chain of length L and total mass M is released from rest with its lower end just touching the top of a table, as shown in Figure P9.65a. Find the force exerted by the table on the chain after the chain has fallen through a distance x, as shown in Figure P9.65b. (Assume each link comes to rest the instant it reaches the table.)

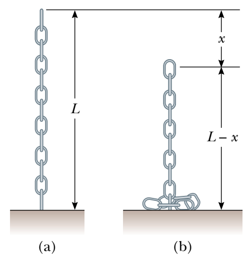


Figure P9.65

66. Two gliders are set in motion on an air track. A spring of force constant k is attached to the near side of one glider. The first glider of mass m_1 has a velocity of $\mathbf{v}_1$, and the second glider of mass m_2 has a velocity of $\mathbf{v}_2$, as shown in Figure P9.66 ($v_1 > v_2$). When m_1 collides with the spring attached to m_2 and compresses the spring to its maximum compression x_m, the velocity of the gliders is $\mathbf{v}$. In terms of $\mathbf{v}_1$, $\mathbf{v}_2$, m_1, m_2, and k, find (a) the velocity $\mathbf{v}$ at maximum compression, (b) the maximum compression x_m, and (c) the velocities of each glider after m_1 has lost contact with the spring.

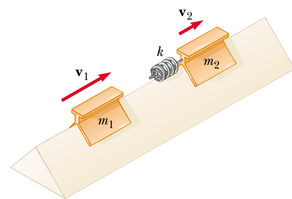


Figure P9.66

67. Sand from a stationary hopper falls onto a moving conveyor belt at the rate of 5.00 kg/s, as shown in Figure P9.67. The conveyor belt is supported by frictionless rollers and moves at a constant speed of 0.750 m/s under the action of a constant horizontal external force $\mathbf{F}_{\text{ext}}$ supplied by the motor that drives the belt. Find (a) the sand's rate of change of momentum in the horizontal direction, (b) the force of friction exerted by the belt on the sand, (c) the external force $\mathbf{F}_{\text{ext}}$, (d) the work done by $\mathbf{F}_{\text{ext}}$ in 1 s, and (e) the kinetic energy acquired by the falling sand each second due to the change in its horizontal motion. (f) Why are the answers to parts (d) and (e) different?

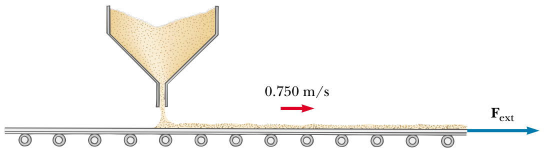


Figure P9.67

68. A rocket has total mass $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest, turns on its engine at time $t = 0$, and puts out exhaust with a relative speed of $v_e = 1\,500$ m/s at the constant rate $k = 2.50$ kg/s. Although the fuel will last for an actual burn time of 330 kg/(2.5 kg/s) = 132 s, define a "projected depletion time" as $T_p = M_i/k =$ 360 kg/(2.5 kg/s) = 144 s. (This would be the burn time if the rocket could use its payload, fuel tanks, and even the walls of the combustion chamber as fuel.)
(a) Show that during the burn the velocity of the rocket is given as a function of time by

$$v(t) = -v_e \ln(1 - t/T_p)$$

(b) Make a graph of the velocity of the rocket as a function of time for times running from 0 to 132 s. (c) Show that the acceleration of the rocket is

$$a(t) = v_e/(T_p - t)$$

(d) Graph the acceleration as a function of time.
(e) Show that the displacement of the rocket from its initial position at $t = 0$ is

$$x(t) = v_e(T_p - t)\ln(1 - t/T_p) + v_e t$$

(f) Graph the displacement during the burn.

69. A 40.0-kg child stands at one end of a 70.0-kg boat that is 4.00 m in length (Fig. P9.69). The boat is initially 3.00 m from the pier. The child notices a turtle on a rock near the far end of the boat and proceeds to walk to that end to catch the turtle. Neglecting friction between the boat and the water, (a) describe the subsequent motion of the system (child plus boat). (b) Where is the child *relative to the pier* when he reaches the far end of the boat? (c) Will he catch the turtle? (Assume he can reach out 1.00 m from the end of the boat.)

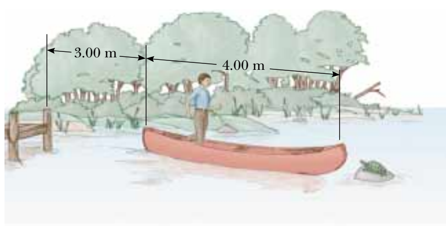


Figure P9.69

70. A student performs a ballistic pendulum experiment, using an apparatus similar to that shown in Figure 9.11b. She obtains the following average data: $h =$ 8.68 cm, $m_1 = 68.8$ g, and $m_2 = 263$ g. The symbols refer to the quantities in Figure 9.11a. (a) Determine the initial speed v_{1i} of the projectile. (b) In the second part of her experiment she is to obtain v_{1i} by firing the same projectile horizontally (with the pendulum removed from the path) and measuring its horizontal displacement x and vertical displacement y (Fig. P9.70). Show that the initial speed of the projectile is related to x and y through the relationship

$$v_{1i} = \frac{x}{\sqrt{2y/g}}$$

What numerical value does she obtain for v_{1i} on the basis of her measured values of $x = 257$ cm and $y =$ 85.3 cm? What factors might account for the difference in this value compared with that obtained in part (a)?

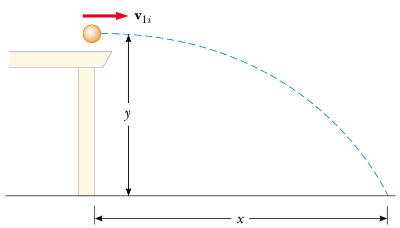


Figure P9.70

71. A 5.00-g bullet moving with an initial speed of 400 m/s is fired into and passes through a 1.00-kg block, as shown in Figure P9.71. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring of force constant 900 N/m. If the block moves 5.00 cm to the right after impact, find (a) the speed at which the bullet emerges from the block and (b) the energy lost in the collision.

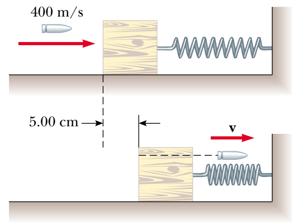


Figure P9.71

72. Two masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_i. Mass m is traveling to the left, while mass $3m$ is traveling to the right. They undergo a head-on elastic collision and each rebounds along the same line as it approached. Find the final speeds of the masses.

73. Two masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_i. Mass m is traveling to the left, while mass $3m$ is traveling to the right. They undergo an elastic glancing collision such that mass m is moving downward after the collision at right angles from its initial direction. (a) Find the final speeds of the two masses. (b) What is the angle θ at which the mass $3m$ is scattered?

74. Review Problem. There are (one can say) three coequal theories of motion: Newton's second law, stating that the total force on an object causes its acceleration; the work–kinetic energy theorem, stating that the total work on an object causes its change in kinetic energy; and the impulse–momentum theorem, stating that the total impulse on an object causes its change in momentum. In this problem, you compare predictions of the three theories in one particular case. A 3.00-kg object has a velocity of 7.00**j** m/s. Then, a total force 12.0**i** N acts on the object for 5.00 s. (a) Calculate the object's final velocity, using the impulse–momentum theorem. (b) Calculate its acceleration from $\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_i)/t$. (c) Calculate its acceleration from $\mathbf{a} = \Sigma\mathbf{F}/m$. (d) Find the object's vector displacement from $\mathbf{r} = \mathbf{v}_i t + \frac{1}{2}\mathbf{a}\,t^2$. (e) Find the work done on the object from $W = \mathbf{F}\cdot\mathbf{r}$. (f) Find the final kinetic energy from $\frac{1}{2}mv_f^2 = \frac{1}{2}m\mathbf{v}_f\cdot\mathbf{v}_f$. (g) Find the final kinetic energy from $\frac{1}{2}mv_i^2 + W$.

75. A rocket has a total mass of $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest. Its engine is turned on at time $t = 0$, and it puts out exhaust with a relative speed of $v_e = 1\,500$ m/s at the constant rate 2.50 kg/s. The burn lasts until the fuel runs out at time 330 kg/(2.5 kg/s) = 132 s. Set up and carry out a computer analysis of the motion according to Euler's method. Find (a) the final velocity of the rocket and (b) the distance it travels during the burn.

Answers to Quick Quizzes

9.1 (d). Two identical objects ($m_1 = m_2$) traveling in the same direction at the same speed ($v_1 = v_2$) have the same kinetic energies and the same momenta. However, this is not true if the two objects are moving at the same speed but in different directions. In the latter case, $K_1 = K_2$, but the differing velocity directions indicate that $\mathbf{p}_1 \neq \mathbf{p}_2$ because momentum is a vector quantity.
It also is possible for particular combinations of masses and velocities to satisfy $K_1 = K_2$ but not $p_1 = p_2$. For example, a 1-kg object moving at 2 m/s has the same kinetic energy as a 4-kg object moving at 1 m/s, but the two clearly do not have the same momenta.

9.2 (b), (c), (a). The slower the ball, the easier it is to catch. If the momentum of the medicine ball is the same as the momentum of the baseball, the speed of the medicine ball must be 1/10 the speed of the baseball because the medicine ball has 10 times the mass. If the kinetic energies are the same, the speed of the medicine ball must be $1/\sqrt{10}$ the speed of the baseball because of the squared speed term in the formula for K. The medicine ball is hardest to catch when it has the same speed as the baseball.

9.3 (c) and (e). Object 2 has a greater acceleration because of its smaller mass. Therefore, it takes less time to travel the distance d. Thus, even though the force applied to objects 1 and 2 is the same, the change in momentum is less for object 2 because Δt is smaller. Therefore, because the initial momenta were the same (both zero), $p_1 > p_2$. The work $W = Fd$ done on both objects is the same because both F and d are the same in the two cases. Therefore, $K_1 = K_2$.

9.4 Because the passenger is brought from the car's initial speed to a full stop, the change in momentum (the impulse) is the same regardless of whether the passenger is stopped by dashboard, seatbelt, or airbag. However, the dashboard stops the passenger very quickly in a front-end collision. The seatbelt takes somewhat more time. Used along with the seatbelt, the airbag can extend the passenger's stopping time further, notably for his head, which would otherwise snap forward. Therefore, the dashboard applies the greatest force, the seatbelt an intermediate force, and the airbag the least force. Airbags are designed to work in conjunction with seatbelts. Make sure you wear your seatbelt at all times while in a moving vehicle.

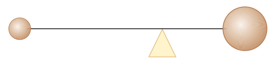
9.5 If we define the ball as our system, momentum is not conserved. The ball's speed—and hence its momentum—continually increase. This is consistent with the fact that the gravitational force is external to this chosen system. However, if we define our system as the ball and the Earth, momentum is conserved, for the Earth also has momentum because the ball exerts a gravitational force on it. As the ball falls, the Earth moves up to meet it (although the Earth's speed is on the order of 10^{25} times less than that of the ball!). This upward movement changes the Earth's momentum. The change in the Earth's momentum is numerically equal to the change in the ball's momentum but is in the opposite direction. Therefore, the total momentum of the Earth–ball system is conserved. Because the Earth's mass is so great, its upward motion is negligibly small.

9.6 (c). The greatest impulse (greatest change in momentum) is imparted to the Frisbee when the skater reverses its momentum vector by catching it and throwing it back. Since this is when the skater imparts the greatest impulse to the Frisbee, then this also is when the Frisbee imparts the greatest impulse to her.

9.7 Both are equally bad. Imagine watching the collision from a safer location alongside the road. As the "crush zones" of the two cars are compressed, you will see that the actual point of contact is stationary. You would see the same thing if your car were to collide with a solid wall.

9.8 No, such movement can never occur if we assume the collisions are elastic. The momentum of the system before the collision is mv, where m is the mass of ball 1 and v is its speed just before the collision. After the collision, we would have two balls, each of mass m and moving with a speed of $v/2$. Thus, the total momentum of the system after the collision would be $m(v/2) + m(v/2) = mv$. Thus, momentum is conserved. However, the kinetic energy just before the collision is $K_i = \frac{1}{2}mv^2$, and that after the collision is $K_f = \frac{1}{2}m(v/2)^2 + \frac{1}{2}m(v/2)^2 = \frac{1}{4}mv^2$. Thus, kinetic energy is *not* conserved. Both momentum and kinetic energy are conserved only when one ball moves out when one ball is released, two balls move out when two are released, and so on.

9.9 No they will not! The piece with the handle will have less mass than the piece made up of the end of the bat. To see why this is so, take the origin of coordinates as the center of mass before the bat was cut. Replace each cut piece by a small sphere located at the center of mass for each piece. The sphere representing the handle piece is farther from the origin, but the product of lesser mass and greater distance balances the product of greater mass and lesser distance for the end piece:



PUZZLER

Did you know that the CD inside this player spins at different speeds, depending on which song is playing? Why would such a strange characteristic be incorporated into the design of every CD player? *(George Semple)*

chapter

10 Rotation of a Rigid Object About a Fixed Axis

Chapter Outline

When an extended object, such as a wheel, rotates about its axis, the motion cannot be analyzed by treating the object as a particle because at any given time different parts of the object have different linear velocities and linear accelerations. For this reason, it is convenient to consider an extended object as a large number of particles, each of which has its own linear velocity and linear acceleration.

In dealing with a rotating object, analysis is greatly simplified by assuming that the object is rigid. A **rigid object** is one that is nondeformable—that is, it is an object in which the separations between all pairs of particles remain constant. All real bodies are deformable to some extent; however, our rigid-object model is useful in many situations in which deformation is negligible.

Rigid object

In this chapter, we treat the rotation of a rigid object about a fixed axis, which is commonly referred to as *pure rotational motion.*

10.1 ANGULAR DISPLACEMENT, VELOCITY, AND ACCELERATION

Figure 10.1 illustrates a planar (flat), rigid object of arbitrary shape confined to the xy plane and rotating about a fixed axis through O. The axis is perpendicular to the plane of the figure, and O is the origin of an xy coordinate system. Let us look at the motion of only one of the millions of "particles" making up this object. A particle at P is at a fixed distance r from the origin and rotates about it in a circle of radius r. (In fact, *every* particle on the object undergoes circular motion about O.) It is convenient to represent the position of P with its polar coordinates (r, θ), where r is the distance from the origin to P and θ is measured *counterclockwise* from some preferred direction—in this case, the positive x axis. In this representation, the only coordinate that changes in time is the angle θ; r remains constant. (In cartesian coordinates, both x and y vary in time.) As the particle moves along the circle from the positive x axis ($\theta = 0$) to P, it moves through an arc of length s, which is related to the angular position θ through the relationship

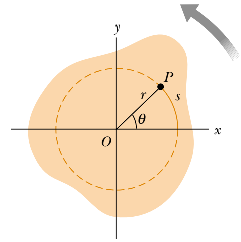


Figure 10.1 A rigid object rotating about a fixed axis through O perpendicular to the plane of the figure. (In other words, the axis of rotation is the z axis.) A particle at P rotates in a circle of radius r centered at O.

$$s = r\theta \qquad \textbf{(10.1a)}$$

$$\theta = \frac{s}{r} \qquad \textbf{(10.1b)}$$

It is important to note the units of θ in Equation 10.1b. Because θ is the ratio of an arc length and the radius of the circle, it is a pure number. However, we commonly give θ the artificial unit **radian** (rad), where

> one radian is the angle subtended by an arc length equal to the radius of the arc.

Radian

Because the circumference of a circle is $2\pi r$, it follows from Equation 10.1b that 360° corresponds to an angle of $2\pi r/r$ rad $= 2\pi$ rad (one revolution). Hence, 1 rad $= 360°/2\pi \approx 57.3°$. To convert an angle in degrees to an angle in radians, we use the fact that 2π rad $= 360°$:

$$\theta\ (\text{rad}) = \frac{\pi}{180°}\,\theta\ (\text{deg})$$

For example, 60° equals $\pi/3$ rad, and 45° equals $\pi/4$ rad.

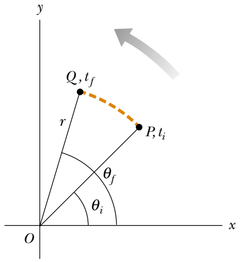
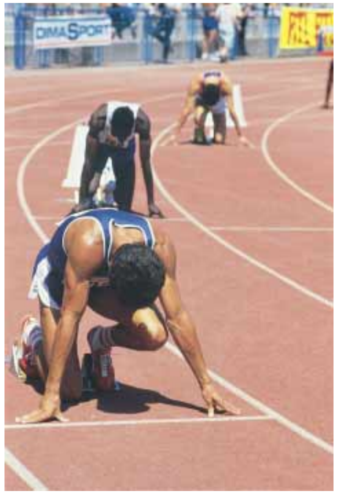


Figure 10.2 A particle on a rotating rigid object moves from P to Q along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radius vector sweeps out an angle $\Delta\theta = \theta_f - \theta_i$.



In a short track event, such as a 200-m or 400-m sprint, the runners begin from staggered positions on the track. Why don't they all begin from the same line?

As the particle in question on our rigid object travels from position P to position Q in a time Δt as shown in Figure 10.2, the radius vector sweeps out an angle $\Delta\theta = \theta_f - \theta_i$. This quantity $\Delta\theta$ is defined as the **angular displacement** of the particle:

$$\Delta\theta = \theta_f - \theta_i \qquad \textbf{(10.2)}$$

We define the **average angular speed** $\bar{\omega}$ (omega) as the ratio of this angular displacement to the time interval Δt:

Average angular speed

$$\bar{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad \textbf{(10.3)}$$

In analogy to linear speed, the **instantaneous angular speed** $\boldsymbol{\omega}$ is defined as the limit of the ratio $\Delta\theta/\Delta t$ as Δt approaches zero:

Instantaneous angular speed

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \textbf{(10.4)}$$

Angular speed has units of radians per second (rad/s), or rather second^{-1} (s^{-1}) because radians are not dimensional. We take ω to be positive when θ is increasing (counterclockwise motion) and negative when θ is decreasing (clockwise motion).

If the instantaneous angular speed of an object changes from ω_i to ω_f in the time interval Δt, the object has an angular acceleration. The **average angular acceleration** $\bar{\alpha}$ (alpha) of a rotating object is defined as the ratio of the change in the angular speed to the time interval Δt:

Average angular acceleration

$$\bar{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \qquad \textbf{(10.5)}$$

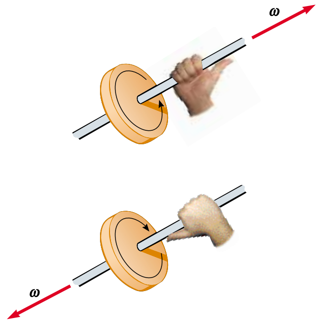


Figure 10.3 The right-hand rule for determining the direction of the angular velocity vector.

In analogy to linear acceleration, the **instantaneous angular acceleration** is defined as the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero:

Instantaneous angular acceleration

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad \textbf{(10.6)}$$

Angular acceleration has units of radians per second squared (rad/s^2), or just second^{-2} (s^{-2}). Note that α is positive when the rate of counterclockwise rotation is increasing or when the rate of clockwise rotation is decreasing.

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and the same angular acceleration. That is, the quantities θ, ω, and α characterize the rotational motion of the entire rigid object. Using these quantities, we can greatly simplify the analysis of rigid-body rotation.

Angular position (θ), angular speed (ω), and angular acceleration (α) are analogous to linear position (x), linear speed (v), and linear acceleration (a). The variables θ, ω, and α differ dimensionally from the variables x, v, and a only by a factor having the unit of length.

We have not specified any direction for ω and α. Strictly speaking, these variables are the magnitudes of the angular velocity and the angular acceleration vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, respectively, and they should always be positive. Because we are considering rotation about a fixed axis, however, we can indicate the directions of the vectors by assigning a positive or negative sign to ω and α, as discussed earlier with regard to Equations 10.4 and 10.6. For rotation about a fixed axis, the only direction that uniquely specifies the rotational motion is the direction along the axis of rotation. Therefore, the directions of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are along this axis. If an object rotates in the xy plane as in Figure 10.1, the direction of $\boldsymbol{\omega}$ is out of the plane of the diagram when the rotation is counterclockwise and into the plane of the diagram when the rotation is clockwise. To illustrate this convention, it is convenient to use the *right-hand rule* demonstrated in Figure 10.3. When the four fingers of the right hand are wrapped in the direction of rotation, the extended right thumb points in the direction of $\boldsymbol{\omega}$. The direction of $\boldsymbol{\alpha}$ follows from its definition $d\boldsymbol{\omega}/dt$. It is the same as the direction of $\boldsymbol{\omega}$ if the angular speed is increasing in time, and it is antiparallel to $\boldsymbol{\omega}$ if the angular speed is decreasing in time.

Quick Quiz 10.1

Describe a situation in which $\omega < 0$ and $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are antiparallel.

10.2 ROTATIONAL KINEMATICS: ROTATIONAL MOTION WITH CONSTANT ANGULAR ACCELERATION

7.2 In our study of linear motion, we found that the simplest form of accelerated motion to analyze is motion under constant linear acceleration. Likewise, for rotational motion about a fixed axis, the simplest accelerated motion to analyze is motion under constant angular acceleration. Therefore, we next develop kinematic relationships for this type of motion. If we write Equation 10.6 in the form $d\omega = \alpha\, dt$, and let $t_i = 0$ and $t_f = t$, we can integrate this expression directly:

$$\omega_f = \omega_i + \alpha t \qquad \text{(for constant } \alpha\text{)} \tag{10.7}$$

Substituting Equation 10.7 into Equation 10.4 and integrating once more we obtain

Rotational kinematic equations

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \qquad \text{(for constant } \alpha\text{)} \tag{10.8}$$

If we eliminate t from Equations 10.7 and 10.8, we obtain

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \qquad \text{(for constant } \alpha\text{)} \tag{10.9}$$

Notice that these kinematic expressions for rotational motion under constant angular acceleration are of the same form as those for linear motion under constant linear acceleration with the substitutions $x \rightarrow \theta$, $v \rightarrow \omega$, and $a \rightarrow \alpha$. Table 10.1 compares the kinematic equations for rotational and linear motion.

EXAMPLE 10.1 Rotating Wheel

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, (a) through what angle does the wheel rotate in 2.00 s?

Solution We can use Figure 10.2 to represent the wheel, and so we do not need a new drawing. This is a straightforward application of an equation from Table 10.1:

$$\theta_f - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2 = (2.00\ \text{rad/s})(2.00\ \text{s}) + \tfrac{1}{2}(3.50\ \text{rad/s}^2)(2.00\ \text{s})^2$$

$$= 11.0\ \text{rad} = (11.0\ \text{rad})(57.3°/\text{rad}) = \boxed{630°}$$

$$= \frac{630°}{360°/\text{rev}} = \boxed{1.75\ \text{rev}}$$

(b) What is the angular speed at $t = 2.00$ s?

Solution Because the angular acceleration and the angular speed are both positive, we can be sure our answer must be greater than 2.00 rad/s.

$$\omega_f = \omega_i + \alpha t = 2.00\ \text{rad/s} + (3.50\ \text{rad/s}^2)(2.00\ \text{s})$$

$$= \boxed{9.00\ \text{rad/s}}$$

We could also obtain this result using Equation 10.9 and the results of part (a). Try it! You also may want to see if you can formulate the linear motion analog to this problem.

Exercise Find the angle through which the wheel rotates between $t = 2.00$ s and $t = 3.00$ s.

Answer 10.8 rad.

TABLE 10.1 Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About a Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$

10.3 ANGULAR AND LINEAR QUANTITIES

In this section we derive some useful relationships between the angular speed and acceleration of a rotating rigid object and the linear speed and acceleration of an arbitrary point in the object. To do so, we must keep in mind that when a rigid object rotates about a fixed axis, as in Figure 10.4, every particle of the object moves in a circle whose center is the axis of rotation.

We can relate the angular speed of the rotating object to the tangential speed of a point P on the object. Because point P moves in a circle, the linear velocity vector $\mathbf{v}$ is always tangent to the circular path and hence is called *tangential velocity*. The magnitude of the tangential velocity of the point P is by definition the tangential speed $v = ds/dt$, where s is the distance traveled by this point measured along the circular path. Recalling that $s = r\theta$ (Eq. 10.1a) and noting that r is constant, we obtain

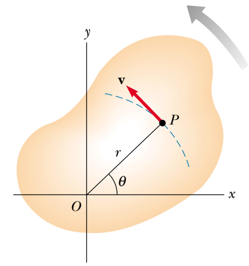


Figure 10.4 As a rigid object rotates about the fixed axis through O, the point P has a linear velocity $\mathbf{v}$ that is always tangent to the circular path of radius r.

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Because $d\theta/dt = \omega$ (see Eq. 10.4), we can say

Relationship between linear and angular speed

$$v = r\omega \tag{10.10}$$

That is, the tangential speed of a point on a rotating rigid object equals the perpendicular distance of that point from the axis of rotation multiplied by the angular speed. Therefore, although every point on the rigid object has the same *angular* speed, not every point has the same *linear* speed because r is not the same for all points on the object. Equation 10.10 shows that the linear speed of a point on the rotating object increases as one moves outward from the center of rotation, as we would intuitively expect. The outer end of a swinging baseball bat moves much faster than the handle.

QuickLab

Spin a tennis ball or basketball and watch it gradually slow down and stop. Estimate α and a_t as accurately as you can.

We can relate the angular acceleration of the rotating rigid object to the tangential acceleration of the point P by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

Relationship between linear and angular acceleration

$$a_t = r\alpha \tag{10.11}$$

That is, the tangential component of the linear acceleration of a point on a rotating rigid object equals the point's distance from the axis of rotation multiplied by the angular acceleration.

In Section 4.4 we found that a point rotating in a circular path undergoes a centripetal, or radial, acceleration $\mathbf{a}_r$ of magnitude v^2/r directed toward the center of rotation (Fig. 10.5). Because $v = r\omega$ for a point P on a rotating object, we can express the radial acceleration of that point as

$$a_r = \frac{v^2}{r} = r\omega^2 \tag{10.12}$$

The total linear acceleration vector of the point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$. ($\mathbf{a}_t$ describes the change in how fast the point is moving, and $\mathbf{a}_r$ represents the change in its direction of travel.) Because $\mathbf{a}$ is a vector having a radial and a tangential component, the magnitude of $\mathbf{a}$ for the point P on the rotating rigid object is

$$a = \sqrt{a_t^{\,2} + a_r^{\,2}} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \tag{10.13}$$

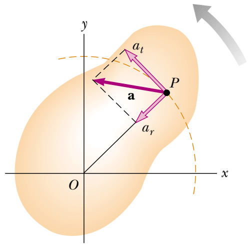


Figure 10.5 As a rigid object rotates about a fixed axis through O, the point P experiences a tangential component of linear acceleration a_t and a radial component of linear acceleration a_r. The total linear acceleration of this point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$.

Quick Quiz 10.2

When a wheel of radius R rotates about a fixed axis, do all points on the wheel have (a) the same angular speed and (b) the same linear speed? If the angular speed is constant and equal to ω, describe the linear speeds and linear accelerations of the points located at (c) $r = 0$, (d) $r = R/2$, and (e) $r = R$, all measured from the center of the wheel.

EXAMPLE 10.2 CD Player

On a compact disc, audio information is stored in a series of pits and flat areas on the surface of the disc. The information is stored digitally, and the alternations between pits and flat areas on the surface represent binary ones and zeroes to be read by the compact disc player and converted back to sound waves. The pits and flat areas are detected by a system consisting of a laser and lenses. The length of a certain number of ones and zeroes is the same everywhere on the disc, whether the information is near the center of the disc or near its outer edge. In order that this length of ones and zeroes always passes by the laser–lens system in the same time period, the linear speed of the disc surface at the location of the lens must be constant. This requires, according to Equation 10.10, that the angular speed vary as the laser–lens system moves radially along the disc. In a typical compact disc player, the disc spins counterclockwise (Fig. 10.6), and the constant speed of the surface at the point of the laser–lens system is 1.3 m/s. (a) Find the angular speed of the disc in revolutions per minute when information is being read from the innermost first track (r = 23 mm) and the outermost final track (r = 58 mm).

Solution Using Equation 10.10, we can find the angular speed; this will give us the required linear speed at the position of the inner track,

$$\omega_i = \frac{v}{r_i} = \frac{1.3 \text{ m/s}}{2.3 \times 10^{-2} \text{ m}} = 56.5 \text{ rad/s}$$

$$= (56.5 \text{ rad/s})\left(\frac{1}{2\pi} \text{ rev/rad}\right)(60 \text{ s/min})$$

$$= 5.4 \times 10^2 \text{ rev/min}$$

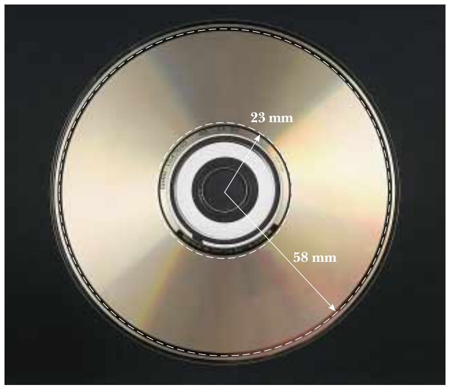


Figure 10.6 A compact disc.

For the outer track,

$$\omega_f = \frac{v}{r_f} = \frac{1.3 \text{ m/s}}{5.8 \times 10^{-2} \text{ m}} = 22.4 \text{ rad/s}$$

$$= 2.1 \times 10^2 \text{ rev/min}$$

The player adjusts the angular speed ω of the disc within this range so that information moves past the objective lens at a constant rate. These angular velocity values are positive because the direction of rotation is counterclockwise.

(b) The maximum playing time of a standard music CD is 74 minutes and 33 seconds. How many revolutions does the disc make during that time?

Solution We know that the angular speed is always decreasing, and we assume that it is decreasing steadily, with α constant. The time interval t is (74 min)(60 s/min) + 33 s = 4 473 s. We are looking for the angular position θ_f, where we set the initial angular position $\theta_i = 0$. We can use Equation 10.3, replacing the average angular speed $\bar{\omega}$ with its mathematical equivalent $(\omega_i + \omega_f)/2$:

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)t$$
$$= 0 + \tfrac{1}{2}(540 \text{ rev/min} + 210 \text{ rev/min})(1 \text{ min}/60 \text{ s})(4\,473 \text{ s})$$
$$= 2.8 \times 10^4 \text{ rev}$$

(c) What total length of track moves past the objective lens during this time?

Solution Because we know the (constant) linear velocity and the time interval, this is a straightforward calculation:

$$x_f = v_i t = (1.3 \text{ m/s})(4\,473 \text{ s}) = 5.8 \times 10^3 \text{ m}$$

More than 3.6 miles of track spins past the objective lens!

(d) What is the angular acceleration of the CD over the 4 473-s time interval? Assume that α is constant.

Solution We have several choices for approaching this problem. Let us use the most direct approach by utilizing Equation 10.5, which is based on the definition of the term we are seeking. We should obtain a negative number for the angular acceleration because the disc spins more and more slowly in the positive direction as time goes on. Our answer should also be fairly small because it takes such a long time—more than an hour—for the change in angular speed to be accomplished:

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{22.4 \text{ rad/s} - 56.5 \text{ rad/s}}{4\,473 \text{ s}}$$

$$= -7.6 \times 10^{-3} \text{ rad/s}^2$$

The disc experiences a very gradual decrease in its rotation rate, as expected.

10.4 ROTATIONAL ENERGY

7.3

Let us now look at the kinetic energy of a rotating rigid object, considering the object as a collection of particles and assuming it rotates about a fixed z axis with an angular speed ω (Fig. 10.7). Each particle has kinetic energy determined by its mass and linear speed. If the mass of the ith particle is m_i and its linear speed is v_i, its kinetic energy is

$$K_i = \tfrac{1}{2} m_i v_i^{\,2}$$

To proceed further, we must recall that although every particle in the rigid object has the same angular speed ω, the individual linear speeds depend on the distance r_i from the axis of rotation according to the expression $v_i = r_i\omega$ (see Eq. 10.10). The *total* kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:

$$K_R = \sum_i K_i = \sum_i \tfrac{1}{2} m_i v_i^{\,2} = \tfrac{1}{2}\sum_i m_i r_i^{\,2}\omega^2$$

We can write this expression in the form

$$K_R = \tfrac{1}{2}\left(\sum_i m_i r_i^{\,2}\right)\omega^2 \tag{10.14}$$

where we have factored ω^2 from the sum because it is common to every particle.

web If you want to learn more about the physics of CD players, visit the Special Interest Group on CD Applications and Technology at **www.sigcat.org**

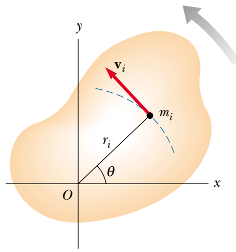


Figure 10.7 A rigid object rotating about a z axis with angular speed ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2}m_i v_i^{\,2}$. The total kinetic energy of the object is called its rotational kinetic energy.

We simplify this expression by defining the quantity in parentheses as the **moment of inertia I:**

Moment of inertia

$$I \equiv \sum_i m_i r_i^2 \qquad \textbf{(10.15)}$$

From the definition of moment of inertia, we see that it has dimensions of ML^2 ($kg \cdot m^2$ in SI units).[1] With this notation, Equation 10.14 becomes

Rotational kinetic energy

$$K_R = \tfrac{1}{2} I \omega^2 \qquad \textbf{(10.16)}$$

Although we commonly refer to the quantity $\frac{1}{2}I\omega^2$ as **rotational kinetic energy,** it is not a new form of energy. It is ordinary kinetic energy because it is derived from a sum over individual kinetic energies of the particles contained in the rigid object. However, the mathematical form of the kinetic energy given by Equation 10.16 is a convenient one when we are dealing with rotational motion, provided we know how to calculate I.

It is important that you recognize the analogy between kinetic energy associated with linear motion $\frac{1}{2}mv^2$ and rotational kinetic energy $\frac{1}{2}I\omega^2$. The quantities I and ω in rotational motion are analogous to m and v in linear motion, respectively. (In fact, I takes the place of m every time we compare a linear-motion equation with its rotational counterpart.) The moment of inertia is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion. Note, however, that mass is an intrinsic property of an object, whereas I depends on the physical arrangement of that mass. Can you think of a situation in which an object's moment of inertia changes even though its mass does not?

EXAMPLE 10.3 The Oxygen Molecule

Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is 2.66×10^{-26} kg, and at room temperature the average separation between the two atoms is $d = 1.21 \times 10^{-10}$ m (the atoms are treated as point masses). (a) Calculate the moment of inertia of the molecule about the z axis.

Solution This is a straightforward application of the definition of I. Because each atom is a distance $d/2$ from the z axis, the moment of inertia about the axis is

$$I = \sum_i m_i r_i^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \tfrac{1}{2}md^2$$
$$= \tfrac{1}{2}(2.66 \times 10^{-26}\ \text{kg})(1.21 \times 10^{-10}\ \text{m})^2$$
$$= \boxed{1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2}$$

This is a very small number, consistent with the minuscule masses and distances involved.

(b) If the angular speed of the molecule about the z axis is 4.60×10^{12} rad/s, what is its rotational kinetic energy?

Solution We apply the result we just calculated for the moment of inertia in the formula for K_R:

$$K_R = \tfrac{1}{2}I\omega^2$$
$$= \tfrac{1}{2}(1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2)(4.60 \times 10^{12}\ \text{rad/s})^2$$
$$= \boxed{2.06 \times 10^{-21}\ \text{J}}$$

[1] Civil engineers use moment of inertia to characterize the elastic properties (rigidity) of such structures as loaded beams. Hence, it is often useful even in a nonrotational context.

EXAMPLE 10.4 Four Rotating Masses

Four tiny spheres are fastened to the corners of a frame of negligible mass lying in the xy plane (Fig. 10.8). We shall assume that the spheres' radii are small compared with the dimensions of the frame. (a) If the system rotates about the y axis with an angular speed ω, find the moment of inertia and the rotational kinetic energy about this axis.

Solution First, note that the two spheres of mass m, which lie on the y axis, do not contribute to I_y (that is, $r_i = 0$ for these spheres about this axis). Applying Equation 10.15, we obtain

$$I_y = \sum_i m_i r_i^2 = Ma^2 + Ma^2 = \boxed{2Ma^2}$$

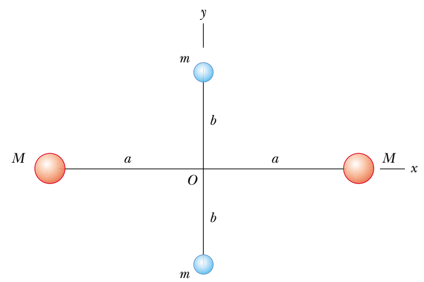


Figure 10.8 The four spheres are at a fixed separation as shown. The moment of inertia of the system depends on the axis about which it is evaluated.

Therefore, the rotational kinetic energy about the y axis is

$$K_R = \tfrac{1}{2}I_y\omega^2 = \tfrac{1}{2}(2Ma^2)\omega^2 = \boxed{Ma^2\omega^2}$$

The fact that the two spheres of mass m do not enter into this result makes sense because they have no motion about the axis of rotation; hence, they have no rotational kinetic energy. By similar logic, we expect the moment of inertia about the x axis to be $I_x = 2mb^2$ with a rotational kinetic energy about that axis of $K_R = mb^2\omega^2$.

(b) Suppose the system rotates in the xy plane about an axis through O (the z axis). Calculate the moment of inertia and rotational kinetic energy about this axis.

Solution Because r_i in Equation 10.15 is the *perpendicular* distance to the axis of rotation, we obtain

$$I_z = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = \boxed{2Ma^2 + 2mb^2}$$

$$K_R = \tfrac{1}{2}I_z\omega^2 = \tfrac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = \boxed{(Ma^2 + mb^2)\omega^2}$$

Comparing the results for parts (a) and (b), we conclude that the moment of inertia and therefore the rotational kinetic energy associated with a given angular speed depend on the axis of rotation. In part (b), we expect the result to include all four spheres and distances because all four spheres are rotating in the xy plane. Furthermore, the fact that the rotational kinetic energy in part (a) is smaller than that in part (b) indicates that it would take less effort (work) to set the system into rotation about the y axis than about the z axis.

10.5 CALCULATION OF MOMENTS OF INERTIA

7.5 We can evaluate the moment of inertia of an extended rigid object by imagining the object divided into many small volume elements, each of which has mass Δm. We use the definition $I = \sum_i r_i^2 \Delta m_i$ and take the limit of this sum as $\Delta m \to 0$. In this limit, the sum becomes an integral over the whole object:

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2\, dm \qquad \textbf{(10.17)}$$

It is usually easier to calculate moments of inertia in terms of the volume of the elements rather than their mass, and we can easily make that change by using Equation 1.1, $\rho = m/V$, where ρ is the density of the object and V is its volume. We want this expression in its differential form $\rho = dm/dV$ because the volumes we are dealing with are very small. Solving for $dm = \rho\, dV$ and substituting the result into Equation 10.17 gives

$$I = \int \rho r^2 \, dV$$

If the object is homogeneous, then ρ is constant and the integral can be evaluated for a known geometry. If ρ is not constant, then its variation with position must be known to complete the integration.

The density given by $\rho = m/V$ sometimes is referred to as *volume density* for the obvious reason that it relates to volume. Often we use other ways of expressing density. For instance, when dealing with a sheet of uniform thickness t, we can define a *surface density* $\sigma = \rho t$, which signifies *mass per unit area.* Finally, when mass is distributed along a uniform rod of cross-sectional area A, we sometimes use *linear density* $\lambda = M/L = \rho A$, which is the *mass per unit length.*

EXAMPLE 10.5 Uniform Hoop

Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center (Fig. 10.9).

Solution All mass elements dm are the same distance $r = R$ from the axis, and so, applying Equation 10.17, we obtain for the moment of inertia about the z axis through O:

$$I_z = \int r^2 \, dm = R^2 \int dm = \boxed{MR^2}$$

Note that this moment of inertia is the same as that of a single particle of mass M located a distance R from the axis of rotation.

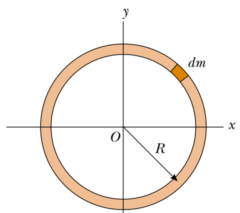


Figure 10.9 The mass elements dm of a uniform hoop are all the same distance from O.

Quick Quiz 10.3

(a) Based on what you have learned from Example 10.5, what do you expect to find for the moment of inertia of two particles, each of mass $M/2$, located anywhere on a circle of radius R around the axis of rotation? (b) How about the moment of inertia of four particles, each of mass $M/4$, again located a distance R from the rotation axis?

EXAMPLE 10.6 Uniform Rigid Rod

Calculate the moment of inertia of a uniform rigid rod of length L and mass M (Fig. 10.10) about an axis perpendicular to the rod (the y axis) and passing through its center of mass.

Solution The shaded length element dx has a mass dm equal to the mass per unit length λ multiplied by dx:

$$dm = \lambda \, dx = \frac{M}{L} \, dx$$

Substituting this expression for dm into Equation 10.17, with $r = x$, we obtain

$$I_y = \int r^2 \, dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} \, dx = \frac{M}{L} \int_{-L/2}^{L/2} x^2 \, dx$$

$$= \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{L/2} = \boxed{\tfrac{1}{12} ML^2}$$

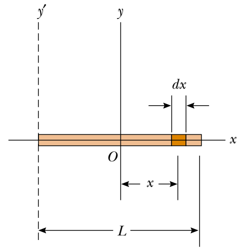


Figure 10.10 A uniform rigid rod of length L. The moment of inertia about the y axis is less than that about the y' axis. The latter axis is examined in Example 10.8.

EXAMPLE 10.7 Uniform Solid Cylinder

A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis (the z axis in Fig. 10.11).

Solution It is convenient to divide the cylinder into many cylindrical shells, each of which has radius r, thickness dr, and length L, as shown in Figure 10.11. The volume dV of each shell is its cross-sectional area multiplied by its length: $dV = dA \cdot L = (2\pi r \, dr)L$. If the mass per unit volume is ρ, then the mass of this differential volume element is $dm = \rho dV = \rho 2\pi r L \, dr$. Substituting this expression for dm into Equation 10.17, we obtain

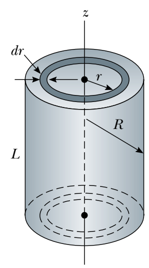


Figure 10.11 Calculating I about the z axis for a uniform solid cylinder.

$$I_z = \int r^2 \, dm = 2\pi\rho L \int_0^R r^3 \, dr = \tfrac{1}{2}\pi\rho L R^4$$

Because the total volume of the cylinder is $\pi R^2 L$, we see that $\rho = M/V = M/\pi R^2 L$. Substituting this value for ρ into the above result gives

$$(1) \qquad I_z = \boxed{\tfrac{1}{2} MR^2}$$

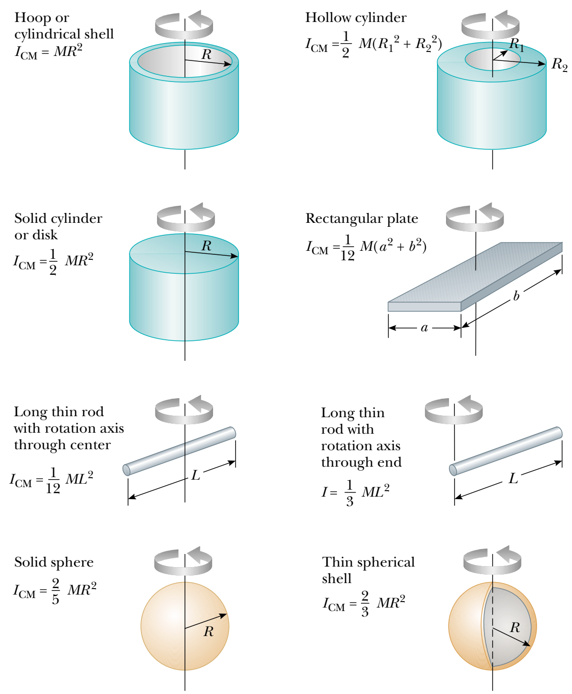
Note that this result does not depend on L, the length of the cylinder. In other words, it applies equally well to a long cylinder and a flat disc. Also note that this is exactly half the value we would expect were all the mass concentrated at the outer edge of the cylinder or disc. (See Example 10.5.)

Table 10.2 gives the moments of inertia for a number of bodies about specific axes. The moments of inertia of rigid bodies with simple geometry (high symmetry) are relatively easy to calculate provided the rotation axis coincides with an axis of symmetry. The calculation of moments of inertia about an arbitrary axis can be cumbersome, however, even for a highly symmetric object. Fortunately, use of an important theorem, called the **parallel-axis theorem,** often simplifies the calculation. Suppose the moment of inertia about an axis through the center of mass of an object is I_{CM}. The parallel-axis theorem states that the moment of inertia about any axis parallel to and a distance D away from this axis is

$$I = I_{CM} + MD^2 \qquad \textbf{(10.18)}$$

Parallel-axis theorem

TABLE 10.2 Moments of Inertia of Homogeneous Rigid Bodies with Different Geometries



Proof of the Parallel-Axis Theorem *(Optional)*. Suppose that an object rotates in the *xy* plane about the *z* axis, as shown in Figure 10.12, and that the coordinates of the center of mass are x_{CM}, y_{CM}. Let the mass element *dm* have coordinates *x*, *y*. Because this element is a distance $r = \sqrt{x^2 + y^2}$ from the *z* axis, the moment of inertia about the *z* axis is

$$I = \int r^2\, dm = \int (x^2 + y^2)\, dm$$

However, we can relate the coordinates *x*, *y* of the mass element *dm* to the coordinates of this same element located in a coordinate system having the object's center of mass as its origin. If the coordinates of the center of mass are x_{CM}, y_{CM} in the original coordinate system centered on *O*, then from Figure 10.12a we see that the relationships between the unprimed and primed coordinates are $x = x' + x_{CM}$

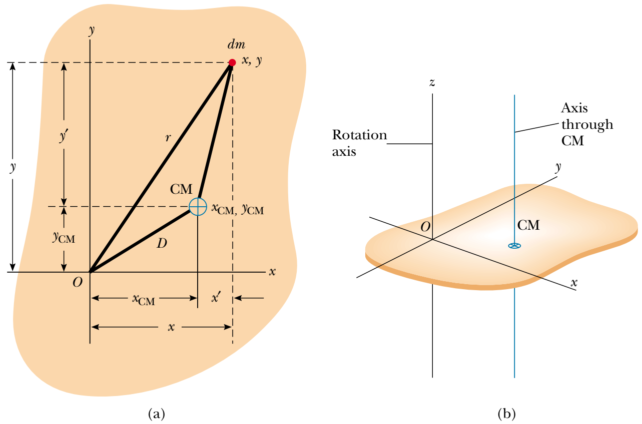


Figure 10.12 (a) The parallel-axis theorem: If the moment of inertia about an axis perpendicular to the figure through the center of mass is I_{CM}, then the moment of inertia about the *z* axis is $I_z = I_{CM} + MD^2$. (b) Perspective drawing showing the *z* axis (the axis of rotation) and the parallel axis through the CM.

and $y = y' + y_{CM}$. Therefore,

$$I = \int [(x' + x_{CM})^2 + (y' + y_{CM})^2]\, dm$$

$$= \int [(x')^2 + (y')^2]\, dm + 2x_{CM} \int x'\, dm + 2y_{CM} \int y'\, dm + (x_{CM}^2 + y_{CM}^2) \int dm$$

The first integral is, by definition, the moment of inertia about an axis that is parallel to the *z* axis and passes through the center of mass. The second two integrals are zero because, by definition of the center of mass, $\int x'\, dm = \int y'\, dm = 0$. The last integral is simply MD^2 because $\int dm = M$ and $D^2 = x_{CM}^2 + y_{CM}^2$. Therefore, we conclude that

$$I = I_{CM} + MD^2$$

EXAMPLE 10.8 Applying the Parallel-Axis Theorem

Consider once again the uniform rigid rod of mass *M* and length *L* shown in Figure 10.10. Find the moment of inertia of the rod about an axis perpendicular to the rod through one end (the *y′* axis in Fig. 10.10).

Solution Intuitively, we expect the moment of inertia to be greater than $I_{CM} = \frac{1}{12}ML^2$ because it should be more difficult to change the rotational motion of a rod spinning about an axis at one end than one that is spinning about its center. Because the distance between the center-of-mass axis and the *y′* axis is $D = L/2$, the parallel-axis theorem gives

$$I = I_{CM} + MD^2 = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2$$

So, it is four times more difficult to change the rotation of a rod spinning about its end than it is to change the motion of one spinning about its center.

Exercise Calculate the moment of inertia of the rod about a perpendicular axis through the point $x = L/4$.

Answer $I = \frac{7}{48} ML^2$.

10.6 TORQUE

7.6 Why are a door's doorknob and hinges placed near opposite edges of the door? This question actually has an answer based on common sense ideas. The harder we push against the door and the farther we are from the hinges, the more likely we are to open or close the door. When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque** $\boldsymbol{\tau}$ (tau).

Consider the wrench pivoted on the axis through O in Figure 10.13. The applied force $\mathbf{F}$ acts at an angle ϕ to the horizontal. We define the magnitude of the torque associated with the force $\mathbf{F}$ by the expression

Definition of torque

$$\tau \equiv rF\sin\phi = Fd \qquad \textbf{(10.19)}$$

where r is the distance between the pivot point and the point of application of $\mathbf{F}$ and d is the perpendicular distance from the pivot point to the line of action of $\mathbf{F}$. (The *line of action* of a force is an imaginary line extending out both ends of the vector representing the force. The dashed line extending from the tail of $\mathbf{F}$ in Figure 10.13 is part of the line of action of $\mathbf{F}$.) From the right triangle in Figure 10.13 that has the wrench as its hypotenuse, we see that $d = r\sin\phi$. This quantity d is called the **moment arm** (or *lever arm*) of $\mathbf{F}$.

Moment arm

It is very important that you recognize that *torque is defined only when a reference axis is specified*. Torque is the product of a force and the moment arm of that force, and moment arm is defined only in terms of an axis of rotation.

In Figure 10.13, the only component of $\mathbf{F}$ that tends to cause rotation is $F\sin\phi$, the component perpendicular to r. The horizontal component $F\cos\phi$, because it passes through O, has no tendency to produce rotation. From the definition of torque, we see that the rotating tendency increases as $\mathbf{F}$ increases and as d increases. This explains the observation that it is easier to close a door if we push at the doorknob rather than at a point close to the hinge. We also want to apply our push as close to perpendicular to the door as we can. Pushing sideways on the doorknob will not cause the door to rotate.

If two or more forces are acting on a rigid object, as shown in Figure 10.14, each tends to produce rotation about the pivot at O. In this example, $\mathbf{F}_2$ tends to

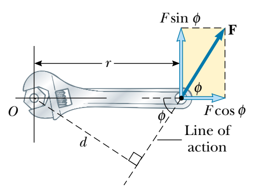


Figure 10.13 The force $\mathbf{F}$ has a greater rotating tendency about O as F increases and as the moment arm d increases. It is the component $F\sin\phi$ that tends to rotate the wrench about O.

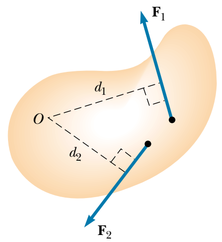


Figure 10.14 The force $\mathbf{F}_1$ tends to rotate the object counterclockwise about O, and $\mathbf{F}_2$ tends to rotate it clockwise.

rotate the object clockwise, and $\mathbf{F}_1$ tends to rotate it counterclockwise. We use the convention that the sign of the torque resulting from a force is positive if the turning tendency of the force is counterclockwise and is negative if the turning tendency is clockwise. For example, in Figure 10.14, the torque resulting from $\mathbf{F}_1$, which has a moment arm d_1, is positive and equal to $+F_1d_1$; the torque from $\mathbf{F}_2$ is negative and equal to $-F_2d_2$. Hence, the net torque about O is

$$\sum\tau = \tau_1 + \tau_2 = F_1d_1 - F_2d_2$$

Torque should not be confused with force. Forces can cause a change in linear motion, as described by Newton's second law. Forces can also cause a change in rotational motion, but the effectiveness of the forces in causing this change depends on both the forces and the moment arms of the forces, in the combination that we call *torque*. Torque has units of force times length—newton · meters in SI units—and should be reported in these units. Do not confuse torque and work, which have the same units but are very different concepts.

EXAMPLE 10.9 The Net Torque on a Cylinder

A one-piece cylinder is shaped as shown in Figure 10.15, with a core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the drawing. A rope wrapped around the drum, which has radius R_1, exerts a force $\mathbf{F}_1$ to the right on the cylinder. A rope wrapped around the core, which has radius R_2, exerts a force $\mathbf{F}_2$ downward on the cylinder. (a) What is the net torque acting on the cylinder about the rotation axis (which is the z axis in Figure 10.15)?

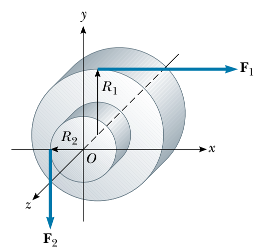


Figure 10.15 A solid cylinder pivoted about the z axis through O. The moment arm of $\mathbf{F}_1$ is R_1, and the moment arm of $\mathbf{F}_2$ is R_2.

Solution The torque due to $\mathbf{F}_1$ is $-R_1F_1$ (the sign is negative because the torque tends to produce clockwise rotation). The torque due to $\mathbf{F}_2$ is $+R_2F_2$ (the sign is positive because the torque tends to produce counterclockwise rotation). Therefore, the net torque about the rotation axis is

$$\sum\tau = \tau_1 + \tau_2 = -R_1F_1 + R_2F_2$$

We can make a quick check by noting that if the two forces are of equal magnitude, the net torque is negative because $R_1 > R_2$. Starting from rest with both forces acting on it, the cylinder would rotate clockwise because $\mathbf{F}_1$ would be more effective at turning it than would $\mathbf{F}_2$.

(b) Suppose $F_1 = 5.0$ N, $R_1 = 1.0$ m, $F_2 = 15.0$ N, and $R_2 = 0.50$ m. What is the net torque about the rotation axis, and which way does the cylinder rotate from rest?

$$\sum\tau = -(5.0\text{ N})(1.0\text{ m}) + (15.0\text{ N})(0.50\text{ m}) = 2.5\text{ N}\cdot\text{m}$$

Because the net torque is positive, if the cylinder starts from rest, it will commence rotating counterclockwise with increasing angular velocity. (If the cylinder's initial rotation is clockwise, it will slow to a stop and then rotate counterclockwise with increasing angular speed.)

10.7 RELATIONSHIP BETWEEN TORQUE AND ANGULAR ACCELERATION

7.6 In this section we show that the angular acceleration of a rigid object rotating about a fixed axis is proportional to the net torque acting about that axis. Before discussing the more complex case of rigid-body rotation, however, it is instructive

Figure 10.16 A particle rotating in a circle under the influence of a tangential force $\mathbf{F}_t$. A force $\mathbf{F}_r$ in the radial direction also must be present to maintain the circular motion.

first to discuss the case of a particle rotating about some fixed point under the influence of an external force.

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force $\mathbf{F}_t$ and a radial force $\mathbf{F}_r$, as shown in Figure 10.16. (As we learned in Chapter 6, the radial force must be present to keep the particle moving in its circular path.) The tangential force provides a tangential acceleration $\mathbf{a}_t$, and

$$F_t = ma_t$$

The torque about the center of the circle due to $\mathbf{F}_t$ is

$$\tau = F_t r = (ma_t)r$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$ (see Eq. 10.11), the torque can be expressed as

$$\tau = (mr\alpha)r = (mr^2)\alpha$$

Recall from Equation 10.15 that mr^2 is the moment of inertia of the rotating particle about the z axis passing through the origin, so that

Relationship between torque and angular acceleration

$$\tau = I\alpha \qquad \textbf{(10.20)}$$

That is, **the torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. It is important to note that $\tau = I\alpha$ is the rotational analog of Newton's second law of motion, $F = ma$.

Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis, as shown in Figure 10.17. The object can be regarded as an infinite number of mass elements dm of infinitesimal size. If we impose a cartesian coordinate system on the object, then each mass element rotates in a circle about the origin, and each has a tangential acceleration $\mathbf{a}_t$ produced by an external tangential force $d\mathbf{F}_t$. For any given element, we know from Newton's second law that

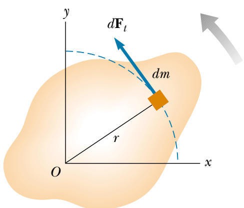


Figure 10.17 A rigid object rotating about an axis through O. Each mass element dm rotates about O with the same angular acceleration α, and the net torque on the object is proportional to α.

$$dF_t = (dm)a_t$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ acts about the origin and is given by

$$d\tau = r\,dF_t = (r\,dm)a_t$$

Because $a_t = r\alpha$, the expression for $d\tau$ becomes

$$d\tau = (r\,dm)r\alpha = (r^2\,dm)\alpha$$

It is important to recognize that although each mass element of the rigid object may have a different linear acceleration $\mathbf{a}_t$, they all have the *same* angular acceleration α. With this in mind, we can integrate the above expression to obtain the net torque about O due to the external forces:

$$\sum \tau = \int (r^2\,dm)\alpha = \alpha \int r^2\,dm$$

where α can be taken outside the integral because it is common to all mass elements. From Equation 10.17, we know that $\int r^2\,dm$ is the moment of inertia of the object about the rotation axis through O, and so the expression for $\sum\tau$ becomes

Torque is proportional to angular acceleration

$$\sum \tau = I\alpha \qquad \textbf{(10.21)}$$

Note that this is the same relationship we found for a particle rotating in a circle (see Eq. 10.20). So, again we see that the net torque about the rotation axis is proportional to the angular acceleration of the object, with the proportionality factor being I, a quantity that depends upon the axis of rotation and upon the size and shape of the object. In view of the complex nature of the system, it is interesting to note that the relationship $\sum\tau = I\alpha$ is strikingly simple and in complete agreement with experimental observations. The simplicity of the result lies in the manner in which the motion is described.

Although each point on a rigid object rotating about a fixed axis may not experience the same force, linear acceleration, or linear speed, each point experiences the same angular acceleration and angular speed at any instant. Therefore, at any instant the rotating rigid object as a whole is characterized by specific values for angular acceleration, net torque, and angular speed.

Every point has the same ω and α

QuickLab

Tip over a child's tall tower of blocks. Try this several times. Does the tower "break" at the same place each time? What affects where the tower comes apart as it tips? If the tower were made of toy bricks that snap together, what would happen? (Refer to Conceptual Example 10.11.)

Finally, note that the result $\sum\tau = I\alpha$ also applies when the forces acting on the mass elements have radial components as well as tangential components. This is because the line of action of all radial components must pass through the axis of rotation, and hence all radial components produce zero torque about that axis.

EXAMPLE 10.10 Rotating Rod

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane, as shown in Figure 10.18. The rod is released from rest in the horizontal position. What is the initial angular acceleration of the rod and the initial linear acceleration of its right end?

Solution We cannot use our kinematic equations to find α or a because the torque exerted on the rod varies with its position, and so neither acceleration is constant. We have enough information to find the torque, however, which we can then use in the torque–angular acceleration relationship (Eq. 10.21) to find α and then a.

The only force contributing to torque about an axis through the pivot is the gravitational force $M\mathbf{g}$ exerted on the rod. (The force exerted by the pivot on the rod has zero torque about the pivot because its moment arm is zero.) To compute the torque on the rod, we can assume that the gravitational force acts at the center of mass of the rod, as shown in Figure 10.18. The torque due to this force about an axis through the pivot is

$$\tau = Mg\left(\frac{L}{2}\right)$$

With $\sum\tau = I\alpha$, and $I = \frac{1}{3}ML^2$ for this axis of rotation (see Table 10.2), we obtain

$$\alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{1/3\,ML^2} = \boxed{\frac{3g}{2L}}$$

All points on the rod have this angular acceleration.

To find the linear acceleration of the right end of the rod, we use the relationship $a_t = r\alpha$ (Eq. 10.11), with $r = L$:

$$a_t = L\alpha = \boxed{\tfrac{3}{2}g}$$

This result—that $a_t > g$ for the free end of the rod—is rather interesting. It means that if we place a coin at the tip of the rod, hold the rod in the horizontal position, and then release the rod, the tip of the rod falls faster than the coin does!

Other points on the rod have a linear acceleration that is less than $\frac{3}{2}g$. For example, the middle of the rod has an acceleration of $\frac{3}{4}g$.

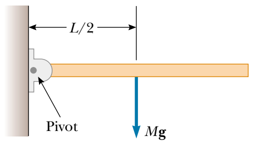


Figure 10.18 The uniform rod is pivoted at the left end.

Conceptual Example 10.11 Falling Smokestacks and Tumbling Blocks

When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground, as shown in Figure 10.19. The same thing happens with a tall tower of children's toy blocks. Why does this happen?

Solution As the smokestack rotates around its base, each higher portion of the smokestack falls with an increasing tangential acceleration. (The tangential acceleration of a given point on the smokestack is proportional to the distance of that portion from the base.) As the acceleration increases, higher portions of the smokestack experience an acceleration greater than that which could result from gravity alone; this is similar to the situation described in Example 10.10. This can happen only if these portions are being pulled downward by a force in addition to the gravitational force. The force that causes this to occur is the shear force from lower portions of the smokestack. Eventually the shear force that provides this acceleration is greater than the smokestack can withstand, and the smokestack breaks.

Figure 10.19 A falling smokestack.

Example 10.12 Angular Acceleration of a Wheel

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle, as shown in Figure 10.20. A light cord wrapped around the wheel supports an object of mass m. Calculate the angular acceleration of the wheel, the linear acceleration of the object, and the tension in the cord.

Solution The torque acting on the wheel about its axis of rotation is $\tau = TR$, where T is the force exerted by the cord on the rim of the wheel. (The gravitational force exerted by the Earth on the wheel and the normal force exerted by the axle on the wheel both pass through the axis of rotation and thus produce no torque.) Because $\sum \tau = I\alpha$, we obtain

$$\sum \tau = I\alpha = TR$$

$$(1) \qquad \alpha = \frac{TR}{I}$$

Now let us apply Newton's second law to the motion of the object, taking the downward direction to be positive:

$$\sum F_y = mg - T = ma$$

$$(2) \qquad a = \frac{mg - T}{m}$$

Equations (1) and (2) have three unknowns, α, a, and T. Because the object and wheel are connected by a string that does not slip, the linear acceleration of the suspended object is equal to the linear acceleration of a point on the rim of the wheel. Therefore, the angular acceleration of the wheel and this linear acceleration are related by $a = R\alpha$. Using this fact together with Equations (1) and (2), we obtain

$$(3) \qquad a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m}$$

$$(4) \qquad T = \frac{mg}{1 + \dfrac{mR^2}{I}}$$

Substituting Equation (4) into Equation (2), and solving for a and α, we find that

$$a = \frac{g}{1 + I/mR^2}$$

$$\alpha = \frac{a}{R} = \frac{g}{R + I/mR}$$

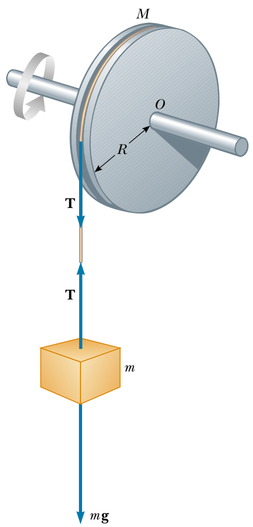


Figure 10.20 The tension in the cord produces a torque about the axle passing through O.

Exercise The wheel in Figure 10.20 is a solid disk of $M = 2.00$ kg, $R = 30.0$ cm, and $I = 0.090\,0$ kg·m². The suspended object has a mass of $m = 0.500$ kg. Find the tension in the cord and the angular acceleration of the wheel.

Answer 3.27 N; 10.9 rad/s².

Example 10.13 Atwood's Machine Revisited

Two blocks having masses m_1 and m_2 are connected to each other by a light cord that passes over two identical, frictionless pulleys, each having a moment of inertia I and radius R, as shown in Figure 10.21a. Find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord. (Assume no slipping between cord and pulleys.)

Solution We shall define the downward direction as positive for m_1 and upward as the positive direction for m_2. This allows us to represent the acceleration of both masses by a single variable a and also enables us to relate a positive a to a positive (counterclockwise) angular acceleration α. Let us write Newton's second law of motion for each block, using the free-body diagrams for the two blocks as shown in Figure 10.21b:

$$(1) \qquad m_1 g - T_1 = m_1 a$$

$$(2) \qquad T_3 - m_2 g = m_2 a$$

Next, we must include the effect of the pulleys on the motion. Free-body diagrams for the pulleys are shown in Figure 10.21c. The net torque about the axle for the pulley on the left is $(T_1 - T_2)R$, while the net torque for the pulley on the right is $(T_2 - T_3)R$. Using the relation $\sum \tau = I\alpha$ for each pulley and noting that each pulley has the same angular acceleration α, we obtain

$$(3) \qquad (T_1 - T_2)R = I\alpha$$

$$(4) \qquad (T_2 - T_3)R = I\alpha$$

We now have four equations with four unknowns: a, T_1, T_2, and T_3. These can be solved simultaneously. Adding Equations (3) and (4) gives

$$(5) \qquad (T_1 - T_3)R = 2I\alpha$$

Adding Equations (1) and (2) gives

$$T_3 - T_1 + m_1 g - m_2 g = (m_1 + m_2)a$$

$$(6) \qquad T_1 - T_3 = (m_1 - m_2)g - (m_1 + m_2)a$$

Substituting Equation (6) into Equation (5), we have

$$[(m_1 - m_2)g - (m_1 + m_2)a]R = 2I\alpha$$

Because $\alpha = a/R$, this expression can be simplified to

$$(m_1 - m_2)g - (m_1 + m_2)a = 2I\frac{a}{R^2}$$

$$(7) \qquad a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2\dfrac{I}{R^2}}$$

This value of a can then be substituted into Equations (1)

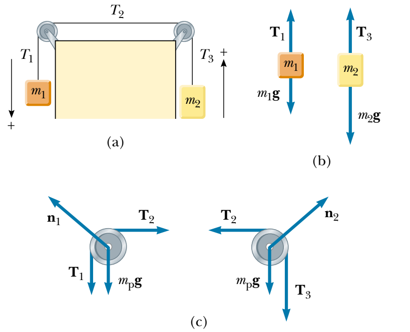


Figure 10.21 (a) Another look at Atwood's machine. (b) Free-body diagrams for the blocks. (c) Free-body diagrams for the pulleys, where $m_p\mathbf{g}$ represents the force of gravity acting on each pulley.

and (2) to give T_1 and T_3. Finally, T_2 can be found from Equation (3) or Equation (4). Note that if $m_1 > m_2$, the acceleration is positive; this means that the left block accelerates downward, the right block accelerates upward, and both pulleys accelerate counterclockwise. If $m_1 < m_2$, then all the values are negative and the motions are reversed. If $m_1 = m_2$, then no acceleration occurs at all. You should compare these results with those found in Example 5.9 on page 129.

10.8 WORK, POWER, AND ENERGY IN ROTATIONAL MOTION

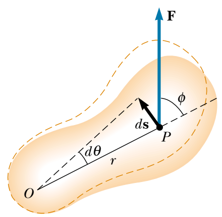


Figure 10.22 A rigid object rotates about an axis through O under the action of an external force **F** applied at P.

In this section, we consider the relationship between the torque acting on a rigid object and its resulting rotational motion in order to generate expressions for the power and a rotational analog to the work–kinetic energy theorem. Consider the rigid object pivoted at O in Figure 10.22. Suppose a single external force **F** is applied at P, where **F** lies in the plane of the page. The work done by **F** as the object rotates through an infinitesimal distance $ds = r\,d\theta$ in a time dt is

$$dW = \mathbf{F}\cdot d\mathbf{s} = (F\sin\phi)\,r\,d\theta$$

where $F\sin\phi$ is the tangential component of **F**, or, in other words, the component of the force along the displacement. Note that *the radial component of* **F** *does no work because it is perpendicular to the displacement.*

Because the magnitude of the torque due to **F** about O is defined as $rF\sin\phi$ by Equation 10.19, we can write the work done for the infinitesimal rotation as

$$dW = \tau\,d\theta \qquad \textbf{(10.22)}$$

The rate at which work is being done by **F** as the object rotates about the fixed axis is

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt}$$

Because dW/dt is the instantaneous power $\mathcal{P}$ (see Section 7.5) delivered by the force, and because $d\theta/dt = \omega$, this expression reduces to

Power delivered to a rigid object

$$\mathcal{P} = \frac{dW}{dt} = \tau\omega \qquad \textbf{(10.23)}$$

This expression is analogous to $\mathcal{P} = Fv$ in the case of linear motion, and the expression $dW = \tau\,d\theta$ is analogous to $dW = F_x\,dx$.

Work and Energy in Rotational Motion

In studying linear motion, we found the energy concept—and, in particular, the work–kinetic energy theorem—extremely useful in describing the motion of a system. The energy concept can be equally useful in describing rotational motion. From what we learned of linear motion, we expect that when a symmetric object rotates about a fixed axis, the work done by external forces equals the change in the rotational energy.

To show that this is in fact the case, let us begin with $\sum\tau = I\alpha$. Using the chain rule from the calculus, we can express the resultant torque as

$$\sum\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\,\omega$$

TABLE 10.3 Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Resultant torque $\Sigma\tau = I\alpha$	Resultant force $\Sigma F = ma$
If α = constant $\begin{cases}\omega_f = \omega_i + \alpha t\\ \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2\\ \omega_f^{\,2} = \omega_i^{\,2} + 2\alpha(\theta_f - \theta_i)\end{cases}$	If a = constant $\begin{cases}v_f = v_i + at\\ x_f - x_i = v_i t + \frac{1}{2}at^2\\ v_f^{\,2} = v_i^{\,2} + 2a(x_f - x_i)\end{cases}$
Work $W = \int_{\theta_i}^{\theta_f}\tau\,d\theta$	Work $W = \int_{x_i}^{x_f}F_x\,dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Resultant torque $\Sigma\tau = dL/dt$	Resultant force $\Sigma F = dp/dt$

Rearranging this expression and noting that $\Sigma\tau\,d\theta = dW$, we obtain

$$\sum\tau\,d\theta = dW = I\omega\,d\omega$$

Integrating this expression, we get for the total work done by the net external force acting on a rotating system

Work–kinetic energy theorem for rotational motion

$$\sum W = \int_{\theta_i}^{\theta_f}\sum\tau\,d\theta = \int_{\omega_i}^{\omega_f}I\omega\,d\omega = \tfrac{1}{2}I\omega_f^{\,2} - \tfrac{1}{2}I\omega_i^{\,2} \qquad \textbf{(10.24)}$$

where the angular speed changes from ω_i to ω_f as the angular position changes from θ_i to θ_f. That is,

> the net work done by external forces in rotating a symmetric rigid object about a fixed axis equals the change in the object's rotational energy.

Table 10.3 lists the various equations we have discussed pertaining to rotational motion, together with the analogous expressions for linear motion. The last two equations in Table 10.3, involving angular momentum L, are discussed in Chapter 11 and are included here only for the sake of completeness.

Quick Quiz 10.4

For a hoop lying in the xy plane, which of the following requires that more work be done by an external agent to accelerate the hoop from rest to an angular speed ω: (a) rotation about the z axis through the center of the hoop, or (b) rotation about an axis parallel to z passing through a point P on the hoop rim?

EXAMPLE 10.14 Rotating Rod Revisited

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end (Fig 10.23). The rod is released from rest in the horizontal position. (a) What is its angular speed when it reaches its lowest position?

Solution The question can be answered by considering the mechanical energy of the system. When the rod is horizontal, it has no rotational energy. The potential energy relative to the lowest position of the center of mass of the rod (O') is $MgL/2$. When the rod reaches its lowest position, the energy is entirely rotational energy, $\frac{1}{2}I\omega^2$, where I is the moment of inertia about the pivot. Because $I = \frac{1}{3}ML^2$ (see Table 10.2) and because mechanical energy is constant, we have $E_i = E_f$ or

$$\tfrac{1}{2}MgL = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(\tfrac{1}{3}ML^2)\omega^2$$

$$\omega = \boxed{\sqrt{\frac{3g}{L}}}$$

(b) Determine the linear speed of the center of mass and the linear speed of the lowest point on the rod when it is in the vertical position.

Solution These two values can be determined from the relationship between linear and angular speeds. We know ω from part (a), and so the linear speed of the center of mass is

$$v_{CM} = r\omega = \frac{L}{2}\,\omega = \boxed{\tfrac{1}{2}\sqrt{3gL}}$$

Because r for the lowest point on the rod is twice what it is for the center of mass, the lowest point has a linear speed equal to

$$2v_{CM} = \boxed{\sqrt{3gL}}$$

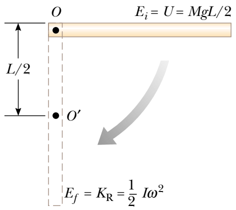


Figure 10.23 A uniform rigid rod pivoted at O rotates in a vertical plane under the action of gravity.

EXAMPLE 10.15 Connected Cylinders

Consider two cylinders having masses m_1 and m_2, where $m_1 \neq m_2$, connected by a string passing over a pulley, as shown in Figure 10.24. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descends through a distance h, and the angular speed of the pulley at this time.

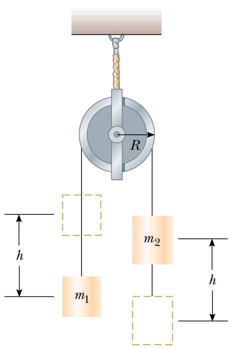


Figure 10.24

Solution We are now able to account for the effect of a massive pulley. Because the string does not slip, the pulley rotates. We neglect friction in the axle about which the pulley rotates for the following reason: Because the axle's radius is small relative to that of the pulley, the frictional torque is much smaller than the torque applied by the two cylinders, provided that their masses are quite different. Mechanical energy is constant; hence, the increase in the system's kinetic energy (the system being the two cylinders, the pulley, and the Earth) equals the decrease in its potential energy. Because $K_i = 0$ (the system is initially at rest), we have

$$\Delta K = K_f - K_i = (\tfrac{1}{2}m_1v_f^2 + \tfrac{1}{2}m_2v_f^2 + \tfrac{1}{2}I\omega_f^2) - 0$$

where v_f is the same for both blocks. Because $v_f = R\omega_f$, this expression becomes

$$\Delta K = \tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^2$$

From Figure 10.24, we see that the system loses potential energy as cylinder 2 descends and gains potential energy as cylinder 1 rises. That is, $\Delta U_2 = -m_2gh$ and $\Delta U_1 = m_1gh$. Applying the principle of conservation of energy in the form $\Delta K + \Delta U_1 + \Delta U_2 = 0$ gives

$$\tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^2 + m_1gh - m_2gh = 0$$

$$\boxed{v_f = \left[\frac{2(m_2 - m_1)gh}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}\right]^{1/2}}$$

Because $v_f = R\omega_f$, the angular speed of the pulley at this instant is

$$\omega_f = \frac{v_f}{R} = \boxed{\frac{1}{R}\left[\frac{2(m_2 - m_1)gh}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}\right]^{1/2}}$$

Exercise Repeat the calculation of v_f, using $\Sigma\tau = I\alpha$ applied to the pulley and Newton's second law applied to the two cylinders. Use the procedures presented in Examples 10.12 and 10.13.

SUMMARY

If a particle rotates in a circle of radius r through an angle θ (measured in radians), the arc length it moves through is $s = r\theta$.

The **angular displacement** of a particle rotating in a circle or of a rigid object rotating about a fixed axis is

$$\Delta\theta = \theta_f - \theta_i \qquad \textbf{(10.2)}$$

The **instantaneous angular speed** of a particle rotating in a circle or of a rigid object rotating about a fixed axis is

$$\omega = \frac{d\theta}{dt} \qquad \textbf{(10.4)}$$

The **instantaneous angular acceleration** of a rotating object is

$$\alpha = \frac{d\omega}{dt} \qquad \textbf{(10.6)}$$

When a rigid object rotates about a fixed axis, every part of the object has the same angular speed and the same angular acceleration.

If a particle or object rotates about a fixed axis under constant angular acceleration, one can apply equations of kinematics that are analogous to those for linear motion under constant linear acceleration:

$$\omega_f = \omega_i + \alpha t \qquad \textbf{(10.7)}$$

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \qquad \textbf{(10.8)}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \qquad \textbf{(10.9)}$$

A useful technique in solving problems dealing with rotation is to visualize a linear version of the same problem.

When a rigid object rotates about a fixed axis, the angular position, angular speed, and angular acceleration are related to the linear position, linear speed, and linear acceleration through the relationships

$$s = r\theta \qquad \textbf{(10.1a)}$$

$$v = r\omega \qquad \textbf{(10.10)}$$

$$a_t = r\alpha \qquad \textbf{(10.11)}$$

You must be able to easily alternate between the linear and rotational variables that describe a given situation.

The **moment of inertia of a system of particles** is

$$I \equiv \sum_i m_i r_i^2 \quad (10.15)$$

If a rigid object rotates about a fixed axis with angular speed ω, its **rotational energy** can be written

$$K_R = \frac{1}{2} I \omega^2 \quad (10.16)$$

where I is the moment of inertia about the axis of rotation.

The **moment of inertia of a rigid object** is

$$I = \int r^2 \, dm \quad (10.17)$$

where r is the distance from the mass element dm to the axis of rotation.

The magnitude of the **torque** associated with a force **F** acting on an object is

$$\tau = Fd \quad (10.19)$$

where d is the moment arm of the force, which is the perpendicular distance from some origin to the line of action of the force. Torque is a measure of the tendency of the force to change the rotation of the object about some axis.

If a rigid object free to rotate about a fixed axis has a **net external torque** acting on it, the object undergoes an angular acceleration α, where

$$\sum \tau = I\alpha \quad (10.21)$$

The rate at which work is done by an external force in rotating a rigid object about a fixed axis, or the **power** delivered, is

$$\mathcal{P} = \tau\omega \quad (10.23)$$

The net work done by external forces in rotating a rigid object about a fixed axis equals the change in the rotational kinetic energy of the object:

$$\sum W = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 \quad (10.24)$$

QUESTIONS

1. What is the angular speed of the second hand of a clock? What is the direction of $\boldsymbol{\omega}$ as you view a clock hanging vertically? What is the magnitude of the angular acceleration vector $\boldsymbol{\alpha}$ of the second hand?
2. A wheel rotates counterclockwise in the xy plane. What is the direction of $\boldsymbol{\omega}$? What is the direction of $\boldsymbol{\alpha}$ if the angular velocity is decreasing in time?
3. Are the kinematic expressions for θ, ω, and α valid when the angular displacement is measured in degrees instead of in radians?
4. A turntable rotates at a constant rate of 45 rev/min. What is its angular speed in radians per second? What is the magnitude of its angular acceleration?
5. Suppose $a = b$ and $M > m$ for the system of particles described in Figure 10.8. About what axis (x, y, or z) does the moment of inertia have the smallest value? the largest value?
6. Suppose the rod in Figure 10.10 has a nonuniform mass distribution. In general, would the moment of inertia about the y axis still equal $ML^2/12$? If not, could the moment of inertia be calculated without knowledge of the manner in which the mass is distributed?
7. Suppose that only two external forces act on a rigid body, and the two forces are equal in magnitude but opposite in direction. Under what condition does the body rotate?
8. Explain how you might use the apparatus described in Example 10.12 to determine the moment of inertia of the wheel. (If the wheel does not have a uniform mass density, the moment of inertia is not necessarily equal to $\frac{1}{2}MR^2$.)

9. Using the results from Example 10.12, how would you calculate the angular speed of the wheel and the linear speed of the suspended mass at $t = 2$ s, if the system is released from rest at $t = 0$? Is the expression $v = R\omega$ valid in this situation?
10. If a small sphere of mass M were placed at the end of the rod in Figure 10.23, would the result for ω be greater than, less than, or equal to the value obtained in Example 10.14?
11. Explain why changing the axis of rotation of an object changes its moment of inertia.
12. Is it possible to change the translational kinetic energy of an object without changing its rotational energy?
13. Two cylinders having the same dimensions are set into rotation about their long axes with the same angular speed. One is hollow, and the other is filled with water. Which cylinder will be easier to stop rotating? Explain your answer.
14. Must an object be rotating to have a nonzero moment of inertia?
15. If you see an object rotating, is there necessarily a net torque acting on it?
16. Can a (momentarily) stationary object have a nonzero angular acceleration?
17. The polar diameter of the Earth is slightly less than the equatorial diameter. How would the moment of inertia of the Earth change if some mass from near the equator were removed and transferred to the polar regions to make the Earth a perfect sphere?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

1. A wheel starts from rest and rotates with constant angular acceleration and reaches an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle (in radians) through which it rotates in this time.
2. What is the angular speed in radians per second of (a) the Earth in its orbit about the Sun and (b) the Moon in its orbit about the Earth?
3. An airliner arrives at the terminal, and its engines are shut off. The rotor of one of the engines has an initial clockwise angular speed of 2 000 rad/s. The engine's rotation slows with an angular acceleration of magnitude 80.0 rad/s^2. (a) Determine the angular speed after 10.0 s. (b) How long does it take for the rotor to come to rest?
4. (a) The positions of the hour and minute hand on a clock face coincide at 12 o'clock. Determine all other times (up to the second) at which the positions of the hands coincide. (b) If the clock also has a second hand, determine all times at which the positions of all three hands coincide, given that they all coincide at 12 o'clock.

WEB 5. An electric motor rotating a grinding wheel at 100 rev/min is switched off. Assuming constant negative acceleration of magnitude 2.00 rad/s^2, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn during the time found in part (a)?

6. A centrifuge in a medical laboratory rotates at a rotational speed of 3 600 rev/min. When switched off, it rotates 50.0 times before coming to rest. Find the constant angular acceleration of the centrifuge.
7. The angular position of a swinging door is described by $\theta = 5.00 + 10.0t + 2.00t^2$ rad. Determine the angular position, angular speed, and angular acceleration of the door (a) at $t = 0$ and (b) at $t = 3.00$ s.
8. The tub of a washer goes into its spin cycle, starting from rest and gaining angular speed steadily for 8.00 s, when it is turning at 5.00 rev/s. At this point the person doing the laundry opens the lid, and a safety switch turns off the washer. The tub smoothly slows to rest in 12.0 s. Through how many revolutions does the tub turn while it is in motion?
9. A rotating wheel requires 3.00 s to complete 37.0 revolutions. Its angular speed at the end of the 3.00-s interval is 98.0 rad/s. What is the constant angular acceleration of the wheel?
10. (a) Find the angular speed of the Earth's rotation on its axis. As the Earth turns toward the east, we see the sky turning toward the west at this same rate.
(b) *The rainy Pleiads wester*
And seek beyond the sea
The head that I shall dream of
That shall not dream of me.

A. E. Housman (*© Robert E. Symons*)

Cambridge, England, is at longitude 0°, and Saskatoon, Saskatchewan, is at longitude 107° west. How much time elapses after the Pleiades set in Cambridge until these stars fall below the western horizon in Saskatoon?

Section 10.3 Angular and Linear Quantities

11. Make an order-of-magnitude estimate of the number of revolutions through which a typical automobile tire

turns in 1 yr. State the quantities you measure or estimate and their values.

12. The diameters of the main rotor and tail rotor of a single-engine helicopter are 7.60 m and 1.02 m, respectively. The respective rotational speeds are 450 rev/min and 4 138 rev/min. Calculate the speeds of the tips of both rotors. Compare these speeds with the speed of sound, 343 m/s.

Figure P10.12 *(Ross Harrrison Koty/Tony Stone Images)*

13. A racing car travels on a circular track with a radius of 250 m. If the car moves with a constant linear speed of 45.0 m/s, find (a) its angular speed and (b) the magnitude and direction of its acceleration.

14. A car is traveling at 36.0 km/h on a straight road. The radius of its tires is 25.0 cm. Find the angular speed of one of the tires, with its axle taken as the axis of rotation.

15. A wheel 2.00 m in diameter lies in a vertical plane and rotates with a constant angular acceleration of 4.00 rad/s^2. The wheel starts at rest at $t = 0$, and the radius vector of point P on the rim makes an angle of 57.3° with the horizontal at this time. At $t = 2.00$ s, find (a) the angular speed of the wheel, (b) the linear speed and acceleration of the point P, and (c) the angular position of the point P.

16. A discus thrower accelerates a discus from rest to a speed of 25.0 m/s by whirling it through 1.25 rev. Assume the discus moves on the arc of a circle 1.00 m in radius. (a) Calculate the final angular speed of the discus. (b) Determine the magnitude of the angular acceleration of the discus, assuming it to be constant. (c) Calculate the acceleration time.

Figure P10.16 *(Bruce Ayers/Tony Stone Images)*

17. A car accelerates uniformly from rest and reaches a speed of 22.0 m/s in 9.00 s. If the diameter of a tire is 58.0 cm, find (a) the number of revolutions the tire makes during this motion, assuming that no slipping occurs. (b) What is the final rotational speed of a tire in revolutions per second?

18. A 6.00-kg block is released from A on the frictionless track shown in Figure P10.18. Determine the radial and tangential components of acceleration for the block at P.

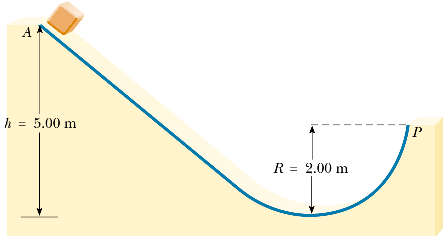


Figure P10.18

WEB **19.** A disc 8.00 cm in radius rotates at a constant rate of 1 200 rev/min about its central axis. Determine (a) its angular speed, (b) the linear speed at a point 3.00 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2.00 s.

20. A car traveling on a flat (unbanked) circular track accelerates uniformly from rest with a tangential acceleration of 1.70 m/s^2. The car makes it one quarter of the way around the circle before it skids off the track. Determine the coefficient of static friction between the car and track from these data.

21. A small object with mass 4.00 kg moves counterclockwise with constant speed 4.50 m/s in a circle of radius 3.00 m centered at the origin. (a) It started at the point with cartesian coordinates (3 m, 0). When its angular displacement is 9.00 rad, what is its position vector, in cartesian unit-vector notation? (b) In what quadrant is the particle located, and what angle does its position vector make with the positive x axis? (c) What is its velocity vector, in unit–vector notation? (d) In what direction is it moving? Make a sketch of the position and velocity vectors. (e) What is its acceleration, expressed in unit–vector notation? (f) What total force acts on the object? (Express your answer in unit vector notation.)

22. A standard cassette tape is placed in a standard cassette player. Each side plays for 30 min. The two tape wheels of the cassette fit onto two spindles in the player. Suppose that a motor drives one spindle at a constant angular speed of ~ 1 rad/s and that the other spindle is free to rotate at any angular speed. Estimate the order of magnitude of the thickness of the tape.

Section 10.4 Rotational Energy

23. Three small particles are connected by rigid rods of negligible mass lying along the y axis (Fig. P10.23). If the system rotates about the x axis with an angular speed of 2.00 rad/s, find (a) the moment of inertia about the x axis and the total rotational kinetic energy evaluated from $\frac{1}{2}I\omega^2$ and (b) the linear speed of each particle and the total kinetic energy evaluated from $\Sigma\frac{1}{2}m_iv_i^2$.

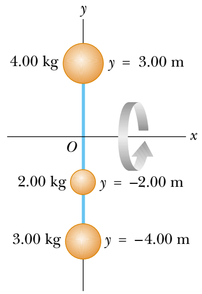


Figure P10.23

24. The center of mass of a pitched baseball (3.80-cm radius) moves at 38.0 m/s. The ball spins about an axis through its center of mass with an angular speed of 125 rad/s. Calculate the ratio of the rotational energy to the translational kinetic energy. Treat the ball as a uniform sphere.

WEB **25.** The four particles in Figure P10.25 are connected by rigid rods of negligible mass. The origin is at the center of the rectangle. If the system rotates in the xy plane about the z axis with an angular speed of 6.00 rad/s, calculate (a) the moment of inertia of the system about the z axis and (b) the rotational energy of the system.

26. The hour hand and the minute hand of Big Ben, the famous Parliament tower clock in London, are 2.70 m long and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively. Calculate the total rotational kinetic energy of the two hands about the axis of rotation. (You may model the hands as long thin rods.)

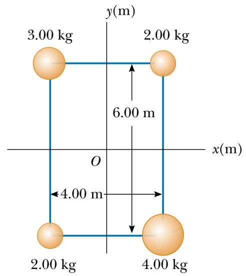


Figure P10.25

Figure P10.26 Problems 26 and 74. *(John Lawrence/Tony Stone Images)*

27. Two masses M and m are connected by a rigid rod of length L and of negligible mass, as shown in Figure P10.27. For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axis passes through the center of mass. Show that this moment of inertia is $I = \mu L^2$, where $\mu = mM/(m + M)$.

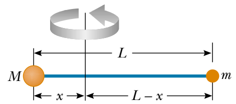


Figure P10.27

Section 10.5 Calculation of Moments of Inertia

28. Three identical thin rods, each of length L and mass m, are welded perpendicular to each other, as shown in Figure P10.28. The entire setup is rotated about an axis

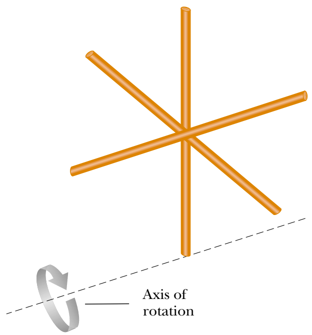


Figure P10.28

that passes through the end of one rod and is parallel to another. Determine the moment of inertia of this arrangement.

29. Figure P10.29 shows a side view of a car tire and its radial dimensions. The rubber tire has two sidewalls of uniform thickness 0.635 cm and a tread wall of uniform thickness 2.50 cm and width 20.0 cm. Suppose its density is uniform, with the value $1.10 \times 10^3\ \text{kg/m}^3$. Find its moment of inertia about an axis through its center perpendicular to the plane of the sidewalls.

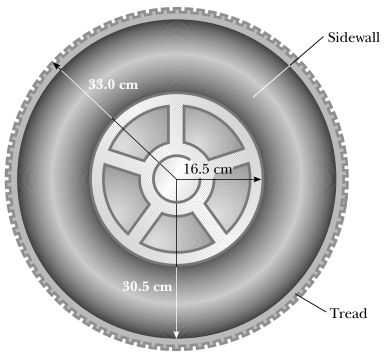


Figure P10.29

30. Use the parallel-axis theorem and Table 10.2 to find the moments of inertia of (a) a solid cylinder about an axis parallel to the center-of-mass axis and passing through the edge of the cylinder and (b) a solid sphere about an axis tangent to its surface.

31. *Attention! About face!* Compute an order-of-magnitude estimate for the moment of inertia of your body as you stand tall and turn around a vertical axis passing through the top of your head and the point halfway between your ankles. In your solution state the quantities you measure or estimate and their values.

Section 10.6 Torque

32. Find the mass m needed to balance the 1 500-kg truck on the incline shown in Figure P10.32. Assume all pulleys are frictionless and massless.

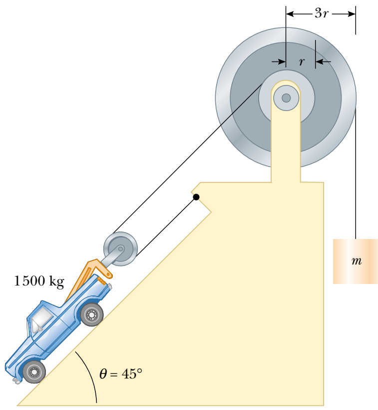


Figure P10.32

WEB **33.** Find the net torque on the wheel in Figure P10.33 about the axle through O if $a = 10.0$ cm and $b = 25.0$ cm.

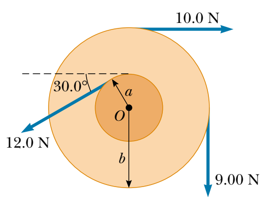


Figure P10.33

34. The fishing pole in Figure P10.34 makes an angle of 20.0° with the horizontal. What is the torque exerted by the fish about an axis perpendicular to the page and passing through the fisher's hand?

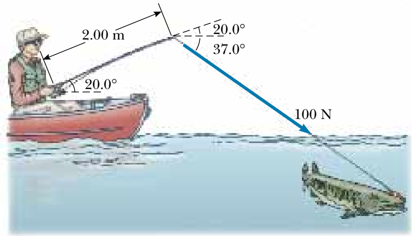


Figure P10.34

35. The tires of a 1 500-kg car are 0.600 m in diameter, and the coefficients of friction with the road surface are $\mu_s = 0.800$ and $\mu_k = 0.600$. Assuming that the weight is evenly distributed on the four wheels, calculate the maximum torque that can be exerted by the engine on a driving wheel such that the wheel does not spin. If you wish, you may suppose that the car is at rest.

36. Suppose that the car in Problem 35 has a disk brake system. Each wheel is slowed by the frictional force between a single brake pad and the disk-shaped rotor. On this particular car, the brake pad comes into contact with the rotor at an average distance of 22.0 cm from the axis. The coefficients of friction between the brake pad and the disk are $\mu_s = 0.600$ and $\mu_k = 0.500$. Calculate the normal force that must be applied to the rotor such that the car slows as quickly as possible.

Section 10.7 Relationship Between Torque and Angular Acceleration

WEB **37.** A model airplane having a mass of 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the linear acceleration of the airplane tangent to its flight path.

38. The combination of an applied force and a frictional force produces a constant total torque of 36.0 N·m on a wheel rotating about a fixed axis. The applied force acts for 6.00 s; during this time the angular speed of the wheel increases from 0 to 10.0 rad/s. The applied force is then removed, and the wheel comes to rest in 60.0 s. Find (a) the moment of inertia of the wheel, (b) the magnitude of the frictional torque, and (c) the total number of revolutions of the wheel.

39. A block of mass $m_1 = 2.00$ kg and a block of mass $m_2 = 6.00$ kg are connected by a massless string over a pulley

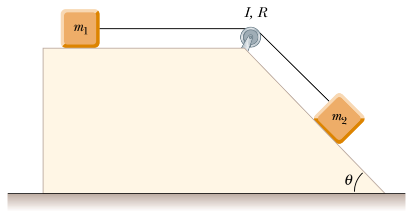


Figure P10.39

in the shape of a disk having radius $R = 0.250$ m and mass $M = 10.0$ kg. These blocks are allowed to move on a fixed block–wedge of angle $\theta = 30.0°$, as shown in Figure P10.39. The coefficient of kinetic friction for both blocks is 0.360. Draw free-body diagrams of both blocks and of the pulley. Determine (a) the acceleration of the two blocks and (b) the tensions in the string on both sides of the pulley.

40. A potter's wheel—a thick stone disk with a radius of 0.500 m and a mass of 100 kg—is freely rotating at 50.0 rev/min. The potter can stop the wheel in 6.00 s by pressing a wet rag against the rim and exerting a radially inward force of 70.0 N. Find the effective coefficient of kinetic friction between the wheel and the rag.

41. A bicycle wheel has a diameter of 64.0 cm and a mass of 1.80 kg. Assume that the wheel is a hoop with all of its mass concentrated on the outside radius. The bicycle is placed on a stationary stand on rollers, and a resistive force of 120 N is applied tangent to the rim of the tire. (a) What force must be applied by a chain passing over a 9.00-cm-diameter sprocket if the wheel is to attain an acceleration of $4.50\ \text{rad/s}^2$? (b) What force is required if the chain shifts to a 5.60-cm-diameter sprocket?

Section 10.8 Work, Power, and Energy in Rotational Motion

42. A cylindrical rod 24.0 cm long with a mass of 1.20 kg and a radius of 1.50 cm has a ball with a diameter of 8.00 cm and a mass of 2.00 kg attached to one end. The arrangement is originally vertical and stationary, with the ball at the top. The apparatus is free to pivot about the bottom end of the rod. (a) After it falls through 90°, what is its rotational kinetic energy? (b) What is the angular speed of the rod and ball? (c) What is the linear speed of the ball? (d) How does this compare with the speed if the ball had fallen freely through the same distance of 28 cm?

43. A 15.0-kg mass and a 10.0-kg mass are suspended by a pulley that has a radius of 10.0 cm and a mass of 3.00 kg (Fig. P10.43). The cord has a negligible mass and causes the pulley to rotate without slipping. The pulley rotates without friction. The masses start from rest 3.00 m apart. Treating the pulley as a uniform disk, determine the speeds of the two masses as they pass each other.

44. A mass m_1 and a mass m_2 are suspended by a pulley that has a radius R and a mass M (see Fig. P10.43). The cord has a negligible mass and causes the pulley to rotate without slipping. The pulley rotates without friction. The masses start from rest a distance d apart. Treating the pulley as a uniform disk, determine the speeds of the two masses as they pass each other.

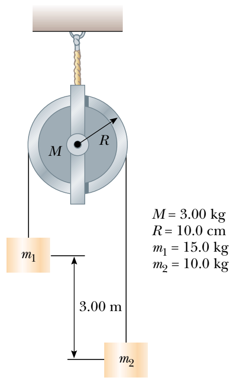


Figure P10.43 Problems 43 and 44.

45. A weight of 50.0 N is attached to the free end of a light string wrapped around a reel with a radius of 0.250 m and a mass of 3.00 kg. The reel is a solid disk, free to rotate in a vertical plane about the horizontal axis passing through its center. The weight is released 6.00 m above the floor. (a) Determine the tension in the string, the acceleration of the mass, and the speed with which the weight hits the floor. (b) Find the speed calculated in part (a), using the principle of conservation of energy.

46. A constant torque of 25.0 N·m is applied to a grindstone whose moment of inertia is 0.130 kg·m^2. Using energy principles, find the angular speed after the grindstone has made 15.0 revolutions. (Neglect friction.)

47. This problem describes one experimental method of determining the moment of inertia of an irregularly shaped object such as the payload for a satellite. Figure P10.47 shows a mass m suspended by a cord wound around a spool of radius r, forming part of a turntable supporting the object. When the mass is released from rest, it descends through a distance h, acquiring a speed v. Show that the moment of inertia I of the equipment (including the turntable) is $mr^2(2gh/v^2 - 1)$.

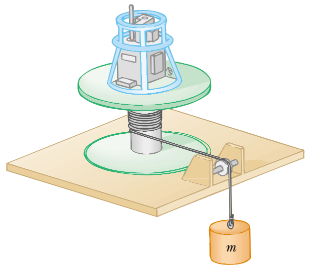


Figure P10.47

48. A bus is designed to draw its power from a rotating flywheel that is brought up to its maximum rate of rotation (3 000 rev/min) by an electric motor. The flywheel is a solid cylinder with a mass of 1 000 kg and a diameter of 1.00 m. If the bus requires an average power of 10.0 kW, how long does the flywheel rotate?

49. (a) A uniform, solid disk of radius R and mass M is free to rotate on a frictionless pivot through a point on its rim (Fig. P10.49). If the disk is released from rest in the position shown by the blue circle, what is the speed of its center of mass when the disk reaches the position indicated by the dashed circle? (b) What is the speed of the lowest point on the disk in the dashed position? (c) Repeat part (a), using a uniform hoop.

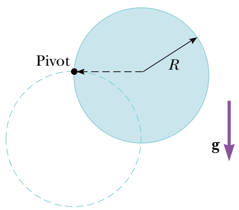


Figure P10.49

50. A horizontal 800-N merry-go-round is a solid disk of radius 1.50 m and is started from rest by a constant horizontal force of 50.0 N applied tangentially to the cylinder. Find the kinetic energy of the solid cylinder after 3.00 s.

ADDITIONAL PROBLEMS

51. Toppling chimneys often break apart in mid-fall (Fig. P10.51) because the mortar between the bricks cannot withstand much shear stress. As the chimney begins to fall, shear forces must act on the topmost sections to accelerate them tangentially so that they can keep up with the rotation of the lower part of the stack. For simplicity, let us model the chimney as a uniform rod of length ℓ pivoted at the lower end. The rod starts at rest in a vertical position (with the frictionless pivot at the bottom) and falls over under the influence of gravity. What fraction of the length of the rod has a tangential acceleration greater than $g \sin\theta$, where θ is the angle the chimney makes with the vertical?

Figure P10.51 A building demolition site in Baltimore, MD. At the left is a chimney, mostly concealed by the building, that has broken apart on its way down. Compare with Figure 10.19. *(Jerry Wachter/Photo Researchers, Inc.)*

52. Review Problem. A mixing beater consists of three thin rods: Each is 10.0 cm long, diverges from a central hub, and is separated from the others by 120°. All turn in the same plane. A ball is attached to the end of each rod. Each ball has a cross-sectional area of 4.00 cm^2 and is shaped so that it has a drag coefficient of 0.600. Calculate the power input required to spin the beater at 1 000 rev/min (a) in air and (b) in water.

53. A grinding wheel is in the form of a uniform solid disk having a radius of 7.00 cm and a mass of 2.00 kg. It starts from rest and accelerates uniformly under the action of the constant torque of 0.600 N·m that the motor

exerts on the wheel. (a) How long does the wheel take to reach its final rotational speed of 1 200 rev/min? (b) Through how many revolutions does it turn while accelerating?

54. The density of the Earth, at any distance r from its center, is approximately

$$\rho = [14.2 - 11.6\,r/R] \times 10^3 \text{ kg/m}^3$$

where R is the radius of the Earth. Show that this density leads to a moment of inertia $I = 0.330MR^2$ about an axis through the center, where M is the mass of the Earth.

55. A 4.00-m length of light nylon cord is wound around a uniform cylindrical spool of radius 0.500 m and mass 1.00 kg. The spool is mounted on a frictionless axle and is initially at rest. The cord is pulled from the spool with a constant acceleration of magnitude 2.50 m/s^2. (a) How much work has been done on the spool when it reaches an angular speed of 8.00 rad/s? (b) Assuming that there is enough cord on the spool, how long does it take the spool to reach this angular speed? (c) Is there enough cord on the spool?

56. A flywheel in the form of a heavy circular disk of diameter 0.600 m and mass 200 kg is mounted on a frictionless bearing. A motor connected to the flywheel accelerates it from rest to 1 000 rev/min. (a) What is the moment of inertia of the flywheel? (b) How much work is done on it during this acceleration? (c) When the angular speed reaches 1 000 rev/min, the motor is disengaged. A friction brake is used to slow the rotational rate to 500 rev/min. How much energy is dissipated as internal energy in the friction brake?

57. A shaft is turning at 65.0 rad/s at time zero. Thereafter, its angular acceleration is given by

$$\alpha = -10 \text{ rad/s}^2 - 5t \text{ rad/s}^3$$

where t is the elapsed time. (a) Find its angular speed at $t = 3.00$ s. (b) How far does it turn in these 3 s?

58. For any given rotational axis, the *radius of gyration* K of a rigid body is defined by the expression $K^2 = I/M$, where M is the total mass of the body and I is its moment of inertia. Thus, the radius of gyration is equal to the distance between an imaginary point mass M and the axis of rotation such that I for the point mass about that axis is the same as that for the rigid body. Find the radius of gyration of (a) a solid disk of radius R, (b) a uniform rod of length L, and (c) a solid sphere of radius R, all three of which are rotating about a central axis.

59. A long, uniform rod of length L and mass M is pivoted about a horizontal, frictionless pin passing through one end. The rod is released from rest in a vertical position, as shown in Figure P10.59. At the instant the rod is horizontal, find (a) its angular speed, (b) the magnitude of its angular acceleration, (c) the x and y components of the acceleration of its center of mass, and (d) the components of the reaction force at the pivot.

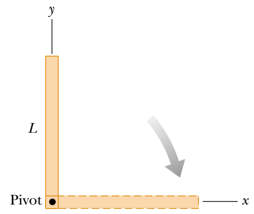


Figure P10.59

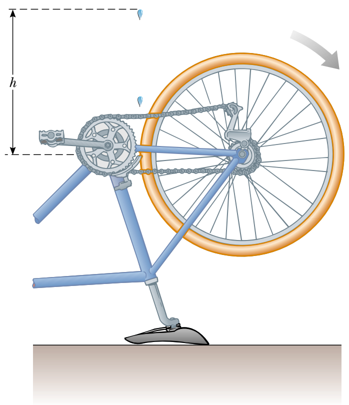


Figure P10.60 Problems 60 and 61.

60. A bicycle is turned upside down while its owner repairs a flat tire. A friend spins the other wheel, of radius 0.381 m, and observes that drops of water fly off tangentially. She measures the height reached by drops moving vertically (Fig. P10.60). A drop that breaks loose from the tire on one turn rises $h = 54.0$ cm above the tangent point. A drop that breaks loose on the next turn rises 51.0 cm above the tangent point. The height to which the drops rise decreases because the angular speed of the wheel decreases. From this information, determine the magnitude of the average angular acceleration of the wheel.

61. A bicycle is turned upside down while its owner repairs a flat tire. A friend spins the other wheel of radius R and observes that drops of water fly off tangentially. She measures the height reached by drops moving vertically (see Fig. P10.60). A drop that breaks loose from the tire on one turn rises a distance h_1 above the tangent point. A drop that breaks loose on the next turn rises a distance $h_2 < h_1$ above the tangent point. The height to which the drops rise decreases because the angular speed of the wheel decreases. From this information, determine the magnitude of the average angular acceleration of the wheel.

62. The top shown in Figure P10.62 has a moment of inertia of 4.00×10^{-4} kg·m^2 and is initially at rest. It is free to rotate about the stationary axis AA'. A string, wrapped around a peg along the axis of the top, is pulled in such a manner that a constant tension of 5.57 N is maintained. If the string does not slip while it is unwound from the peg, what is the angular speed of the top after 80.0 cm of string has been pulled off the peg?

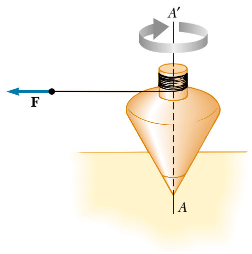


Figure P10.62

63. A cord is wrapped around a pulley of mass m and of radius r. The free end of the cord is connected to a block of mass M. The block starts from rest and then slides down an incline that makes an angle θ with the horizontal. The coefficient of kinetic friction between block and incline is μ. (a) Use energy methods to show that the block's speed as a function of displacement d down the incline is

$$v = [4gdM(m + 2M)^{-1}(\sin\theta - \mu\cos\theta)]^{1/2}$$

(b) Find the magnitude of the acceleration of the block in terms of μ, m, M, g, and θ.

64. (a) What is the rotational energy of the Earth about its spin axis? The radius of the Earth is 6 370 km, and its mass is 5.98×10^{24} kg. Treat the Earth as a sphere of moment of inertia $\frac{2}{5}MR^2$. (b) The rotational energy of the Earth is decreasing steadily because of tidal friction. Estimate the change in one day, given that the rotational period increases by about 10 μs each year.

65. The speed of a moving bullet can be determined by allowing the bullet to pass through two rotating paper disks mounted a distance d apart on the same axle (Fig. P10.65). From the angular displacement $\Delta\theta$ of the two bullet holes in the disks and the rotational speed of the disks, we can determine the speed v of the bullet. Find the bullet speed for the following data: $d = 80$ cm, $\omega = 900$ rev/min, and $\Delta\theta = 31.0°$.

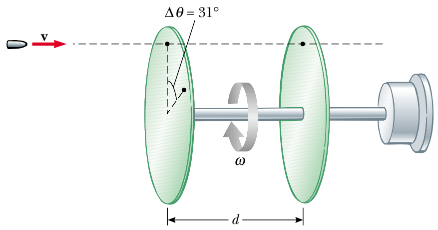


Figure P10.65

66. A wheel is formed from a hoop and n equally spaced spokes extending from the center of the hoop to its rim. The mass of the hoop is M, and the radius of the hoop (and hence the length of each spoke) is R. The mass of each spoke is m. Determine (a) the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel and (b) the moment of inertia of the wheel about an axis through its rim and perpendicular to the plane of the wheel.

67. A uniform, thin, solid door has a height of 2.20 m, a width of 0.870 m, and a mass of 23.0 kg. Find its moment of inertia for rotation on its hinges. Are any of the data unnecessary?

68. A uniform, hollow, cylindrical spool has inside radius $R/2$, outside radius R, and mass M (Fig. P10.68). It is mounted so that it rotates on a massless horizontal axle. A mass m is connected to the end of a string wound around the spool. The mass m falls from rest through a distance y in time t. Show that the torque due to the frictional forces between spool and axle is

$$\tau_f = R[m(g - 2y/t^2) - M(5y/4t^2)]$$

69. An electric motor can accelerate a Ferris wheel of moment of inertia $I = 20\,000$ kg·m^2 from rest to 10.0 rev/min in 12.0 s. When the motor is turned off, friction causes the wheel to slow down from 10.0 to 8.00 rev/min in 10.0 s. Determine (a) the torque generated by the motor to bring the wheel to 10.0 rev/min and (b) the power that would be needed to maintain this rotational speed.

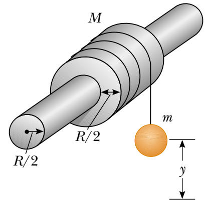


Figure P10.68

70. The pulley shown in Figure P10.70 has radius R and moment of inertia I. One end of the mass m is connected to a spring of force constant k, and the other end is fastened to a cord wrapped around the pulley. The pulley axle and the incline are frictionless. If the pulley is wound counterclockwise so that the spring is stretched a distance d from its unstretched position and is then released from rest, find (a) the angular speed of the pulley when the spring is again unstretched and (b) a numerical value for the angular speed at this point if $I = 1.00$ kg·m^2, $R = 0.300$ m, $k = 50.0$ N/m, $m = 0.500$ kg, $d = 0.200$ m, and $\theta = 37.0°$.

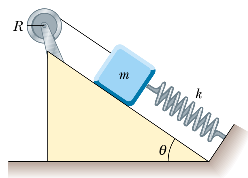


Figure P10.70

71. Two blocks, as shown in Figure P10.71, are connected by a string of negligible mass passing over a pulley of radius 0.250 m and moment of inertia I. The block on the frictionless incline is moving upward with a constant acceleration of 2.00 m/s^2. (a) Determine T_1 and T_2, the tensions in the two parts of the string. (b) Find the moment of inertia of the pulley.

72. A common demonstration, illustrated in Figure P10.72, consists of a ball resting at one end of a uniform board

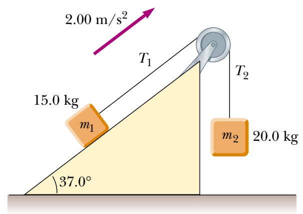


Figure P10.71

of length ℓ, hinged at the other end, and elevated at an angle θ. A light cup is attached to the board at r_c so that it will catch the ball when the support stick is suddenly

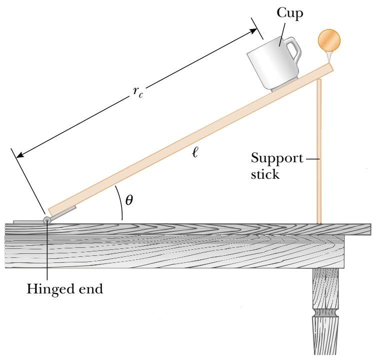


Figure P10.72

removed. (a) Show that the ball will lag behind the falling board when θ is less than 35.3° ; and that (b) the ball will fall into the cup when the board is supported at this limiting angle and the cup is placed at

$$r_c = \frac{2\,\ell}{3\cos\theta}$$

(c) If a ball is at the end of a 1.00-m stick at this critical angle, show that the cup must be 18.4 cm from the moving end.

73. As a result of friction, the angular speed of a wheel changes with time according to the relationship

$$d\theta/dt = \omega_0 e^{-\sigma t}$$

where ω_0 and σ are constants. The angular speed changes from 3.50 rad/s at $t = 0$ to 2.00 rad/s at $t = 9.30$ s. Use this information to determine σ and ω_0. Then, determine (a) the magnitude of the angular acceleration at $t = 3.00$ s, (b) the number of revolutions the wheel makes in the first 2.50 s, and (c) the number of revolutions it makes before coming to rest.

74. The hour hand and the minute hand of Big Ben, the famous Parliament tower clock in London, are 2.70 m long and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively (see Fig. P10.26). (a) Determine the total torque due to the weight of these hands about the axis of rotation when the time reads (i) 3:00, (ii) 5:15, (iii) 6:00, (iv) 8:20, and (v) 9:45. (You may model the hands as long thin rods.) (b) Determine all times at which the total torque about the axis of rotation is zero. Determine the times to the nearest second, solving a transcendental equation numerically.

Answers to Quick Quizzes

10.1 The fact that ω is negative indicates that we are dealing with an object that is rotating in the clockwise direction. We also know that when $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are antiparallel, ω must be decreasing—the object is slowing down. Therefore, the object is spinning more and more slowly (with less and less angular speed) in the clockwise, or negative, direction. This has a linear analogy to a sky diver opening her parachute. The velocity is negative—downward. When the sky diver opens the parachute, a large upward force causes an upward acceleration. As a result, the acceleration and velocity vectors are in opposite directions. Consequently, the parachutist slows down.

10.2 (a) Yes, all points on the wheel have the same angular speed. This is why we use angular quantities to describe rotational motion. (b) No, not all points on the wheel have the same linear speed. (c) $v = 0$, $a = 0$. (d) $v = R\omega/2$, $a = a_r = v^2/(R/2) = R\omega^2/2$ (a_t is zero at all points because ω is constant). (e) $v = R\omega$, $a = R\omega^2$.

10.3 (a) $I = MR^2$. (b) $I = MR^2$. The moment of inertia of a system of masses equidistant from an axis of rotation is always the sum of the masses multiplied by the square of the distance from the axis.

10.4 (b) Rotation about the axis through point P requires more work. The moment of inertia of the hoop about the center axis is $I_{CM} = MR^2$, whereas, by the parallel-axis theorem, the moment of inertia about the axis through point P is $I_P = I_{CM} + MR^2 = MR^2 + MR^2 = 2MR^2$.

PUZZLER

One of the most popular early bicycles was the penny–farthing, introduced in 1870. The bicycle was so named because the size relationship of its two wheels was about the same as the size relationship of the penny and the farthing, two English coins. When the rider looks down at the top of the front wheel, he sees it moving forward faster than he and the handlebars are moving. Yet the center of the wheel does not appear to be moving at all relative to the handlebars. How can different parts of the rolling wheel move at different linear speeds? *(© Steve Lovegrove/Tasmanian Photo Library)*

chapter 11

Rolling Motion and Angular Momentum

Chapter Outline

In the preceding chapter we learned how to treat a rigid body rotating about a fixed axis; in the present chapter, we move on to the more general case in which the axis of rotation is not fixed in space. We begin by describing such motion, which is called *rolling motion.* The central topic of this chapter is, however, angular momentum, a quantity that plays a key role in rotational dynamics. In analogy to the conservation of linear momentum, we find that the angular momentum of a rigid object is always conserved if no external torques act on the object. Like the law of conservation of linear momentum, the law of conservation of angular momentum is a fundamental law of physics, equally valid for relativistic and quantum systems.

11.1 ROLLING MOTION OF A RIGID OBJECT

7.7 In this section we treat the motion of a rigid object rotating about a moving axis. In general, such motion is very complex. However, we can simplify matters by restricting our discussion to a homogeneous rigid object having a high degree of symmetry, such as a cylinder, sphere, or hoop. Furthermore, we assume that the object undergoes rolling motion along a flat surface. We shall see that if an object such as a cylinder rolls without slipping on the surface (we call this *pure rolling motion*), a simple relationship exists between its rotational and translational motions.

Suppose a cylinder is rolling on a straight path. As Figure 11.1 shows, the center of mass moves in a straight line, but a point on the rim moves in a more complex path called a *cycloid.* This means that the axis of rotation remains parallel to its initial orientation in space. Consider a uniform cylinder of radius R rolling without slipping on a horizontal surface (Fig. 11.2). As the cylinder rotates through an angle θ, its center of mass moves a linear distance $s = R\theta$ (see Eq. 10.1a). Therefore, the linear speed of the center of mass for pure rolling motion is given by

$$v_{\text{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \qquad \textbf{(11.1)}$$

where ω is the angular velocity of the cylinder. Equation 11.1 holds whenever a cylinder or sphere rolls without slipping and is the **condition for pure rolling**

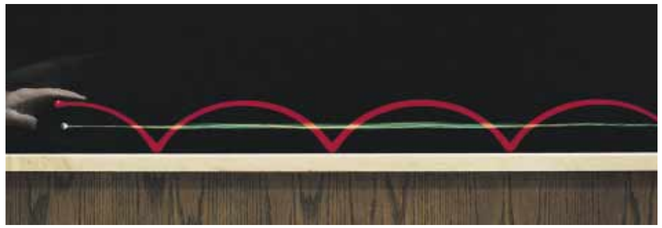


Figure 11.1 One light source at the center of a rolling cylinder and another at one point on the rim illustrate the different paths these two points take. The center moves in a straight line (green line), whereas the point on the rim moves in the path called a *cycloid* (red curve). *(Henry Leap and Jim Lehman)*

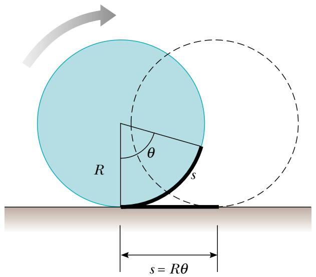


Figure 11.2 In pure rolling motion, as the cylinder rotates through an angle θ, its center of mass moves a linear distance $s = R\theta$.

motion. The magnitude of the linear acceleration of the center of mass for pure rolling motion is

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha \qquad \textbf{(11.2)}$$

where α is the angular acceleration of the cylinder.

The linear velocities of the center of mass and of various points on and within the cylinder are illustrated in Figure 11.3. A short time after the moment shown in the drawing, the rim point labeled P will have rotated from the six o'clock position to, say, the seven o'clock position, the point Q will have rotated from the ten o'clock position to the eleven o'clock position, and so on. Note that the linear velocity of any point is in a direction perpendicular to the line from that point to the contact point P. At any instant, the part of the rim that is at point P is at rest relative to the surface because slipping does not occur.

All points on the cylinder have the same angular speed. Therefore, because the distance from P' to P is twice the distance from P to the center of mass, P' has a speed $2v_{CM} = 2R\omega$. To see why this is so, let us model the rolling motion of the cylinder in Figure 11.4 as a combination of translational (linear) motion and rotational motion. For the pure translational motion shown in Figure 11.4a, imagine that the cylinder does not rotate, so that each point on it moves to the right with speed v_{CM}. For the pure rotational motion shown in Figure 11.4b, imagine that a rotation axis through the center of mass is stationary, so that each point on the cylinder has the same rotational speed ω. The combination of these two motions represents the rolling motion shown in Figure 11.4c. Note in Figure 11.4c that the top of the cylinder has linear speed $v_{CM} + R\omega = v_{CM} + v_{CM} = 2v_{CM}$, which is greater than the linear speed of any other point on the cylinder. As noted earlier, the center of mass moves with linear speed v_{CM} while the contact point between the surface and cylinder has a linear speed of zero.

7.2

We can express the total kinetic energy of the rolling cylinder as

$$K = \tfrac{1}{2}I_P\omega^2 \qquad \textbf{(11.3)}$$

where I_P is the moment of inertia about a rotation axis through P. Applying the parallel-axis theorem, we can substitute $I_P = I_{CM} + MR^2$ into Equation 11.3 to obtain

$$K = \tfrac{1}{2}I_{CM}\omega^2 + \tfrac{1}{2}MR^2\omega^2$$

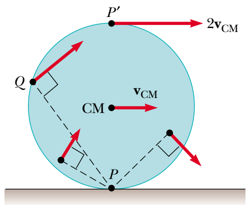


Figure 11.3 All points on a rolling object move in a direction perpendicular to an axis through the instantaneous point of contact P. In other words, all points rotate about P. The center of mass of the object moves with a velocity $\mathbf{v}_{CM}$, and the point P' moves with a velocity $2\mathbf{v}_{CM}$.

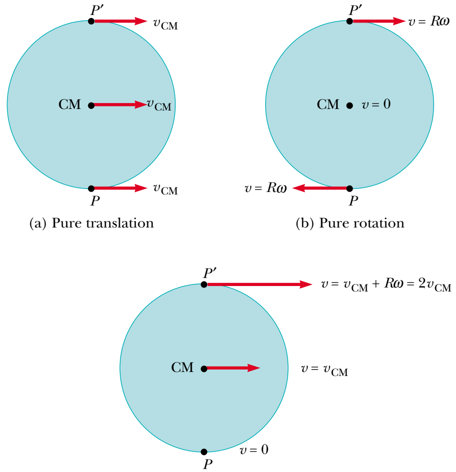


Figure 11.4 The motion of a rolling object can be modeled as a combination of pure translation and pure rotation.

or, because $v_{CM} = R\omega$,

Total kinetic energy of a rolling body

$$K = \tfrac{1}{2}I_{CM}\omega^2 + \tfrac{1}{2}Mv_{CM}^{\ 2} \qquad \textbf{(11.4)}$$

The term $\frac{1}{2}I_{CM}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass, and the term $\frac{1}{2}Mv_{CM}^{\ 2}$ represents the kinetic energy the cylinder would have if it were just translating through space without rotating. Thus, we can say that the **total kinetic energy of a rolling object is the sum of the rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.**

We can use energy methods to treat a class of problems concerning the rolling motion of a sphere down a rough incline. (The analysis that follows also applies to the rolling motion of a cylinder or hoop.) We assume that the sphere in Figure 11.5 rolls without slipping and is released from rest at the top of the incline. Note that accelerated rolling motion is possible only if a frictional force is present between the sphere and the incline to produce a net torque about the center of mass. Despite the presence of friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant. On the other hand, if the sphere were to slip, mechanical energy would be lost as motion progressed.

Using the fact that $v_{CM} = R\omega$ for pure rolling motion, we can express Equation 11.4 as

$$K = \tfrac{1}{2}I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \tfrac{1}{2}Mv_{CM}^{\ 2}$$

$$K = \tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}^{\ 2} \qquad \textbf{(11.5)}$$

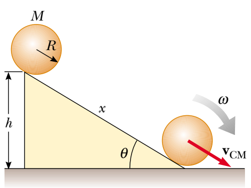


Figure 11.5 A sphere rolling down an incline. Mechanical energy is conserved if no slipping occurs.

By the time the sphere reaches the bottom of the incline, work equal to Mgh has been done on it by the gravitational field, where h is the height of the incline. Because the sphere starts from rest at the top, its kinetic energy at the bottom, given by Equation 11.5, must equal this work done. Therefore, the speed of the center of mass at the bottom can be obtained by equating these two quantities:

$$\frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}^2 = Mgh$$

$$v_{CM} = \left(\frac{2gh}{1 + I_{CM}/MR^2}\right)^{1/2} \qquad \textbf{(11.6)}$$

Quick Quiz 11.1

Imagine that you slide your textbook across a gymnasium floor with a certain initial speed. It quickly stops moving because of friction between it and the floor. Yet, if you were to start a basketball rolling with the same initial speed, it would probably keep rolling from one end of the gym to the other. Why does a basketball roll so far? Doesn't friction affect its motion?

EXAMPLE 11.1 Sphere Rolling Down an Incline

For the solid sphere shown in Figure 11.5, calculate the linear speed of the center of mass at the bottom of the incline and the magnitude of the linear acceleration of the center of mass.

Solution The sphere starts from the top of the incline with potential energy $U_g = Mgh$ and kinetic energy $K = 0$. As we have seen before, if it fell vertically from that height, it would have a linear speed of $\sqrt{2gh}$ at the moment before it hit the floor. After rolling down the incline, the linear speed of the center of mass must be less than this value because some of the initial potential energy is diverted into rotational kinetic energy rather than all being converted into translational kinetic energy. For a uniform solid sphere, $I_{CM} = \frac{2}{5}MR^2$ (see Table 10.2), and therefore Equation 11.6 gives

$$v_{CM} = \left(\frac{2gh}{1 + \frac{2/5MR^2}{MR^2}}\right)^{1/2} = \boxed{\left(\frac{10}{7}gh\right)^{1/2}}$$

which is less than $\sqrt{2gh}$.

To calculate the linear acceleration of the center of mass, we note that the vertical displacement is related to the displacement x along the incline through the relationship $h = x\sin\theta$. Hence, after squaring both sides, we can express the equation above as

$$v_{CM}^2 = \frac{10}{7}gx\sin\theta$$

Comparing this with the expression from kinematics, $v_{CM}^2 = 2a_{CM}x$ (see Eq. 2.12), we see that the acceleration of the center of mass is

$$a_{CM} = \boxed{\frac{5}{7}g\sin\theta}$$

These results are quite interesting in that both the speed and the acceleration of the center of mass are *independent* of the mass and the radius of the sphere! That is, **all homogeneous solid spheres experience the same speed and acceleration on a given incline.**

If we repeated the calculations for a hollow sphere, a solid cylinder, or a hoop, we would obtain similar results in which only the factor in front of $g\sin\theta$ would differ. The constant factors that appear in the expressions for v_{CM} and a_{CM} depend only on the moment of inertia about the center of mass for the specific body. In all cases, the acceleration of the center of mass is *less* than $g\sin\theta$, the value the acceleration would have if the incline were frictionless and no rolling occurred.

EXAMPLE 11.2 Another Look at the Rolling Sphere

In this example, let us use dynamic methods to verify the results of Example 11.1. The free-body diagram for the sphere is illustrated in Figure 11.6.

Solution Newton's second law applied to the center of mass gives

$$(1) \qquad \Sigma F_x = Mg\sin\theta - f = Ma_{CM}$$

$$\Sigma F_y = n - Mg\cos\theta = 0$$

where x is measured along the slanted surface of the incline.

Now let us write an expression for the torque acting on the sphere. A convenient axis to choose is the one that passes

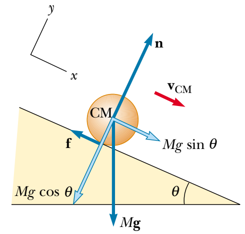


Figure 11.6 Free-body diagram for a solid sphere rolling down an incline.

through the center of the sphere and is perpendicular to the plane of the figure.[1] Because **n** and $M\mathbf{g}$ go through the center of mass, they have zero moment arm about this axis and thus do not contribute to the torque. However, the force of static friction produces a torque about this axis equal to fR in the clockwise direction; therefore, because τ is also in the clockwise direction,

$$\tau_{CM} = fR = I_{CM}\alpha$$

Because $I_{CM} = \frac{2}{5}MR^2$ and $\alpha = a_{CM}/R$, we obtain

$$(2) \qquad f = \frac{I_{CM}\alpha}{R} = \left(\frac{\frac{2}{5}MR^2}{R}\right)\frac{a_{CM}}{R} = \frac{2}{5}Ma_{CM}$$

Substituting Equation (2) into Equation (1) gives

$$a_{CM} = \boxed{\frac{5}{7}g\sin\theta}$$

which agrees with the result of Example 11.1.

Note that $\Sigma\mathbf{F} = m\mathbf{a}$ applies only if $\Sigma\mathbf{F}$ is the net external force exerted on the sphere and **a** is the acceleration of its center of mass. In the case of our sphere rolling down an incline, even though the frictional force does not change the total kinetic energy of the sphere, it does contribute to $\Sigma\mathbf{F}$ and thus decreases the acceleration of the center of mass. As a result, the final translational kinetic energy is less than it would be in the absence of friction. As mentioned in Example 11.1, some of the initial potential energy is converted to rotational kinetic energy.

QuickLab

Hold a basketball and a tennis ball side by side at the top of a ramp and release them at the same time. Which reaches the bottom first? Does the outcome depend on the angle of the ramp? What if the angle were 90° (that is, if the balls were in free fall)?

Quick Quiz 11.2

Which gets to the bottom first: a ball rolling without sliding down incline A or a box sliding down a frictionless incline B having the same dimensions as incline A?

11.2 THE VECTOR PRODUCT AND TORQUE

Consider a force **F** acting on a rigid body at the vector position **r** (Fig. 11.7). **The origin *O* is assumed to be in an inertial frame, so Newton's first law is valid in this case.** As we saw in Section 10.6, the *magnitude* of the torque due to this force relative to the origin is, by definition, $rF\sin\phi$, where ϕ is the angle between **r** and **F**. The axis about which **F** tends to produce rotation is perpendicular to the plane formed by **r** and **F**. If the force lies in the xy plane, as it does in Figure 11.7, the torque $\boldsymbol{\tau}$ is represented by a vector parallel to the z axis. The force in Figure 11.7 creates a torque that tends to rotate the body counterclockwise about the z axis; thus the direction of $\boldsymbol{\tau}$ is toward increasing z, and $\boldsymbol{\tau}$ is therefore in the positive z direction. If we reversed the direction of **F** in Figure 11.7, then $\boldsymbol{\tau}$ would be in the negative z direction.

The torque $\boldsymbol{\tau}$ involves the two vectors **r** and **F**, and its direction is perpendicular to the plane of **r** and **F**. We can establish a mathematical relationship between $\boldsymbol{\tau}$, **r**, and **F**, using a new mathematical operation called the **vector product,** or **cross product:**

Torque

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \qquad \textbf{(11.7)}$$

[1] Although a coordinate system whose origin is at the center of mass of a rolling object is not an inertial frame, the expression $\tau_{CM} = I\alpha$ still applies in the center-of-mass frame.

We now give a formal definition of the vector product. Given any two vectors **A** and **B**, the **vector product A × B** is defined as a third vector **C**, the magnitude of which is $AB\sin\theta$, where θ is the angle between **A** and **B**. That is, if **C** is given by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \qquad \textbf{(11.8)}$$

then its magnitude is

$$C \equiv AB\sin\theta \qquad \textbf{(11.9)}$$

The quantity $AB\sin\theta$ is equal to the area of the parallelogram formed by **A** and **B**, as shown in Figure 11.8. The *direction* of **C** is perpendicular to the plane formed by **A** and **B**, and the best way to determine this direction is to use the right-hand rule illustrated in Figure 11.8. The four fingers of the right hand are pointed along **A** and then "wrapped" into **B** through the angle θ. The direction of the erect right thumb is the direction of **A × B = C.** Because of the notation, **A × B** is often read "**A** cross **B**"; hence, the term *cross product*.

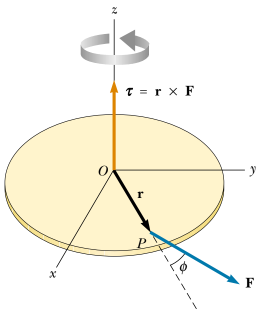


Figure 11.7 The torque vector $\boldsymbol{\tau}$ lies in a direction perpendicular to the plane formed by the position vector **r** and the applied force vector **F**.

Some properties of the vector product that follow from its definition are as follows:

Properties of the vector product

1. Unlike the scalar product, the vector product is *not* commutative. Instead, the order in which the two vectors are multiplied in a cross product is important:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad \textbf{(11.10)}$$

   Therefore, if you change the order of the vectors in a cross product, you must change the sign. You could easily verify this relationship with the right-hand rule.
2. If **A** is parallel to **B** ($\theta = 0°$ or $180°$), then $\mathbf{A} \times \mathbf{B} = 0$; therefore, it follows that $\mathbf{A} \times \mathbf{A} = 0$.
3. If **A** is perpendicular to **B**, then $|\mathbf{A} \times \mathbf{B}| = AB$.
4. The vector product obeys the distributive law:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad \textbf{(11.11)}$$

5. The derivative of the cross product with respect to some variable such as t is

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B} \qquad \textbf{(11.12)}$$

   where it is important to preserve the multiplicative order of **A** and **B**, in view of Equation 11.10.

It is left as an exercise to show from Equations 11.9 and 11.10 and from the definition of unit vectors that the cross products of the rectangular unit vectors **i**, **j**, and **k** obey the following rules:

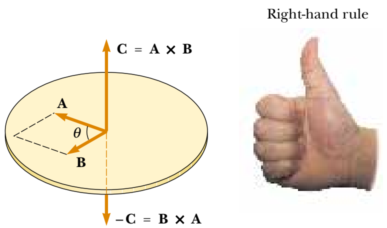


Figure 11.8 The vector product **A × B** is a third vector **C** having a magnitude $AB\sin\theta$ equal to the area of the parallelogram shown. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and this direction is determined by the right-hand rule.

Cross products of unit vectors

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \qquad \textbf{(11.13a)}$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k} \qquad \textbf{(11.13b)}$$

$$\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \qquad \textbf{(11.13c)}$$

$$\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j} \qquad \textbf{(11.13d)}$$

Signs are interchangeable in cross products. For example, $\mathbf{A} \times (-\mathbf{B}) = -\mathbf{A} \times \mathbf{B}$ and $\mathbf{i} \times (-\mathbf{j}) = -\mathbf{i} \times \mathbf{j}$.

The cross product of any two vectors **A** and **B** can be expressed in the following determinant form:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

Expanding these determinants gives the result

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \qquad \textbf{(11.14)}$$

EXAMPLE 11.3 The Cross Product

Two vectors lying in the xy plane are given by the equations $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$. Find $\mathbf{A} \times \mathbf{B}$ and verify that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Solution Using Equations 11.13a through 11.13d, we obtain

$$\mathbf{A} \times \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) \times (-\mathbf{i} + 2\mathbf{j})$$

$$= 2\mathbf{i} \times 2\mathbf{j} + 3\mathbf{j} \times (-\mathbf{i}) = 4\mathbf{k} + 3\mathbf{k} = 7\mathbf{k}$$

(We have omitted the terms containing $\mathbf{i} \times \mathbf{i}$ and $\mathbf{j} \times \mathbf{j}$ because, as Equation 11.13a shows, they are equal to zero.)

We can show that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, since

$$\mathbf{B} \times \mathbf{A} = (-\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j})$$

$$= -\mathbf{i} \times 3\mathbf{j} + 2\mathbf{j} \times 2\mathbf{i} = -3\mathbf{k} - 4\mathbf{k} = -7\mathbf{k}$$

Therefore, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

As an alternative method for finding **A × B,** we could use Equation 11.14, with $A_x = 2$, $A_y = 3$, $A_z = 0$ and $B_x = -1$, $B_y = 2$, $B_z = 0$:

$$\mathbf{A} \times \mathbf{B} = (0)\mathbf{i} - (0)\mathbf{j} + [(2)(2) - (3)(-1)]\mathbf{k} = 7\mathbf{k}$$

Exercise Use the results to this example and Equation 11.9 to find the angle between **A** and **B**.

Answer 60.3°

11.3 ANGULAR MOMENTUM OF A PARTICLE

7.8

Imagine a rigid pole sticking up through the ice on a frozen pond (Fig. 11.9). A skater glides rapidly toward the pole, aiming a little to the side so that she does not hit it. As she approaches a point beside the pole, she reaches out and grabs the pole, an action that whips her rapidly into a circular path around the pole. Just as the idea of linear momentum helps us analyze translational motion, a rotational analog—*angular momentum*—helps us describe this skater and other objects undergoing rotational motion.

To analyze the motion of the skater, we need to know her mass and her velocity, as well as her position relative to the pole. In more general terms, consider a particle of mass m located at the vector position $\mathbf{r}$ and moving with linear velocity $\mathbf{v}$ (Fig. 11.10).

Angular momentum of a particle

The instantaneous angular momentum $\mathbf{L}$ of the particle relative to the origin O is defined as the cross product of the particle's instantaneous position vector $\mathbf{r}$ and its instantaneous linear momentum $\mathbf{p}$:

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{11.15}$$

The SI unit of angular momentum is $\text{kg}\cdot\text{m}^2/\text{s}$. It is important to note that both the magnitude and the direction of $\mathbf{L}$ depend on the choice of origin. Following the right-hand rule, note that the direction of $\mathbf{L}$ is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{p}$. In Figure 11.10, $\mathbf{r}$ and $\mathbf{p}$ are in the xy plane, and so $\mathbf{L}$ points in the z direction. Because $\mathbf{p} = m\mathbf{v}$, the magnitude of $\mathbf{L}$ is

$$L = mvr\sin\phi \tag{11.16}$$

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{p}$. It follows that L is zero when $\mathbf{r}$ is parallel to $\mathbf{p}$ ($\phi = 0$ or $180°$). In other words, when the linear velocity of the particle is along a line that passes through the origin, the particle has zero angular momentum with respect to the origin. On the other hand, if $\mathbf{r}$ is perpendicular to $\mathbf{p}$ ($\phi = 90°$), then $L = mvr$. At that instant, the particle moves exactly as if it were on the rim of a wheel rotating about the origin in a plane defined by $\mathbf{r}$ and $\mathbf{p}$.

Figure 11.9 As the skater passes the pole, she grabs hold of it. This causes her to swing around the pole rapidly in a circular path.

Quick Quiz 11.3

Recall the skater described at the beginning of this section. What would be her angular momentum relative to the pole if she were skating directly toward it?

In describing linear motion, we found that the net force on a particle equals the time rate of change of its linear momentum, $\Sigma\mathbf{F} = d\mathbf{p}/dt$ (see Eq. 9.3). We now show that the net torque acting on a particle equals the time rate of change of its angular momentum. Let us start by writing the net torque on the particle in the form

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{11.17}$$

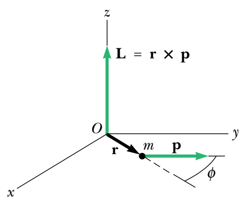


Figure 11.10 The angular momentum $\mathbf{L}$ of a particle of mass m and linear momentum $\mathbf{p}$ located at the vector position $\mathbf{r}$ is a vector given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The value of $\mathbf{L}$ depends on the origin about which it is measured and is a vector perpendicular to both $\mathbf{r}$ and $\mathbf{p}$.

Now let us differentiate Equation 11.15 with respect to time, using the rule given by Equation 11.12:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

Remember, it is important to adhere to the order of terms because $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The last term on the right in the above equation is zero because $\mathbf{v} = d\mathbf{r}/dt$ is parallel to $\mathbf{p} = m\mathbf{v}$ (property 2 of the vector product). Therefore,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{11.18}$$

Comparing Equations 11.17 and 11.18, we see that

The net torque equals time rate of change of angular momentum

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{11.19}$$

which is the rotational analog of Newton's second law, $\Sigma\mathbf{F} = d\mathbf{p}/dt$. Note that torque causes the angular momentum $\mathbf{L}$ to change just as force causes linear momentum $\mathbf{p}$ to change. This rotational result, Equation 11.19, states that

the net torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

It is important to note that Equation 11.19 is valid only if $\Sigma\boldsymbol{\tau}$ and $\mathbf{L}$ are measured about the same origin. (Of course, the same origin must be used in calculating all of the torques.) Furthermore, **the expression is valid for any origin fixed in an inertial frame.**

Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i$$

where the vector sum is over all n particles in the system.

Because individual angular momenta can change with time, so can the total angular momentum. In fact, from Equations 11.18 and 11.19, we find that the time rate of change of the total angular momentum equals the vector sum of all torques acting on the system, both those associated with internal forces between particles and those associated with external forces. However, the net torque associated with all internal forces is zero. To understand this, recall that Newton's third law tells us that internal forces between particles of the system are equal in magnitude and opposite in direction. If we assume that these forces lie along the line of separation of each pair of particles, then the torque due to each action–reaction force pair is zero. That is, the moment arm d from O to the line of action of the forces is equal for both particles. In the summation, therefore, we see that the net internal torque vanishes. We conclude that the total angular momentum of a system can vary with time only if a net external torque is acting on the system, so that we have

$$\sum \boldsymbol{\tau}_{\text{ext}} = \sum_i \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt}\sum_i \mathbf{L}_i = \frac{d\mathbf{L}}{dt} \tag{11.20}$$

That is,

the time rate of change of the total angular momentum of a system about some origin in an inertial frame equals the net external torque acting on the system about that origin.

Note that Equation 11.20 is the rotational analog of Equation 9.38, $\Sigma\mathbf{F}_{\text{ext}} = d\mathbf{p}/dt$, for a system of particles.

EXAMPLE 11.4 Circular Motion

A particle moves in the xy plane in a circular path of radius r, as shown in Figure 11.11. (a) Find the magnitude and direction of its angular momentum relative to O when its linear velocity is $\mathbf{v}$.

Solution You might guess that because the linear momentum of the particle is always changing (in direction, not magnitude), the direction of the angular momentum must also change. In this example, however, this is not the case. The magnitude of $\mathbf{L}$ is given by

$$L = mvr \sin 90° = mvr \quad (\text{for } \mathbf{r} \text{ perpendicular to } \mathbf{v})$$

This value of L is constant because all three factors on the right are constant. The direction of $\mathbf{L}$ also is constant, even though the direction of $\mathbf{p} = m\mathbf{v}$ keeps changing. You can visualize this by sliding the vector $\mathbf{v}$ in Figure 11.11 parallel to itself until its tail meets the tail of $\mathbf{r}$ and by then applying the right-hand rule. (You can use $\mathbf{v}$ to determine the direction of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ because the direction of $\mathbf{p}$ is the same as the direction of $\mathbf{v}$.) Line up your fingers so that they point along $\mathbf{r}$ and wrap your fingers into the vector $\mathbf{v}$. Your thumb points upward and away from the page; this is the direction of $\mathbf{L}$. Hence, we can write the vector expression $\mathbf{L} = (mvr)\mathbf{k}$. If the particle were to move clockwise, $\mathbf{L}$ would point downward and into the page.

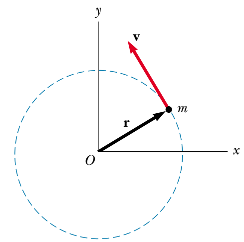


Figure 11.11 A particle moving in a circle of radius r has an angular momentum about O that has magnitude mvr. The vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ points *out* of the diagram.

(b) Find the magnitude and direction of $\mathbf{L}$ in terms of the particle's angular speed ω.

Solution Because $v = r\omega$ for a particle rotating in a circle, we can express L as

$$L = mvr = mr^2\omega = I\omega$$

where I is the moment of inertia of the particle about the z axis through O. Because the rotation is counterclockwise, the direction of $\boldsymbol{\omega}$ is along the z axis (see Section 10.1). The direction of $\mathbf{L}$ is the same as that of $\boldsymbol{\omega}$, and so we can write the angular momentum as $\mathbf{L} = I\boldsymbol{\omega} = I\omega\mathbf{k}$.

Exercise A car of mass 1 500 kg moves with a linear speed of 40 m/s on a circular race track of radius 50 m. What is the magnitude of its angular momentum relative to the center of the track?

Answer $3.0 \times 10^6 \text{ kg}\cdot\text{m}^2/\text{s}$

11.4 ANGULAR MOMENTUM OF A ROTATING RIGID OBJECT

Consider a rigid object rotating about a fixed axis that coincides with the z axis of a coordinate system, as shown in Figure 11.12. Let us determine the angular momentum of this object. Each particle of the object rotates in the xy plane about the z axis with an angular speed ω. The magnitude of the angular momentum of a particle of mass m_i about the origin O is $m_iv_ir_i$. Because $v_i = r_i\omega$, we can express the magnitude of the angular momentum of this particle as

$$L_i = m_ir_i^2\omega$$

The vector $\mathbf{L}_i$ is directed along the z axis, as is the vector $\boldsymbol{\omega}$.

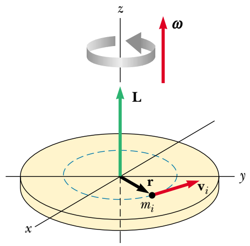


Figure 11.12 When a rigid body rotates about an axis, the angular momentum $\mathbf{L}$ is in the same direction as the angular velocity $\boldsymbol{\omega}$, according to the expression $\mathbf{L} = I\boldsymbol{\omega}$.

We can now find the angular momentum (which in this situation has only a z component) of the whole object by taking the sum of L_i over all particles:

$$L_z = \sum_i m_ir_i^2\omega = \left(\sum_i m_ir_i^2\right)\omega$$

$$L_z = I\omega \tag{11.21}$$

where I is the moment of inertia of the object about the z axis.

Now let us differentiate Equation 11.21 with respect to time, noting that I is constant for a rigid body:

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha \tag{11.22}$$

where α is the angular acceleration relative to the axis of rotation. Because dL_z/dt is equal to the net external torque (see Eq. 11.20), we can express Equation 11.22 as

$$\sum \tau_{\text{ext}} = \frac{dL_z}{dt} = I\alpha \tag{11.23}$$

That is, the net external torque acting on a rigid object rotating about a fixed axis equals the moment of inertia about the rotation axis multiplied by the object's angular acceleration relative to that axis.

Equation 11.23 also is valid for a rigid object rotating about a moving axis provided the moving axis (1) passes through the center of mass and (2) is a symmetry axis.

You should note that if a symmetrical object rotates about a fixed axis passing through its center of mass, you can write Equation 11.21 in vector form as $\mathbf{L} = I\boldsymbol{\omega}$, where $\mathbf{L}$ is the total angular momentum of the object measured with respect to the axis of rotation. Furthermore, the expression is valid for any object, regardless of its symmetry, if $\mathbf{L}$ stands for the component of angular momentum along the axis of rotation.[2]

EXAMPLE 11.5 Bowling Ball

Estimate the magnitude of the angular momentum of a bowling ball spinning at 10 rev/s, as shown in Figure 11.13.

Solution We start by making some estimates of the relevant physical parameters and model the ball as a uniform solid sphere. A typical bowling ball might have a mass of 6 kg and a radius of 12 cm. The moment of inertia of a solid sphere about an axis through its center is, from Table 10.2,

$$I = \tfrac{2}{5}MR^2 = \tfrac{2}{5}(6 \text{ kg})(0.12 \text{ m})^2 = 0.035 \text{ kg}\cdot\text{m}^2$$

Therefore, the magnitude of the angular momentum is

[2] In general, the expression $\mathbf{L} = I\boldsymbol{\omega}$ is not always valid. If a rigid object rotates about an arbitrary axis, $\mathbf{L}$ and $\boldsymbol{\omega}$ may point in different directions. In this case, the moment of inertia cannot be treated as a scalar. Strictly speaking, $\mathbf{L} = I\boldsymbol{\omega}$ applies only to rigid objects of any shape that rotate about one of three mutually perpendicular axes (called *principal axes*) through the center of mass. This is discussed in more advanced texts on mechanics.

$$L = I\omega = (0.035\ \text{kg}\cdot\text{m}^2)(10\ \text{rev/s})(2\pi\ \text{rad/rev})$$
$$= 2.2\ \text{kg}\cdot\text{m}^2/\text{s}$$

Because of the roughness of our estimates, we probably want to keep only one significant figure, and so $L \approx$ $2\ \text{kg}\cdot\text{m}^2/\text{s}$.

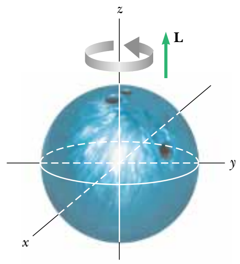


Figure 11.13 A bowling ball that rotates about the z axis in the direction shown has an angular momentum $\mathbf{L}$ in the positive z direction. If the direction of rotation is reversed, $\mathbf{L}$ points in the negative z direction.

EXAMPLE 11.6 Rotating Rod

A rigid rod of mass M and length ℓ is pivoted without friction at its center (Fig. 11.14). Two particles of masses m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed ω. (a) Find an expression for the magnitude of the angular momentum of the system.

Solution This is different from the last example in that we now must account for the motion of more than one object. The moment of inertia of the system equals the sum of the moments of inertia of the three components: the rod and the two particles. Referring to Table 10.2 to obtain the expression for the moment of inertia of the rod, and using the expression $I = mr^2$ for each particle, we find that the total moment of inertia about the z axis through O is

$$I = \frac{1}{12}M\ell^2 + m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2$$
$$= \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)$$

Therefore, the magnitude of the angular momentum is

$$L = I\omega = \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega$$

(b) Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizontal.

Solution If the masses of the two particles are equal, then the system has no angular acceleration because the net torque on the system is zero when $m_1 = m_2$. If the initial angle θ is exactly $\pi/2$ or $-\pi/2$ (vertical position), then the rod will be in equilibrium. To find the angular acceleration of the system at any angle θ, we first calculate the net torque on the system and then use $\Sigma\tau_{\text{ext}} = I\alpha$ to obtain an expression for α.

The torque due to the force m_1g about the pivot is

$$\tau_1 = m_1 g\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_1 \text{ out of page})$$

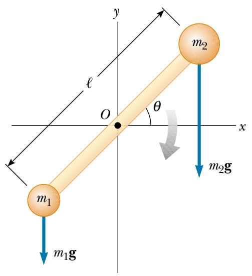


Figure 11.14 Because gravitational forces act on the rotating rod, there is in general a net nonzero torque about O when $m_1 \neq m_2$. This net torque produces an angular acceleration given by $\alpha = \Sigma\tau_{\text{ext}}/I$.

The torque due to the force m_2g about the pivot is

$$\tau_2 = -m_2 g\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_2 \text{ into page})$$

Hence, the net torque exerted on the system about O is

$$\sum \tau_{\text{ext}} = \tau_1 + \tau_2 = \tfrac{1}{2}(m_1 - m_2)g\ell\cos\theta$$

The direction of $\Sigma\boldsymbol{\tau}_{\text{ext}}$ is out of the page if $m_1 > m_2$ and is into the page if $m_2 > m_1$.

To find α, we use $\Sigma\tau_{\text{ext}} = I\alpha$, where I was obtained in part (a):

$$\alpha = \frac{\Sigma\tau_{\text{ext}}}{I} = \frac{2(m_1 - m_2)g\cos\theta}{\ell(M/3 + m_1 + m_2)}$$

Note that α is zero when θ is $\pi/2$ or $-\pi/2$ (vertical position) and is a maximum when θ is 0 or π (horizontal position).

Exercise If $m_2 > m_1$, at what value of θ is ω a maximum?

Answer $\theta = -\pi/2$.

EXAMPLE 11.7 Two Connected Masses

A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley, as shown in Figure 11.15. The radius of the pulley is R, and the moment of inertia about its axle is I. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

Solution We need to determine the angular momentum of the system, which consists of the two objects and the pulley. Let us calculate the angular momentum about an axis that coincides with the axle of the pulley.

At the instant the sphere and block have a common speed v, the angular momentum of the sphere is m_1vR, and that of the block is m_2vR. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Hence, the total angular momentum of the system is

$$\text{(1)} \qquad L = m_1vR + m_2vR + I\frac{v}{R}$$

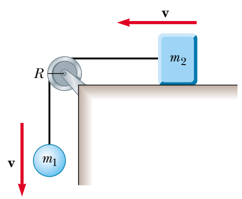


Figure 11.15

Now let us evaluate the total external torque acting on the system about the pulley axle. Because it has a moment arm of zero, the force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on the block is balanced by the force of gravity $m_2\mathbf{g}$, and so these forces do not contribute to the torque. The force of gravity $m_1\mathbf{g}$ acting on the sphere produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. (Note that in this situation, the tension is *not* equal to m_1g.) This is the total external torque about the pulley axle; that is, $\Sigma\tau_{\text{ext}} = m_1gR$. Using this result, together with Equation (1) and Equation 11.23, we find

$$\sum \tau_{\text{ext}} = \frac{dL}{dt}$$

$$m_1gR = \frac{d}{dt}\left[(m_1 + m_2)Rv + I\frac{v}{R}\right]$$

$$\text{(2)} \qquad m_1gR = (m_1 + m_2)R\frac{dv}{dt} + \frac{I}{R}\frac{dv}{dt}$$

Because $dv/dt = a$, we can solve this for a to obtain

$$a = \frac{m_1g}{(m_1 + m_2) + I/R^2}$$

You may wonder why we did not include the forces that the cord exerts on the objects in evaluating the net torque about the axle. The reason is that these forces are internal to the system under consideration, and we analyzed the system as a whole. Only external torques contribute to the change in the system's angular momentum.

11.5 CONSERVATION OF ANGULAR MOMENTUM

7.9

In Chapter 9 we found that the total linear momentum of a system of particles remains constant when the resultant external force acting on the system is zero. We have an analogous conservation law in rotational motion:

Conservation of angular momentum

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

This follows directly from Equation 11.20, which indicates that if

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt} = 0 \qquad \textbf{(11.24)}$$

then

$$\mathbf{L} = \text{constant} \qquad \textbf{(11.25)}$$

For a system of particles, we write this conservation law as $\Sigma\,\mathbf{L}_n = \text{constant}$, where the index n denotes the nth particle in the system.

If the mass of an object undergoes redistribution in some way, then the object's moment of inertia changes; hence, its angular speed must change because $L = I\omega$. In this case we express the conservation of angular momentum in the form

$$\mathbf{L}_i = \mathbf{L}_f = \text{constant} \quad \textbf{(11.26)}$$

If the system is an object rotating about a *fixed* axis, such as the z axis, we can write $L_z = I\omega$, where L_z is the component of $\mathbf{L}$ along the axis of rotation and I is the moment of inertia about this axis. In this case, we can express the conservation of angular momentum as

$$I_i\omega_i = I_f\omega_f = \text{constant} \quad \textbf{(11.27)}$$

This expression is valid both for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system as long as that axis remains parallel to itself. We require only that the net external torque be zero.

Although we do not prove it here, there is an important theorem concerning the angular momentum of an object relative to the object's center of mass:

> The resultant torque acting on an object about an axis through the center of mass equals the time rate of change of angular momentum regardless of the motion of the center of mass.

This theorem applies even if the center of mass is accelerating, provided $\boldsymbol{\tau}$ and $\mathbf{L}$ are evaluated relative to the center of mass.

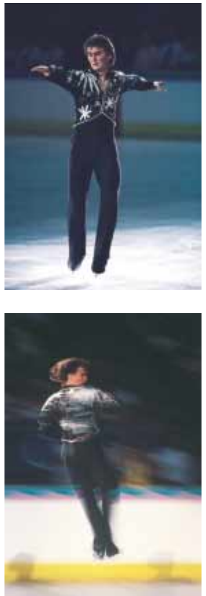
In Equation 11.26 we have a third conservation law to add to our list. We can now state that the energy, linear momentum, and angular momentum of an isolated system all remain constant:

$$\left.\begin{aligned} K_i + U_i &= K_f + U_f \\ \mathbf{p}_i &= \mathbf{p}_f \\ \mathbf{L}_i &= \mathbf{L}_f \end{aligned}\right\} \quad \text{For an isolated system}$$

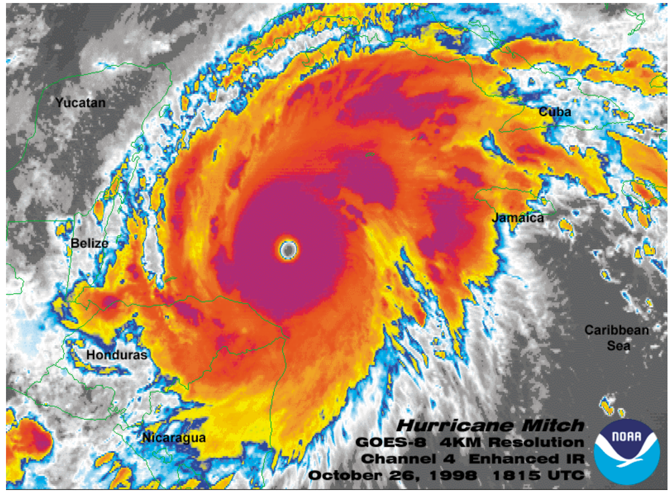
There are many examples that demonstrate conservation of angular momentum. You may have observed a figure skater spinning in the finale of a program. The angular speed of the skater increases when the skater pulls his hands and feet close to his body, thereby decreasing I. Neglecting friction between skates and ice, no external torques act on the skater. The change in angular speed is due to the fact that, because angular momentum is conserved, the product $I\omega$ remains constant, and a decrease in the moment of inertia of the skater causes an increase in the angular speed. Similarly, when divers or acrobats wish to make several somersaults, they pull their hands and feet close to their bodies to rotate at a higher rate. In these cases, the external force due to gravity acts through the center of mass and hence exerts no torque about this point. Therefore, the angular momentum about the center of mass must be conserved—that is, $I_i\omega_i = I_f\omega_f$. For example, when divers wish to double their angular speed, they must reduce their moment of inertia to one-half its initial value.

Quick Quiz 11.4

A particle moves in a straight line, and you are told that the net torque acting on it is zero about some unspecified point. Decide whether the following statements are true or false: (a) The net force on the particle must be zero. (b) The particle's velocity must be constant.



Angular momentum is conserved as figure skater Todd Eldredge pulls his arms toward his body. *(© 1998 David Madison)*



A color-enhanced, infrared image of Hurricane Mitch, which devastated large areas of Honduras and Nicaragua in October 1998. The spiral, nonrigid mass of air undergoes rotation and has angular momentum. *(Courtesy of NOAA)*

EXAMPLE 11.8 Formation of a Neutron Star

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4 km, collapses into a neutron star of radius 3.0 km. Determine the period of rotation of the neutron star.

Solution The same physics that makes a skater spin faster with his arms pulled in describes the motion of the neutron star. Let us assume that during the collapse of the stellar core, (1) no torque acts on it, (2) it remains spherical, and (3) its mass remains constant. Also, let us use the symbol T for the period, with T_i being the initial period of the star and T_f being the period of the neutron star. The period is the length of time a point on the star's equator takes to make one complete circle around the axis of rotation. The angular speed of a star is given by $\omega = 2\pi/T$. Therefore, because I is proportional to r^2, Equation 11.27 gives

$$T_f = T_i\left(\frac{r_f}{r_i}\right)^2 = (30 \text{ days})\left(\frac{3.0 \text{ km}}{1.0 \times 10^4 \text{ km}}\right)^2$$

$$= 2.7 \times 10^{-6} \text{ days} = \boxed{0.23 \text{ s}}$$

Thus, the neutron star rotates about four times each second; this result is approximately the same as that for a spinning figure skater.

EXAMPLE 11.9 The Merry-Go-Round

A horizontal platform in the shape of a circular disk rotates in a horizontal plane about a frictionless vertical axle (Fig. 11.16). The platform has a mass $M = 100$ kg and a radius $R = 2.0$ m. A student whose mass is $m = 60$ kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2.0 rad/s when the student is at the rim, what is the angular speed when he has reached a point $r = 0.50$ m from the center?

Solution The speed change here is similar to the increase in angular speed of the spinning skater when he pulls his arms inward. Let us denote the moment of inertia of the platform as I_p and that of the student as I_s. Treating the student as a point mass, we can write the initial moment of inertia I_i of the system (student plus platform) about the axis of rotation:

$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2$$

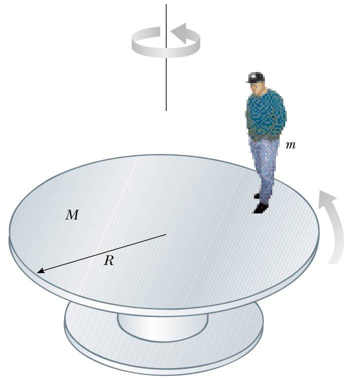


Figure 11.16 As the student walks toward the center of the rotating platform, the angular speed of the system increases because the angular momentum must remain constant.

When the student has walked to the position $r < R$, the moment of inertia of the system reduces to

$$I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$

Note that we still use the greater radius R when calculating I_{pf} because the radius of the platform has not changed. Because no external torques act on the system about the axis of rotation, we can apply the law of conservation of angular momentum:

$$I_i\omega_i = I_f\omega_f$$

$$\left(\tfrac{1}{2}MR^2 + mR^2\right)\omega_i = \left(\tfrac{1}{2}MR^2 + mr^2\right)\omega_f$$

$$\omega_f = \left(\frac{\tfrac{1}{2}MR^2 + mR^2}{\tfrac{1}{2}MR^2 + mr^2}\right)\omega_i$$

$$\omega_f = \left(\frac{200 + 240}{200 + 15}\right)(2.0\text{ rad/s}) = \boxed{4.1\text{ rad/s}}$$

As expected, the angular speed has increased.

Exercise Calculate the initial and final rotational energies of the system.

Answer $K_i = 880$ J; $K_f = 1.8 \times 10^3$ J.

Quick Quiz 11.5

Note that the rotational energy of the system described in Example 11.9 increases. What accounts for this increase in energy?

Example 11.10 The Spinning Bicycle Wheel

In a favorite classroom demonstration, a student holds the axle of a spinning bicycle wheel while seated on a stool that is free to rotate (Fig. 11.17). The student and stool are initially at rest while the wheel is spinning in a horizontal plane with an initial angular momentum $\mathbf{L}_i$ that points upward. When the wheel is inverted about its center by 180°, the student and stool start rotating. In terms of $\mathbf{L}_i$, what are the magnitude and the direction of $\mathbf{L}$ for the student plus stool?

Figure 11.17 The wheel is initially spinning when the student is at rest. What happens when the wheel is inverted?

Solution The system consists of the student, the wheel, and the stool. Initially, the total angular momentum of the system $\mathbf{L}_i$ comes entirely from the spinning wheel. As the wheel is inverted, the student applies a torque to the wheel, but this torque is internal to the system. No external torque is acting on the system about the vertical axis. Therefore, the angular momentum of the system is conserved. Initially, we have

$$\mathbf{L}_{\text{system}} = \mathbf{L}_i = \mathbf{L}_{\text{wheel}} \qquad \text{(upward)}$$

After the wheel is inverted, we have $\mathbf{L}_{\text{inverted wheel}} = -\mathbf{L}_i$. For angular momentum to be conserved, some other part of the system has to start rotating so that the total angular momentum remains the initial angular momentum $\mathbf{L}_i$. That other part of the system is the student plus the stool she is sitting on. So, we can now state that

$$\mathbf{L}_f = \mathbf{L}_i = \mathbf{L}_{\text{student+stool}} - \mathbf{L}_i$$

$$\mathbf{L}_{\text{student+stool}} = \boxed{2\mathbf{L}_i}$$

Example 11.11 Disk and Stick

A 2.0-kg disk traveling at 3.0 m/s strikes a 1.0-kg stick that is lying flat on nearly frictionless ice, as shown in Figure 11.18. Assume that the collision is elastic. Find the translational speed of the disk, the translational speed of the stick, and the rotational speed of the stick after the collision. The moment of inertia of the stick about its center of mass is 1.33 kg·m².

Solution Because the disk and stick form an isolated system, we can assume that total energy, linear momentum, and angular momentum are all conserved. We have three unknowns, and so we need three equations to solve simultaneously. The first comes from the law of the conservation of linear momentum:

$$p_i = p_f$$

$$m_d v_{di} = m_d v_{df} + m_s v_s$$

$$(2.0\text{ kg})(3.0\text{ m/s}) = (2.0\text{ kg})v_{df} + (1.0\text{ kg})v_s$$

$$(1) \qquad 6.0\text{ kg}\cdot\text{m/s} - (2.0\text{ kg})v_{df} = (1.0\text{ kg})v_s$$

Now we apply the law of conservation of angular momentum, using the initial position of the center of the stick as our reference point. We know that the component of angular momentum of the disk along the axis perpendicular to the plane of the ice is negative (the right-hand rule shows that $\mathbf{L}_d$ points into the ice).

$$L_i = L_f$$

$$-rm_d v_{di} = -rm_d v_{df} + I\omega$$

$$-(2.0\text{ m})(2.0\text{ kg})(3.0\text{ m/s}) = -(2.0\text{ m})(2.0\text{ kg})v_{df} + (1.33\text{ kg}\cdot\text{m}^2)\omega$$

$$-12\text{ kg}\cdot\text{m}^2/\text{s} = -(4.0\text{ kg}\cdot\text{m})v_{df} + (1.33\text{ kg}\cdot\text{m}^2)\omega$$

$$(2) \qquad -9.0\text{ rad/s} + (3.0\text{ rad/m})v_{df} = \omega$$

We used the fact that radians are dimensionless to ensure consistent units for each term.

Finally, the elastic nature of the collision reminds us that kinetic energy is conserved; in this case, the kinetic energy consists of translational and rotational forms:

$$K_i = K_f$$

$$\tfrac{1}{2}m_d v_{di}^2 = \tfrac{1}{2}m_d v_{df}^2 + \tfrac{1}{2}m_s v_s^2 + \tfrac{1}{2}I\omega^2$$

$$\tfrac{1}{2}(2.0\text{ kg})(3.0\text{ m/s})^2 = \tfrac{1}{2}(2.0\text{ kg})v_{df}^2 + \tfrac{1}{2}(1.0\text{ kg})v_s^2 + \tfrac{1}{2}(1.33\text{ kg}\cdot\text{m}^2/\text{s})\omega^2$$

$$(3) \qquad 54\text{ m}^2/\text{s}^2 = 6.0v_{df}^2 + 3.0v_s^2 + (4.0\text{ m}^2)\omega^2$$

In solving Equations (1), (2), and (3) simultaneously, we find that $v_{df} = 2.3$ m/s, $v_s = 1.3$ m/s, and $\omega = -2.0$ rad/s. These values seem reasonable. The disk is moving more slowly than it was before the collision, and the stick has a small translational speed. Table 11.1 summarizes the initial and final values of variables for the disk and the stick and verifies the conservation of linear momentum, angular momentum, and kinetic energy.

Exercise Verify the values in Table 11.1.

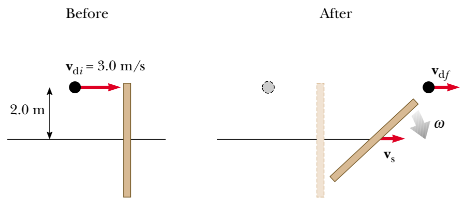


Figure 11.18 Overhead view of a disk striking a stick in an elastic collision, which causes the stick to rotate.

TABLE 11.1 Comparison of Values in Example 11.11 Before and After the Collision[a]

	v (m/s)	ω (rad/s)	p (kg·m/s)	L (kg·m²/s)	K_{trans} (J)	K_{rot} (J)
Before						
Disk	3.0	—	6.0	−12	9.0	—
Stick	0	0	0	0	0	0
Total	—	—	6.0	−12	9.0	0
After						
Disk	2.3	—	4.7	−9.3	5.4	—
Stick	1.3	−2.0	1.3	−2.7	0.9	2.7
Total	—	—	6.0	−12	6.3	2.7

[a] Notice that linear momentum, angular momentum, and total kinetic energy are conserved.

Optional Section

11.6 THE MOTION OF GYROSCOPES AND TOPS

A very unusual and fascinating type of motion you probably have observed is that of a top spinning about its axis of symmetry, as shown in Figure 11.19a. If the top spins very rapidly, the axis rotates about the z axis, sweeping out a cone (see Fig. 11.19b). The motion of the axis about the vertical—known as **precessional motion**—is usually slow relative to the spin motion of the top.

Precessional motion

It is quite natural to wonder why the top does not fall over. Because the center of mass is not directly above the pivot point O, a net torque is clearly acting on the top about O—a torque resulting from the force of gravity $M\mathbf{g}$. The top would certainly fall over if it were not spinning. Because it is spinning, however, it has an angular momentum $\mathbf{L}$ directed along its symmetry axis. As we shall show, the motion of this symmetry axis about the z axis (the precessional motion) occurs because the torque produces a change in the *direction* of the symmetry axis. This is an excellent example of the importance of the directional nature of angular momentum.

The two forces acting on the top are the downward force of gravity $M\mathbf{g}$ and the normal force $\mathbf{n}$ acting upward at the pivot point O. The normal force produces no torque about the pivot because its moment arm through that point is zero. However, the force of gravity produces a torque $\boldsymbol{\tau} = \mathbf{r} \times M\mathbf{g}$ about O, where the direction of $\boldsymbol{\tau}$ is perpendicular to the plane formed by $\mathbf{r}$ and $M\mathbf{g}$. By necessity, the vector $\boldsymbol{\tau}$ lies in a horizontal xy plane perpendicular to the angular momentum vector. The net torque and angular momentum of the top are related through Equation 11.19:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

From this expression, we see that the nonzero torque produces a change in angular momentum $d\mathbf{L}$—a change that is in the same direction as $\boldsymbol{\tau}$. Therefore, like the torque vector, $d\mathbf{L}$ must also be at right angles to $\mathbf{L}$. Figure 11.19b illustrates the resulting precessional motion of the symmetry axis of the top. In a time Δt, the change in angular momentum is $\Delta\mathbf{L} = \mathbf{L}_f - \mathbf{L}_i = \boldsymbol{\tau}\,\Delta t$. Because $\Delta\mathbf{L}$ is perpendicular to $\mathbf{L}$, the magnitude of $\mathbf{L}$ does not change ($|\mathbf{L}_i| = |\mathbf{L}_f|$). Rather, what is changing is the *direction* of $\mathbf{L}$. Because the change in angular momentum $\Delta\mathbf{L}$ is in the direction of $\boldsymbol{\tau}$, which lies in the xy plane, the top undergoes precessional motion.

The essential features of precessional motion can be illustrated by considering the simple gyroscope shown in Figure 11.20a. This device consists of a wheel free to spin about an axle that is pivoted at a distance h from the center of mass of the wheel. When given an angular velocity $\boldsymbol{\omega}$ about the axle, the wheel has an angular momentum $\mathbf{L} = I\boldsymbol{\omega}$ directed along the axle as shown. Let us consider the torque acting on the wheel about the pivot O. Again, the force $\mathbf{n}$ exerted by the support on the axle produces no torque about O, and the force of gravity $M\mathbf{g}$ produces a torque of magnitude Mgh about O, where the axle is perpendicular to the support. The direction of this torque is perpendicular to the axle (and perpendicular to $\mathbf{L}$), as shown in Figure 11.20a. This torque causes the angular momentum to change in the direction perpendicular to the axle. Hence, the axle moves in the direction of the torque—that is, in the horizontal plane.

To simplify the description of the system, we must make an assumption: The total angular momentum of the precessing wheel is the sum of the angular momentum $I\boldsymbol{\omega}$ due to the spinning and the angular momentum due to the motion of

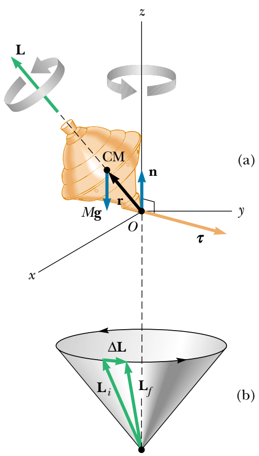


Figure 11.19 Precessional motion of a top spinning about its symmetry axis. (a) The only external forces acting on the top are the normal force $\mathbf{n}$ and the force of gravity $M\mathbf{g}$. The direction of the angular momentum $\mathbf{L}$ is along the axis of symmetry. The right-hand rule indicates that $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times M\mathbf{g}$ is in the xy plane. (b). The direction of $\Delta\mathbf{L}$ is parallel to that of $\boldsymbol{\tau}$ in part (a). The fact that $\mathbf{L}_f = \Delta\mathbf{L} + \mathbf{L}_i$ indicates that the top precesses about the z axis.

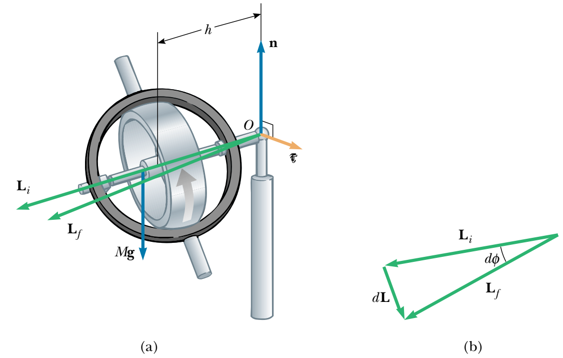
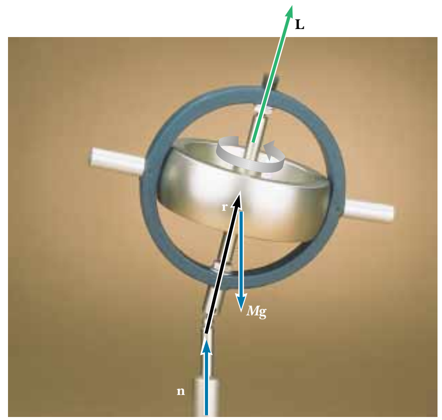


Figure 11.20 (a) The motion of a simple gyroscope pivoted a distance h from its center of mass. The force of gravity $M\mathbf{g}$ produces a torque about the pivot, and this torque is perpendicular to the axle. (b) This torque results in a change in angular momentum $d\mathbf{L}$ in a direction perpendicular to the axle. The axle sweeps out an angle $d\phi$ in a time dt.



This toy gyroscope undergoes precessional motion about the vertical axis as it spins about its axis of symmetry. The only forces acting on it are the force of gravity $M\mathbf{g}$ and the upward force of the pivot $\mathbf{n}$. The direction of its angular momentum $\mathbf{L}$ is along the axis of symmetry. The torque and $\Delta\mathbf{L}$ are directed into the page. *(Courtesy of Central Scientific Company)*

the center of mass about the pivot. In our treatment, we shall neglect the contribution from the center-of-mass motion and take the total angular momentum to be just $I\boldsymbol{\omega}$. In practice, this is a good approximation if $\boldsymbol{\omega}$ is made very large.

In a time dt, the torque due to the gravitational force changes the angular momentum of the system by $d\mathbf{L} = \boldsymbol{\tau}\, dt$. When added vectorially to the original total angular momentum $I\boldsymbol{\omega}$, this additional angular momentum causes a shift in the direction of the total angular momentum.

The vector diagram in Figure 11.20b shows that in the time dt, the angular momentum vector rotates through an angle $d\phi$, which is also the angle through which the axle rotates. From the vector triangle formed by the vectors $\mathbf{L}_i$, $\mathbf{L}_f$, and $d\mathbf{L}$, we see that

$$\sin(d\phi) \approx d\phi = \frac{dL}{L} = \frac{(Mgh)\,dt}{L}$$

where we have used the fact that, for small values of any angle θ, $\sin\theta \approx \theta$. Dividing through by dt and using the relationship $L = I\omega$, we find that the rate at which the axle rotates about the vertical axis is

$$\omega_p = \frac{d\phi}{dt} = \frac{Mgh}{I\omega} \tag{11.28}$$

Precessional frequency

The angular speed ω_p is called the **precessional frequency.** This result is valid only when $\omega_p \ll \omega$. Otherwise, a much more complicated motion is involved. As you can see from Equation 11.28, the condition $\omega_p \ll \omega$ is met when $I\omega$ is great compared with Mgh. Furthermore, note that the precessional frequency decreases as ω increases—that is, as the wheel spins faster about its axis of symmetry.

Quick Quiz 11.6

How much work is done by the force of gravity when a top precesses through one complete circle?

Optional Section

11.7 ANGULAR MOMENTUM AS A FUNDAMENTAL QUANTITY

We have seen that the concept of angular momentum is very useful for describing the motion of macroscopic systems. However, the concept also is valid on a submicroscopic scale and has been used extensively in the development of modern theories of atomic, molecular, and nuclear physics. In these developments, it was found that the angular momentum of a system is a fundamental quantity. The word *fundamental* in this context implies that angular momentum is an intrinsic property of atoms, molecules, and their constituents, a property that is a part of their very nature.

To explain the results of a variety of experiments on atomic and molecular systems, we rely on the fact that the angular momentum has discrete values. These discrete values are multiples of the fundamental unit of angular momentum $\hbar = h/2\pi$, where h is called Planck's constant:

$$\text{Fundamental unit of angular momentum} = \hbar = 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Let us accept this postulate without proof for the time being and show how it can be used to estimate the angular speed of a diatomic molecule. Consider the O_2 molecule as a rigid rotor, that is, two atoms separated by a fixed distance d and rotating about the center of mass (Fig. 11.21). Equating the angular momentum to the fundamental unit $\hbar$, we can estimate the lowest angular speed:

$$I_{\text{CM}}\omega \approx \hbar \qquad \text{or} \qquad \omega \approx \frac{\hbar}{I_{\text{CM}}}$$

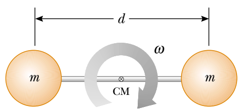


Figure 11.21 The rigid-rotor model of a diatomic molecule. The rotation occurs about the center of mass in the plane of the page.

In Example 10.3, we found that the moment of inertia of the O_2 molecule about this axis of rotation is $1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2$. Therefore,

$$\omega \approx \frac{\hbar}{I_{\text{CM}}} = \frac{1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2} = 5.41 \times 10^{11}\ \text{rad/s}$$

Actual angular speeds are multiples of this smallest possible value.

This simple example shows that certain classical concepts and models, when properly modified, might be useful in describing some features of atomic and molecular systems. A wide variety of phenomena on the submicroscopic scale can be explained only if we assume discrete values of the angular momentum associated with a particular type of motion.

The Danish physicist Niels Bohr (1885–1962) accepted and adopted this radical idea of discrete angular momentum values in developing his theory of the hydrogen atom. Strictly classical models were unsuccessful in describing many properties of the hydrogen atom. Bohr postulated that the electron could occupy only those circular orbits about the proton for which the orbital angular momentum was equal to $n\hbar$, where n is an integer. That is, he made the bold assumption that orbital angular momentum is quantized. From this simple model, the rotational frequencies of the electron in the various orbits can be estimated (see Problem 43).

SUMMARY

The **total kinetic energy** of a rigid object rolling on a rough surface without slipping equals the rotational kinetic energy about its center of mass, $\frac{1}{2}I_{\text{CM}}\omega^2$, plus the translational kinetic energy of the center of mass, $\frac{1}{2}Mv_{\text{CM}}{}^2$:

$$K = \tfrac{1}{2}I_{\text{CM}}\omega^2 + \tfrac{1}{2}Mv_{\text{CM}}{}^2 \tag{11.4}$$

The **torque** $\boldsymbol{\tau}$ due to a force $\mathbf{F}$ about an origin in an inertial frame is defined to be

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \tag{11.7}$$

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, the **cross product** $\mathbf{A} \times \mathbf{B}$ is a vector $\mathbf{C}$ having a magnitude

$$C \equiv AB\sin\theta \tag{11.9}$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. The direction of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$, and this direction is determined by the right-hand rule.

The **angular momentum L** of a particle having linear momentum $\mathbf{p} = m\mathbf{v}$ is

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{11.15}$$

where $\mathbf{r}$ is the vector position of the particle relative to an origin in an inertial frame.

The **net external torque** acting on a particle or rigid object is equal to the time rate of change of its angular momentum:

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt} \tag{11.20}$$

The *z component* of **angular momentum** of a rigid object rotating about a fixed z axis is

$$L_z = I\omega \tag{11.21}$$

where I is the moment of inertia of the object about the axis of rotation and ω is its angular speed.

The **net external torque** acting on a rigid object equals the product of its moment of inertia about the axis of rotation and its angular acceleration:

$$\sum \tau_{\text{ext}} = I\alpha \qquad \textbf{(11.23)}$$

If the net external torque acting on a system is zero, then the total angular momentum of the system is constant. Applying this **law of conservation of angular momentum** to a system whose moment of inertia changes gives

$$I_i\omega_i = I_f\omega_f = \text{constant} \qquad \textbf{(11.27)}$$

QUESTIONS

1. Is it possible to calculate the torque acting on a rigid body without specifying a center of rotation? Is the torque independent of the location of the center of rotation?
2. Is the triple product defined by $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ a scalar or a vector quantity? Explain why the operation $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ has no meaning.
3. In some motorcycle races, the riders drive over small hills, and the motorcycles become airborne for a short time. If a motorcycle racer keeps the throttle open while leaving the hill and going into the air, the motorcycle tends to nose upward. Why does this happen?
4. If the torque acting on a particle about a certain origin is zero, what can you say about its angular momentum about that origin?
5. Suppose that the velocity vector of a particle is completely specified. What can you conclude about the direction of its angular momentum vector with respect to the direction of motion?
6. If a single force acts on an object, and the torque caused by that force is nonzero about some point, is there any other point about which the torque is zero?
7. If a system of particles is in motion, is it possible for the total angular momentum to be zero about some origin? Explain.
8. A ball is thrown in such a way that it does not spin about its own axis. Does this mean that the angular momentum is zero about an arbitrary origin? Explain.
9. In a tape recorder, the tape is pulled past the read-and-write heads at a constant speed by the drive mechanism. Consider the reel from which the tape is pulled—as the tape is pulled off it, the radius of the roll of remaining tape decreases. How does the torque on the reel change with time? How does the angular speed of the reel change with time? If the tape mechanism is suddenly turned on so that the tape is quickly pulled with a great force, is the tape more likely to break when pulled from a nearly full reel or a nearly empty reel?
10. A scientist at a hotel sought assistance from a bellhop to carry a mysterious suitcase. When the unaware bellhop rounded a corner carrying the suitcase, it suddenly moved away from him for some unknown reason. At this point, the alarmed bellhop dropped the suitcase and ran off. What do you suppose might have been in the suitcase?
11. When a cylinder rolls on a horizontal surface as in Figure 11.3, do any points on the cylinder have only a vertical component of velocity at some instant? If so, where are they?
12. Three objects of uniform density—a solid sphere, a solid cylinder, and a hollow cylinder—are placed at the top of an incline (Fig. Q11.12). If they all are released from rest at the same elevation and roll without slipping, which object reaches the bottom first? Which reaches it last? You should try this at home and note that the result is independent of the masses and the radii of the objects.

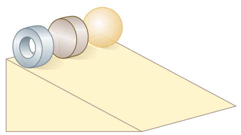


Figure Q11.12 Which object wins the race?

13. A mouse is initially at rest on a horizontal turntable mounted on a frictionless vertical axle. If the mouse begins to walk around the perimeter, what happens to the turntable? Explain.
14. Stars originate as large bodies of slowly rotating gas. Because of gravity, these regions of gas slowly decrease in size. What happens to the angular speed of a star as it shrinks? Explain.
15. Often, when a high diver wants to execute a flip in midair, she draws her legs up against her chest. Why does this make her rotate faster? What should she do when she wants to come out of her flip?
16. As a tether ball winds around a thin pole, what happens to its angular speed? Explain.

17. Two solid spheres—a large, massive sphere and a small sphere with low mass—are rolled down a hill. Which sphere reaches the bottom of the hill first? Next, a large, low-density sphere and a small, high-density sphere having the same mass are rolled down the hill. Which one reaches the bottom first in this case?
18. Suppose you are designing a car for a coasting race—the cars in this race have no engines; they simply coast down a hill. Do you want to use large wheels or small wheels? Do you want to use solid, disk-like wheels or hoop-like wheels? Should the wheels be heavy or light?
19. Why do tightrope walkers carry a long pole to help themselves keep their balance?
20. Two balls have the same size and mass. One is hollow, whereas the other is solid. How would you determine which is which without breaking them apart?
21. A particle is moving in a circle with constant speed. Locate one point about which the particle's angular momentum is constant and another about which it changes with time.
22. If global warming occurs over the next century, it is likely that some polar ice will melt and the water will be distributed closer to the equator. How would this change the moment of inertia of the Earth? Would the length of the day (one revolution) increase or decrease?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging   ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/**   = Computer useful in solving problem   = Interactive Physics
☐ = paired numerical/symbolic problems

Section 11.1 Rolling Motion of a Rigid Object

WEB 1. A cylinder of mass 10.0 kg rolls without slipping on a horizontal surface. At the instant its center of mass has a speed of 10.0 m/s, determine (a) the translational kinetic energy of its center of mass, (b) the rotational energy about its center of mass, and (c) its total energy.
2. A bowling ball has a mass of 4.00 kg, a moment of inertia of 1.60×10^{-2} kg·m², and a radius of 0.100 m. If it rolls down the lane without slipping at a linear speed of 4.00 m/s, what is its total energy?
3. A bowling ball has a mass M, a radius R, and a moment of inertia $\frac{2}{5}MR^2$. If it starts from rest, how much work must be done on it to set it rolling without slipping at a linear speed v? Express the work in terms of M and v.
4. A uniform solid disk and a uniform hoop are placed side by side at the top of an incline of height h. If they are released from rest and roll without slipping, determine their speeds when they reach the bottom. Which object reaches the bottom first?
5. (a) Determine the acceleration of the center of mass of a uniform solid disk rolling down an incline making angle θ with the horizontal. Compare this acceleration with that of a uniform hoop. (b) What is the minimum coefficient of friction required to maintain pure rolling motion for the disk?
6. A ring of mass 2.40 kg, inner radius 6.00 cm, and outer radius 8.00 cm rolls (without slipping) up an inclined plane that makes an angle of $\theta = 36.9°$ (Fig. P11.6). At the moment the ring is at position $x = 2.00$ m up the plane, its speed is 2.80 m/s. The ring continues up the plane for some additional distance and then rolls back down. It does not roll off the top end. How far up the plane does it go?

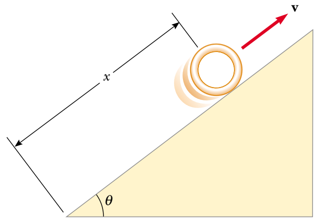


Figure P11.6

7. A metal can containing condensed mushroom soup has a mass of 215 g, a height of 10.8 cm, and a diameter of 6.38 cm. It is placed at rest on its side at the top of a 3.00-m-long incline that is at an angle of 25.0° to the horizontal and is then released to roll straight down. Assuming energy conservation, calculate the moment of inertia of the can if it takes 1.50 s to reach the bottom of the incline. Which pieces of data, if any, are unnecessary for calculating the solution?
8. A tennis ball is a hollow sphere with a thin wall. It is set rolling without slipping at 4.03 m/s on the horizontal section of a track, as shown in Figure P11.8. It rolls around the inside of a vertical circular loop 90.0 cm in diameter and finally leaves the track at a point 20.0 cm below the horizontal section. (a) Find the speed of the ball at the top of the loop. Demonstrate that it will not fall from the track. (b) Find its speed as it leaves the track. (c) Suppose that static friction between the ball and the track was negligible, so that the ball slid instead of rolling. Would its speed

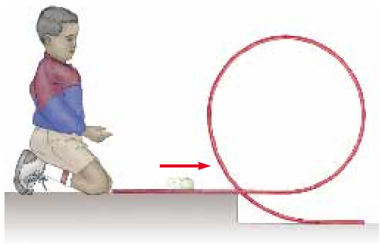


Figure P11.8

then be higher, lower, or the same at the top of the loop? Explain.

Section 11.2 The Vector Product and Torque

9. Given $\mathbf{M} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{N} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, calculate the vector product $\mathbf{M} \times \mathbf{N}$.

10. The vectors 42.0 cm at 15.0° and 23.0 cm at 65.0° both start from the origin. Both angles are measured counterclockwise from the *x* axis. The vectors form two sides of a parallelogram. (a) Find the area of the parallelogram. (b) Find the length of its longer diagonal.

WEB **11.** Two vectors are given by $\mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between **A** and **B**.

12. For the vectors $\mathbf{A} = -3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 10\mathbf{j} + 9\mathbf{k}$, evaluate the expressions (a) $\cos^{-1}(\mathbf{A}\cdot\mathbf{B}/AB)$ and (b) $\sin^{-1}(|\mathbf{A} \times \mathbf{B}|/AB)$. (c) Which give(s) the angle between the vectors?

13. A force of $\mathbf{F} = 2.00\mathbf{i} + 3.00\mathbf{j}$ N is applied to an object that is pivoted about a fixed axis aligned along the *z* coordinate axis. If the force is applied at the point $\mathbf{r} = (4.00\mathbf{i} + 5.00\mathbf{j} + 0\mathbf{k})$ m, find (a) the magnitude of the net torque about the *z* axis and (b) the direction of the torque vector $\boldsymbol{\tau}$.

14. A student claims that she has found a vector **A** such that $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{A} = (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$. Do you believe this claim? Explain.

15. Vector **A** is in the negative *y* direction, and vector **B** is in the negative *x* direction. What are the directions of (a) $\mathbf{A} \times \mathbf{B}$ and (b) $\mathbf{B} \times \mathbf{A}$?

16. A particle is located at the vector position $\mathbf{r} = (\mathbf{i} + 3\mathbf{j})$ m, and the force acting on it is $\mathbf{F} = (3\mathbf{i} + 2\mathbf{j})$ N. What is the torque about (a) the origin and (b) the point having coordinates (0, 6) m?

17. If $|\mathbf{A} \times \mathbf{B}| = \mathbf{A}\cdot\mathbf{B}$, what is the angle between **A** and **B**?

18. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act along the two sides of an equilateral triangle, as shown in Figure P11.18. Point *O* is the intersection of the altitudes of the triangle. Find a third force $\mathbf{F}_3$ to be applied at *B* and along *BC* that will make the total torque about the point *O* be zero. Will the total torque change if $\mathbf{F}_3$ is applied not at *B*, but rather at any other point along *BC*?

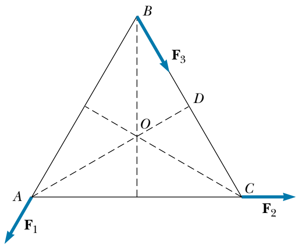


Figure P11.18

Section 11.3 Angular Momentum of a Particle

19. A light, rigid rod 1.00 m in length joins two particles—with masses 4.00 kg and 3.00 kg—at its ends. The combination rotates in the *xy* plane about a pivot through the center of the rod (Fig. P11.19). Determine the angular momentum of the system about the origin when the speed of each particle is 5.00 m/s.

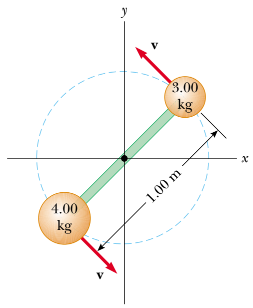


Figure P11.19

20. A 1.50-kg particle moves in the *xy* plane with a velocity of $\mathbf{v} = (4.20\mathbf{i} - 3.60\mathbf{j})$ m/s. Determine the particle's angular momentum when its position vector is $\mathbf{r} = (1.50\mathbf{i} + 2.20\mathbf{j})$ m.

WEB **21.** The position vector of a particle of mass 2.00 kg is given as a function of time by $\mathbf{r} = (6.00\mathbf{i} + 5.00t\mathbf{j})$ m. Determine the angular momentum of the particle about the origin as a function of time.

22. A conical pendulum consists of a bob of mass *m* in motion in a circular path in a horizontal plane, as shown in Figure P11.22. During the motion, the supporting wire of length ℓ maintains the constant angle θ with the vertical. Show that the magnitude of the angular momentum of the mass about the center of the circle is

$$L = (m^2 g \ell^3 \sin^4\theta/\cos\theta)^{1/2}$$

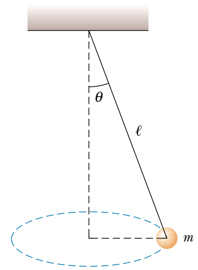


Figure P11.22

23. A particle of mass *m* moves in a circle of radius *R* at a constant speed *v*, as shown in Figure P11.23. If the motion begins at point *Q*, determine the angular momentum of the particle about point *P* as a function of time.

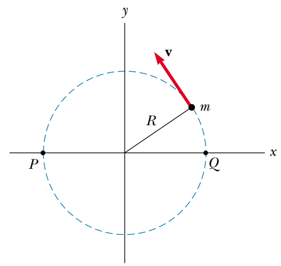


Figure P11.23

24. A 4.00-kg mass is attached to a light cord that is wound around a pulley (see Fig. 10.20). The pulley is a uniform solid cylinder with a radius of 8.00 cm and a mass of 2.00 kg. (a) What is the net torque on the system about the point *O*? (b) When the mass has a speed *v*, the pulley has an angular speed $\omega = v/R$. Determine the total angular momentum of the system about *O*. (c) Using the fact that $\boldsymbol{\tau} = d\mathbf{L}/dt$ and your result from part (b), calculate the acceleration of the mass.

25. A particle of mass *m* is shot with an initial velocity $\mathbf{v}_i$ and makes an angle θ with the horizontal, as shown in Figure P11.25. The particle moves in the gravitational field of the Earth. Find the angular momentum of the particle about the origin when the particle is (a) at the origin, (b) at the highest point of its trajectory, and (c) just about to hit the ground. (d) What torque causes its angular momentum to change?

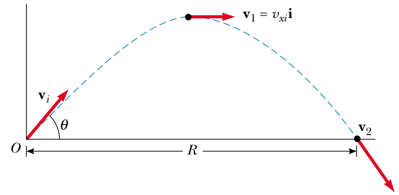


Figure P11.25

26. Heading straight toward the summit of Pike's Peak, an airplane of mass 12 000 kg flies over the plains of Kansas at a nearly constant altitude of 4.30 km and with a constant velocity of 175 m/s westward. (a) What is the airplane's vector angular momentum relative to a wheat farmer on the ground directly below the airplane? (b) Does this value change as the airplane continues its motion along a straight line? (c) What is its angular momentum relative to the summit of Pike's Peak?

27. A ball of mass *m* is fastened at the end of a flagpole connected to the side of a tall building at point *P*, as shown in Figure P11.27. The length of the flagpole is ℓ, and θ is the angle the flagpole makes with the horizontal. Suppose that the ball becomes loose and starts to fall. Determine the angular momentum (as a function of time) of the ball about point *P*. Neglect air resistance.

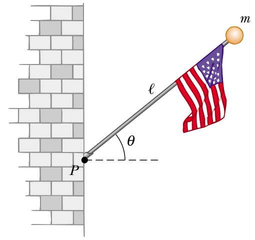


Figure P11.27

28. A fireman clings to a vertical ladder and directs the nozzle of a hose horizontally toward a burning building. The rate of water flow is 6.31 kg/s, and the nozzle speed is 12.5 m/s. The hose passes between the fireman's feet, which are 1.30 m vertically below the nozzle. Choose the origin to be inside the hose between the fireman's feet. What torque must the fireman exert on the hose? That is, what is the rate of change of angular momentum of the water?

Section 11.4 Angular Momentum of a Rotating Rigid Object

29. A uniform solid sphere with a radius of 0.500 m and a mass of 15.0 kg turns counterclockwise about a vertical axis through its center. Find its vector angular momentum when its angular speed is 3.00 rad/s.

30. A uniform solid disk with a mass of 3.00 kg and a radius of 0.200 m rotates about a fixed axis perpendicular to its face. If the angular speed is 6.00 rad/s, calculate the angular momentum of the disk when the axis of rotation (a) passes through its center of mass and (b) passes through a point midway between the center and the rim.

31. A particle with a mass of 0.400 kg is attached to the 100-cm mark of a meter stick with a mass of 0.100 kg. The meter stick rotates on a horizontal, frictionless table with an angular speed of 4.00 rad/s. Calculate the angular momentum of the system when the stick is pivoted about an axis (a) perpendicular to the table through the 50.0-cm mark and (b) perpendicular to the table through the 0-cm mark.

32. The hour and minute hands of Big Ben, the famous Parliament Building tower clock in London, are 2.70 m and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively. Calculate the total angular momentum of these hands about the center point. Treat the hands as long thin rods.

Section 11.5 Conservation of Angular Momentum

33. A cylinder with a moment of inertia of I_1 rotates about a vertical, frictionless axle with angular velocity ω_i. A second cylinder that has a moment of inertia of I_2 and initially is not rotating drops onto the first cylinder (Fig. P11.33). Because of friction between the surfaces, the two eventually reach the same angular speed ω_f. (a) Calculate ω_f. (b) Show that the kinetic energy of the system decreases in this interaction and calculate the ratio of the final rotational energy to the initial rotational energy.

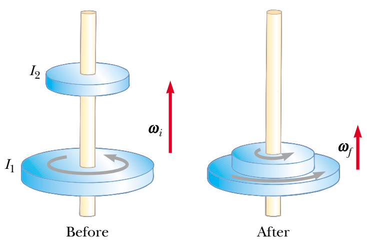


Figure P11.33

34. A playground merry-go-round of radius $R = 2.00$ m has a moment of inertia of $I = 250\ \text{kg}\cdot\text{m}^2$ and is rotating at 10.0 rev/min about a frictionless vertical axle. Facing the axle, a 25.0-kg child hops onto the merry-go-round and manages to sit down on its edge. What is the new angular speed of the merry-go-round?

35. A student sits on a freely rotating stool holding two weights, each of which has a mass of 3.00 kg. When his arms are extended horizontally, the weights are 1.00 m from the axis of rotation and he rotates with an angular speed of 0.750 rad/s. The moment of inertia of the student plus stool is $3.00\ \text{kg}\cdot\text{m}^2$ and is assumed to be constant. The student pulls the weights inward horizontally to a position 0.300 m from the rotation axis. (a) Find the new angular speed of the student. (b) Find the kinetic energy of the rotating system before and after he pulls the weights inward.

36. A uniform rod with a mass of 100 g and a length of 50.0 cm rotates in a horizontal plane about a fixed, vertical, frictionless pin passing through its center. Two small beads, each having a mass 30.0 g, are mounted on the rod so that they are able to slide without friction along its length. Initially, the beads are held by catches at positions 10.0 cm on each side of center; at this time, the system rotates at an angular speed of 20.0 rad/s. Suddenly, the catches are released, and the small beads slide outward along the rod. Find (a) the angular speed of the system at the instant the beads reach the ends of the rod and (b) the angular speed of the rod after the beads fly off the rod's ends.

WEB **37.** A 60.0-kg woman stands at the rim of a horizontal turntable having a moment of inertia of $500\ \text{kg}\cdot\text{m}^2$ and a radius of 2.00 m. The turntable is initially at rest and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim clockwise (as viewed from above the system) at a constant speed of 1.50 m/s relative to the Earth. (a) In what direction and with what angular speed does the turntable rotate? (b) How much work does the woman do to set herself and the turntable into motion?

38. A puck with a mass of 80.0 g and a radius of 4.00 cm slides along an air table at a speed of 1.50 m/s, as shown in Figure P11.38a. It makes a glancing collision with a second puck having a radius of 6.00 cm and a mass of 120 g (initially at rest) such that their rims just touch. Because their rims are coated with instant-acting glue, the pucks stick together and spin after the collision (Fig. P11.38b). (a) What is the angular momentum of the system relative to the center of mass? (b) What is the angular speed about the center of mass?

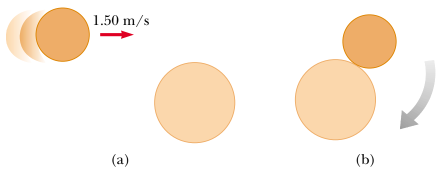


Figure P11.38

39. A wooden block of mass M resting on a frictionless horizontal surface is attached to a rigid rod of length ℓ and of negligible mass (Fig. P11.39). The rod is pivoted at the other end. A bullet of mass m traveling parallel to the horizontal surface and normal to the rod with speed v hits the block and becomes embedded in it. (a) What is the angular momentum of the bullet–block system? (b) What fraction of the original kinetic energy is lost in the collision?

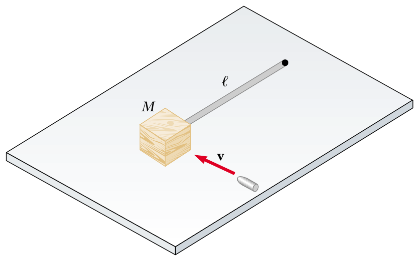


Figure P11.39

40. A space station shaped like a giant wheel has a radius of 100 m and a moment of inertia of $5.00 \times 10^8\ \text{kg}\cdot\text{m}^2$. A crew of 150 are living on the rim, and the station's rotation causes the crew to experience an acceleration of $1g$ (Fig. P11.40). When 100 people move to the center of the station for a union meeting, the angular speed changes. What acceleration is experienced by the managers remaining at the rim? Assume that the average mass of each inhabitant is 65.0 kg.

41. A wad of sticky clay of mass m and velocity $\mathbf{v}_i$ is fired at a solid cylinder of mass M and radius R (Fig. P11.41). The cylinder is initially at rest and is mounted on a fixed horizontal axle that runs through the center of mass. The line of motion of the projectile is perpendicular to the axle and at a distance d, less than R, from the center. (a) Find the angular speed of the system just after the clay strikes and sticks to the surface of the cylinder. (b) Is mechanical energy conserved in this process? Explain your answer.

42. Suppose a meteor with a mass of 3.00×10^{13} kg is moving at 30.0 km/s relative to the center of the Earth and strikes the Earth. What is the order of magnitude of the maximum possible decrease in the angular speed of the Earth due to this collision? Explain your answer.

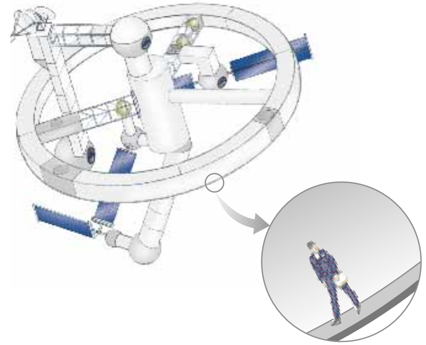


Figure P11.40

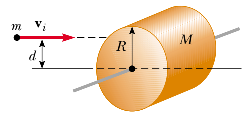


Figure P11.41

(Optional)
Section 11.7 Angular Momentum as a Fundamental Quantity

43. In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 0.529×10^{-10} m around the proton. Assuming that the orbital angular momentum of the electron is equal to $h/2\pi$, calculate (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the angular speed of the electron's motion.

ADDITIONAL PROBLEMS

44. **Review Problem.** A rigid, massless rod has three equal masses attached to it, as shown in Figure P11.44. The rod is free to rotate in a vertical plane about a frictionless axle perpendicular to the rod through the point P, and it is released from rest in the horizontal position at $t = 0$. Assuming m and d are known, find (a) the moment of inertia of the system about the pivot, (b) the torque acting on the system at $t = 0$, (c) the angular acceleration of the system at $t = 0$, (d) the linear acceleration of the mass labeled "3" at $t = 0$, (e) the maximum kinetic energy of the system, (f) the maximum angular speed attained by the rod, (g) the maximum angular momentum of the system, and (h) the maximum speed attained by the mass labeled "2."

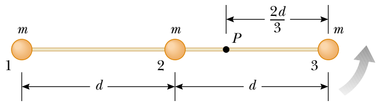


Figure P11.44

45. A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl having a much greater radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping (Fig. P11.45). Determine the angular speed of the sphere when it reaches the bottom of the bowl.

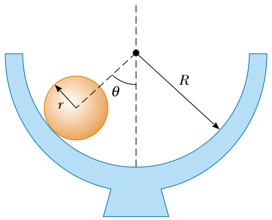


Figure P11.45

46. A 100-kg uniform horizontal disk of radius 5.50 m turns without friction at 2.50 rev/s on a vertical axis through its center, as shown in Figure P11.46. A feedback mechanism senses the angular speed of the disk, and a drive motor at A ensures that the angular speed remains constant. While the disk turns, a 1.20-kg mass on top of the disk slides outward in a radial slot. The 1.20-kg mass starts at the center of the disk at time $t = 0$ and moves outward with a constant speed of 1.25 cm/s relative to the disk until it reaches the edge at $t = 440$ s. The sliding mass experiences no friction. Its motion is constrained by a brake at B so that its radial speed remains constant. The constraint produces tension in a light string tied to the mass. (a) Find the torque as a function of time that the drive motor must provide while the mass is sliding. (b) Find the value of this torque at $t = 440$ s, just before the sliding mass finishes its motion. (c) Find the power that the drive motor must deliver as a function of time. (d) Find the value of the power when the sliding mass is just reaching the end of the slot. (e) Find the string tension as a function of time. (f) Find the work done by the drive motor during the 440-s motion. (g) Find the work done by the string brake on the sliding mass. (h) Find the total work done on the system consisting of the disk and the sliding mass.

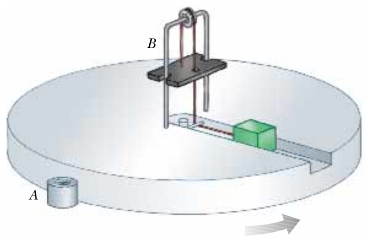


Figure P11.46

47. A string is wound around a uniform disk of radius R and mass M. The disk is released from rest when the string is vertical and its top end is tied to a fixed bar (Fig. P11.47). Show that (a) the tension in the string is one-third the weight of the disk, (b) the magnitude of the acceleration of the center of mass is $2g/3$, and (c) the speed of the center of mass is $(4gh/3)^{1/2}$ as the disk descends. Verify your answer to part (c) using the energy approach.

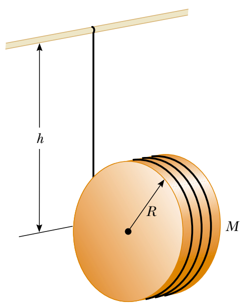


Figure P11.47

48. Comet Halley moves about the Sun in an elliptical orbit, with its closest approach to the Sun being about 0.590 AU and its greatest distance from the Sun being 35.0 AU (1 AU = the average Earth–Sun distance). If the comet's speed at its closest approach is 54.0 km/s, what is its speed when it is farthest from the Sun? The angular momentum of the comet about the Sun is conserved because no torque acts on the comet. The gravitational force exerted by the Sun on the comet has a moment arm of zero.

49. A constant horizontal force **F** is applied to a lawn roller having the form of a uniform solid cylinder of radius R and mass M (Fig. P11.49). If the roller rolls without slipping on the horizontal surface, show that (a) the acceleration of the center of mass is $2\mathbf{F}/3M$ and that (b) the minimum coefficient of friction necessary to prevent slipping is $F/3Mg$. (*Hint:* Consider the torque with respect to the center of mass.)

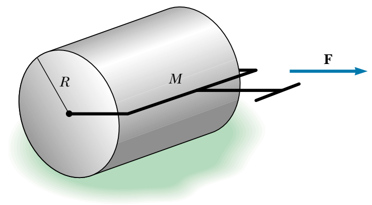


Figure P11.49

50. A light rope passes over a light, frictionless pulley. A bunch of bananas of mass M is fastened at one end, and a monkey of mass M clings to the other (Fig. P11.50).

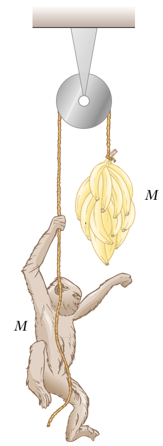


Figure P11.50

The monkey climbs the rope in an attempt to reach the bananas. (a) Treating the system as consisting of the monkey, bananas, rope, and pulley, evaluate the net torque about the pulley axis. (b) Using the results to part (a), determine the total angular momentum about the pulley axis and describe the motion of the system. Will the monkey reach the bananas?

51. A solid sphere of mass m and radius r rolls without slipping along the track shown in Figure P11.51. The sphere starts from rest with its lowest point at height h above the bottom of a loop of radius R, which is much larger than r. (a) What is the minimum value that h can have (in terms of R) if the sphere is to complete the loop? (b) What are the force components on the sphere at point P if $h = 3R$?

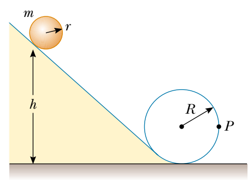


Figure P11.51

52. A thin rod with a mass of 0.630 kg and a length of 1.24 m is at rest, hanging vertically from a strong fixed hinge at its top end. Suddenly, a horizontal impulsive force (14.7**i**) N is applied to it. (a) Suppose that the force acts at the bottom end of the rod. Find the acceleration of the rod's center of mass and the horizontal force that the hinge exerts. (b) Suppose that the force acts at the midpoint of the rod. Find the acceleration of this point and the horizontal hinge reaction. (c) Where can the impulse be applied so that the hinge exerts no horizontal force? (This point is called the *center of percussion.*)

53. At one moment, a bowling ball is both sliding and spinning on a horizontal surface such that its rotational kinetic energy equals its translational kinetic energy. Let v_{CM} represent the ball's center-of-mass speed relative to the surface. Let v_r represent the speed of the topmost point on the ball's surface relative to the center of mass. Find the ratio v_{CM}/v_r.

54. A projectile of mass m moves to the right with speed v_i (Fig. P11.54a). The projectile strikes and sticks to the end of a stationary rod of mass M and length d that is pivoted about a frictionless axle through its center (Fig. P11.54b). (a) Find the angular speed of the system right after the collision. (b) Determine the fractional loss in mechanical energy due to the collision.

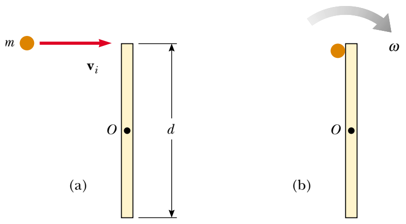


Figure P11.54

55. A mass m is attached to a cord passing through a small hole in a frictionless, horizontal surface (Fig. P11.55). The mass is initially orbiting with speed v_i in a circle of radius r_i. The cord is then slowly pulled from below, and the radius of the circle decreases to r. (a) What is the speed of the mass when the radius is r? (b) Find the tension in the cord as a function of r. (c) How much work W is done in moving m from r_i to r? (*Note:* The tension depends on r.) (d) Obtain numerical values for v, T, and W when $r = 0.100$ m, $m = 50.0$ g, $r_i = 0.300$ m, and $v_i = 1.50$ m/s.

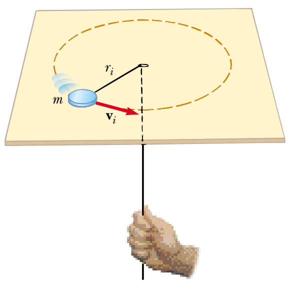


Figure P11.55

56. A bowler releases a bowling ball with no spin, sending it sliding straight down the alley toward the pins. The ball continues to slide for some distance before its motion becomes rolling without slipping; of what order of magnitude is this distance? State the quantities you take as data, the values you measure or estimate for them, and your reasoning.

57. Following Thanksgiving dinner, your uncle falls into a deep sleep while sitting straight up and facing the television set. A naughty grandchild balances a small spherical grape at the top of his bald head, which itself has the shape of a sphere. After all of the children have had time to giggle, the grape starts from rest and rolls down your uncle's head without slipping. It loses contact with your uncle's scalp when the radial line joining it to the center of curvature makes an angle θ with the vertical. What is the measure of angle θ?

58. A thin rod of length h and mass M is held vertically with its lower end resting on a frictionless horizontal surface. The rod is then released to fall freely. (a) Determine the speed of its center of mass just before it hits the horizontal surface. (b) Now suppose that the rod has a fixed pivot at its lower end. Determine the speed of the rod's center of mass just before it hits the surface.

WEB 59. Two astronauts (Fig. P11.59), each having a mass of 75.0 kg, are connected by a 10.0-m rope of negligible mass. They are isolated in space, orbiting their center of mass at speeds of 5.00 m/s. (a) Treating the astronauts as particles, calculate the magnitude of the angular momentum and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to 5.00 m. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronaut in shortening the rope?

60. Two astronauts (see Fig. P11.59), each having a mass M, are connected by a rope of length d having negligible mass. They are isolated in space, orbiting their center of mass at speeds v. Treating the astronauts as particles, calculate (a) the magnitude of the angular momentum and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to $d/2$. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronaut in shortening the rope?

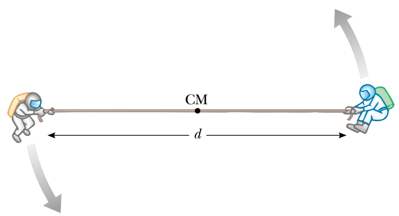


Figure P11.59 Problems 59 and 60.

61. A solid cube of wood of side $2a$ and mass M is resting on a horizontal surface. The cube is constrained to rotate about an axis AB (Fig. P11.61). A bullet of mass m and speed v is shot at the face opposite $ABCD$ at a height of $4a/3$. The bullet becomes embedded in the cube. Find the minimum value of v required to tip the cube so that it falls on face $ABCD$. Assume $m \ll M$.

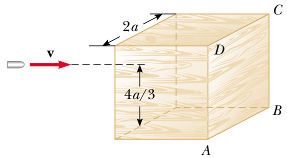


Figure P11.61

62. A large, cylindrical roll of paper of initial radius R lies on a long, horizontal surface with the open end of the paper nailed to the surface. The roll is given a slight shove ($v_i \approx 0$) and begins to unroll. (a) Determine the speed of the center of mass of the roll when its radius has diminished to r. (b) Calculate a numerical value for this speed at $r = 1.00$ mm, assuming $R = 6.00$ m. (c) What happens to the energy of the system when the paper is completely unrolled? (*Hint:* Assume that the roll has a uniform density and apply energy methods.)

63. A spool of wire of mass M and radius R is unwound under a constant force $\mathbf{F}$ (Fig. P11.63). Assuming that the spool is a uniform solid cylinder that does not slip, show that (a) the acceleration of the center of mass is $4\mathbf{F}/3M$ and that (b) the force of friction is to the *right* and is equal in magnitude to $F/3$. (c) If the cylinder starts from rest and rolls without slipping, what is the speed of its center of mass after it has rolled through a distance d?

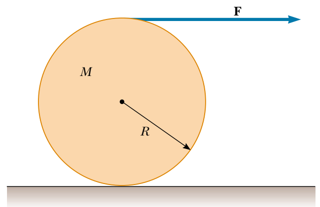


Figure P11.63

64. A uniform solid disk is set into rotation with an angular speed ω_i about an axis through its center. While still rotating at this speed, the disk is placed into contact with a horizontal surface and released, as shown in Figure P11.64. (a) What is the angular speed of the disk once pure rolling takes place? (b) Find the fractional loss in kinetic energy from the time the disk is released until the time pure rolling occurs. (*Hint:* Consider torques about the center of mass.)

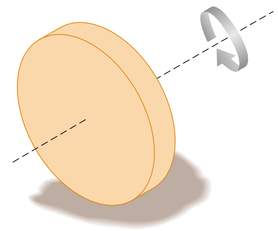


Figure P11.64 Problems 64 and 65.

65. Suppose a solid disk of radius R is given an angular speed ω_i about an axis through its center and is then lowered to a horizontal surface and released, as shown in Problem 64 (see Fig. P11.64). Furthermore, assume that the coefficient of friction between the disk and the surface is μ. (a) Show that the time it takes for pure rolling motion to occur is $R\omega_i/3\mu g$. (b) Show that the distance the disk travels before pure rolling occurs is $R^2\omega_i^2/18\mu g$.

66. A solid cube of side $2a$ and mass M is sliding on a frictionless surface with uniform velocity $\mathbf{v}$, as shown in Figure P11.66a. It hits a small obstacle at the end of the table; this causes the cube to tilt, as shown in Figure

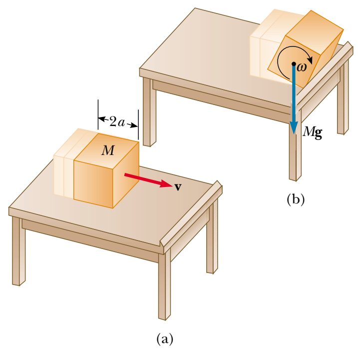


Figure P11.66

P11.66b. Find the minimum value of **v** such that the cube falls off the table. Note that the moment of inertia of the cube about an axis along one of its edges is $8Ma^2/3$. (*Hint:* The cube undergoes an inelastic collision at the edge.)

67. A plank with a mass $M = 6.00$ kg rides on top of two identical solid cylindrical rollers that have $R = 5.00$ cm and $m = 2.00$ kg (Fig. P11.67). The plank is pulled by a constant horizontal force of magnitude $F = 6.00$ N applied to the end of the plank and perpendicular to the axes of the cylinders (which are parallel). The cylinders roll without slipping on a flat surface. Also, no slipping occurs between the cylinders and the plank. (a) Find the acceleration of the plank and that of the rollers. (b) What frictional forces are acting?

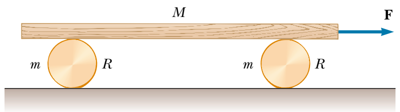


Figure P11.67

68. A spool of wire rests on a horizontal surface as in Figure P11.68. As the wire is pulled, the spool does not slip at the contact point P. On separate trials, each one of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ is applied to the spool. For each one of these forces, determine the direction in which the spool will roll. Note that the line of action of $\mathbf{F}_2$ passes through P.

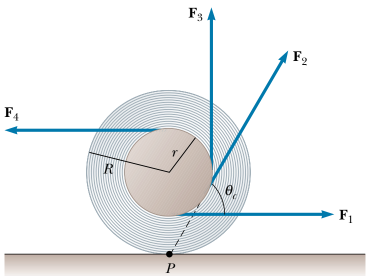


Figure P11.68 Problems 68 and 69.

69. The spool of wire shown in Figure P11.68 has an inner radius r and an outer radius R. The angle θ between the applied force and the horizontal can be varied. Show that the critical angle for which the spool does not slip and remains stationary is

$$\cos\theta_c = \frac{r}{R}$$

(*Hint:* At the critical angle, the line of action of the applied force passes through the contact point.)

70. In a demonstration that employs a ballistics cart, a ball is projected vertically upward from a cart moving with constant velocity along the horizontal direction. The ball lands in the catching cup of the cart because both the cart and the ball have the same horizontal component of velocity. Now consider a ballistics cart on an incline making an angle θ with the horizontal, as shown in Figure P11.70. The cart (including its wheels) has a mass M, and the moment of inertia of each of the two wheels is $mR^2/2$. (a) Using conservation of energy considerations (assuming that there is no friction between the cart and the axles) and assuming pure rolling motion (that is, no slipping), show that the acceleration of the cart along the incline is

$$a_x = \left(\frac{M}{M + 2m}\right) g\sin\theta$$

(b) Note that the x component of acceleration of the ball released by the cart is $g\sin\theta$. Thus, the x component of the cart's acceleration is *smaller* than that of the ball by the factor $M/(M + 2m)$. Use this fact and kinematic equations to show that the ball overshoots the cart by an amount Δx, where

$$\Delta x = \left(\frac{4m}{M + 2m}\right)\left(\frac{\sin\theta}{\cos^2\theta}\right)\frac{{v_{yi}}^2}{g}$$

and v_{yi} is the initial speed of the ball imparted to it by the spring in the cart. (c) Show that the distance d that the ball travels measured along the incline is

$$d = \frac{2{v_{yi}}^2}{g}\,\frac{\sin\theta}{\cos^2\theta}$$

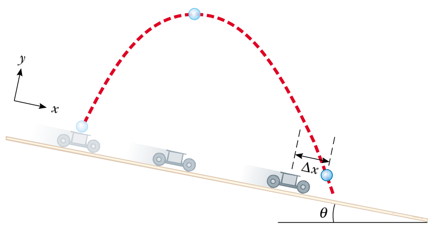


Figure P11.70

Answers to Quick Quizzes

11.1 There is very little resistance to motion that can reduce the kinetic energy of the rolling ball. Even though there is friction between the ball and the floor (if there were not, then no rotation would occur, and the ball would slide), there is no relative motion of the two surfaces (by the definition of "rolling"), and so kinetic friction cannot reduce K. (Air drag and friction associated with deformation of the ball eventually stop the ball.)

11.2 The box. Because none of the box's initial potential energy is converted to rotational kinetic energy, at any time $t > 0$ its translational kinetic energy is greater than that of the rolling ball.

11.3 Zero. If she were moving directly toward the pole, **r** and **p** would be antiparallel to each other, and the sine of the angle between them is zero; therefore, $L = 0$.

11.4 Both (a) and (b) are false. The net force is not necessarily zero. If the line of action of the net force passes through the point, then the net torque about an axis passing through that point is zero even though the net force is not zero. Because the net force is not necessarily zero, you cannot conclude that the particle's velocity is constant.

11.5 The student does work as he walks from the rim of the platform toward its center.

11.6 Because it is perpendicular to the precessional motion of the top, the force of gravity does no work. This is a situation in which a force causes motion but does no work.

PUZZLER

This one-bottle wine holder is an interesting example of a mechanical system that seems to defy gravity. The system (holder plus bottle) is balanced when its center of gravity is directly over the lowest support point. What two conditions are necessary for an object to exhibit this kind of stability? *(Charles D. Winters)*

chapter 12

Static Equilibrium and Elasticity

Chapter Outline

In Chapters 10 and 11 we studied the dynamics of rigid objects—that is, objects whose parts remain at a fixed separation with respect to each other when subjected to external forces. Part of this chapter addresses the conditions under which a rigid object is in equilibrium. The term *equilibrium* implies either that the object is at rest or that its center of mass moves with constant velocity. We deal here only with the former case, in which the object is described as being in *static equilibrium.* Static equilibrium represents a common situation in engineering practice, and the principles it involves are of special interest to civil engineers, architects, and mechanical engineers. If you are an engineering student you will undoubtedly take an advanced course in statics in the future.

The last section of this chapter deals with how objects deform under load conditions. Such deformations are usually elastic and do not affect the conditions for equilibrium. An *elastic* object returns to its original shape when the deforming forces are removed. Several elastic constants are defined, each corresponding to a different type of deformation.

12.1 THE CONDITIONS FOR EQUILIBRIUM

In Chapter 5 we stated that one necessary condition for equilibrium is that the net force acting on an object be zero. If the object is treated as a particle, then this is the only condition that must be satisfied for equilibrium. The situation with real (extended) objects is more complex, however, because these objects cannot be treated as particles. For an extended object to be in static equilibrium, a second condition must be satisfied. This second condition involves the net torque acting on the extended object. Note that equilibrium does not require the absence of motion. For example, a rotating object can have constant angular velocity and still be in equilibrium.

Consider a single force **F** acting on a rigid object, as shown in Figure 12.1. The effect of the force depends on its point of application P. If **r** is the position vector of this point relative to O, the torque associated with the force **F** about O is given by Equation 11.7:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

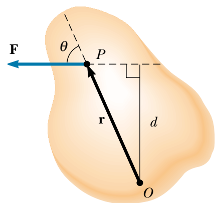


Figure 12.1 A single force **F** acts on a rigid object at the point P.

Recall from the discussion of the vector product in Section 11.2 that the vector $\boldsymbol{\tau}$ is perpendicular to the plane formed by **r** and **F**. You can use the right-hand rule to determine the direction of $\boldsymbol{\tau}$: Curl the fingers of your right hand in the direction of rotation that **F** tends to cause about an axis through O: your thumb then points in the direction of $\boldsymbol{\tau}$. Hence, in Figure 12.1 $\boldsymbol{\tau}$ is directed toward you out of the page.

As you can see from Figure 12.1, the tendency of **F** to rotate the object about an axis through O depends on the moment arm d, as well as on the magnitude of **F**. Recall that the magnitude of $\boldsymbol{\tau}$ is Fd (see Eq. 10.19). Now suppose a rigid object is acted on first by force $\mathbf{F}_1$ and later by force $\mathbf{F}_2$. If the two forces have the same magnitude, they will produce the same effect on the object only if they have the same direction and the same line of action. In other words,

Equivalent forces

two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are **equivalent** if and only if $F_1 = F_2$ and if and only if the two produce the same torque about any axis.

The two forces shown in Figure 12.2 are equal in magnitude and opposite in direction. They are *not* equivalent. The force directed to the right tends to rotate the object clockwise about an axis perpendicular to the diagram through O, whereas the force directed to the left tends to rotate it counterclockwise about that axis.

Suppose an object is pivoted about an axis through its center of mass, as shown in Figure 12.3. Two forces of equal magnitude act in opposite directions along parallel lines of action. A pair of forces acting in this manner form what is called a **couple.** (The two forces shown in Figure 12.2 also form a couple.) Do not make the mistake of thinking that the forces in a couple are a result of Newton's third law. They cannot be third-law forces because they act on the same object. Third-law force pairs act on different objects. Because each force produces the same torque Fd, the net torque has a magnitude of $2Fd$. Clearly, the object rotates clockwise and undergoes an angular acceleration about the axis. With respect to rotational motion, this is a nonequilibrium situation. The net torque on the object gives rise to an angular acceleration α according to the relationship $\sum\tau = 2Fd = I\alpha$ (see Eq. 10.21).

In general, an object is in rotational equilibrium only if its angular acceleration $\alpha = 0$. Because $\sum\tau = I\alpha$ for rotation about a fixed axis, our second necessary condition for equilibrium is that **the net torque about any axis must be zero.** We now have two necessary conditions for equilibrium of an object:

Conditions for equilibrium

1. The resultant external force must equal zero. $$\sum \mathbf{F} = 0 \qquad \textbf{(12.1)}$$

2. The resultant external torque about *any* axis must be zero. $$\sum \tau = 0 \qquad \textbf{(12.2)}$$

The first condition is a statement of translational equilibrium; it tells us that the linear acceleration of the center of mass of the object must be zero when viewed from an inertial reference frame. The second condition is a statement of rotational equilibrium and tells us that the angular acceleration about any axis must be zero. In the special case of **static equilibrium,** which is the main subject of this chapter, the object is at rest and so has no linear or angular speed (that is, $v_{\mathrm{CM}} = 0$ and $\omega = 0$).

Quick Quiz 12.1

(a) Is it possible for a situation to exist in which Equation 12.1 is satisfied while Equation 12.2 is not? (b) Can Equation 12.2 be satisfied while Equation 12.1 is not?

The two vector expressions given by Equations 12.1 and 12.2 are equivalent, in general, to six scalar equations: three from the first condition for equilibrium, and three from the second (corresponding to x, y, and z components). Hence, in a complex system involving several forces acting in various directions, you could be faced with solving a set of equations with many unknowns. Here, we restrict our discussion to situations in which all the forces lie in the xy plane. (Forces whose vector representations are in the same plane are said to be *coplanar.*) With this restriction, we must deal with only three scalar equations. Two of these come from balancing the forces in the x and y directions. The third comes from the torque equation—namely, that the net torque about *any* point in the xy plane must be zero. Hence, the two conditions of equilibrium provide the equations

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau_z = 0 \qquad \textbf{(12.3)}$$

where the axis of the torque equation is arbitrary, as we now show.

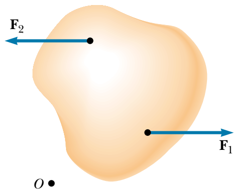


Figure 12.2 The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are not equivalent because they do not produce the same torque about some axis, even though they are equal in magnitude and opposite in direction.

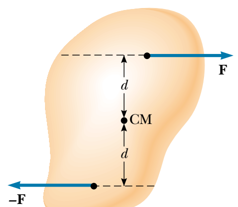


Figure 12.3 Two forces of equal magnitude form a couple if their lines of action are different parallel lines. In this case, the object rotates clockwise. The net torque about any axis is $2Fd$.

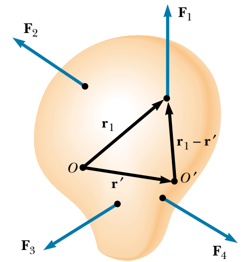


Figure 12.4 Construction showing that if the net torque is zero about origin O, it is also zero about any other origin, such as O'.

Regardless of the number of forces that are acting, if an object is in translational equilibrium and if the net torque is zero about one axis, then the net torque must also be zero about any other axis. The point can be inside or outside the boundaries of the object. Consider an object being acted on by several forces such that the resultant force $\Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = 0$. Figure 12.4 describes this situation (for clarity, only four forces are shown). The point of application of $\mathbf{F}_1$ relative to O is specified by the position vector $\mathbf{r}_1$. Similarly, the points of application of $\mathbf{F}_2, \mathbf{F}_3, \ldots$ are specified by $\mathbf{r}_2, \mathbf{r}_3, \ldots$ (not shown). The net torque about an axis through O is

$$\sum \tau_O = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \cdots$$

Now consider another arbitrary point O' having a position vector $\mathbf{r}'$ relative to O. The point of application of $\mathbf{F}_1$ relative to O' is identified by the vector $\mathbf{r}_1 - \mathbf{r}'$. Likewise, the point of application of $\mathbf{F}_2$ relative to O' is $\mathbf{r}_2 - \mathbf{r}'$, and so forth. Therefore, the torque about an axis through O' is

$$\begin{aligned}\sum \tau_{O'} &= (\mathbf{r}_1 - \mathbf{r}') \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}') \times \mathbf{F}_2 + (\mathbf{r}_3 - \mathbf{r}') \times \mathbf{F}_3 + \cdots \\ &= \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \cdots - \mathbf{r}' \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots)\end{aligned}$$

Because the net force is assumed to be zero (given that the object is in translational equilibrium), the last term vanishes, and we see that the torque about O' is equal to the torque about O. Hence, **if an object is in translational equilibrium and the net torque is zero about one point, then the net torque must be zero about any other point.**

12.2 MORE ON THE CENTER OF GRAVITY

We have seen that the point at which a force is applied can be critical in determining how an object responds to that force. For example, two equal-magnitude but oppositely directed forces result in equilibrium if they are applied at the same point on an object. However, if the point of application of one of the forces is moved, so that the two forces no longer act along the same line of action, then a force couple results and the object undergoes an angular acceleration. (This is the situation shown in Figure 12.3.)

Whenever we deal with a rigid object, one of the forces we must consider is the force of gravity acting on it, and we must know the point of application of this force. As we learned in Section 9.6, on every object is a special point called its center of gravity. All the various gravitational forces acting on all the various mass elements of the object are equivalent to a single gravitational force acting through this point. Thus, to compute the torque due to the gravitational force on an object of mass M, we need only consider the force $M\mathbf{g}$ acting at the center of gravity of the object.

How do we find this special point? As we mentioned in Section 9.6, if we assume that $\mathbf{g}$ is uniform over the object, then the center of gravity of the object coincides with its center of mass. To see that this is so, consider an object of arbitrary shape lying in the xy plane, as illustrated in Figure 12.5. Suppose the object is divided into a large number of particles of masses $m_1, m_2, m_3, \ldots$ having coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$. In

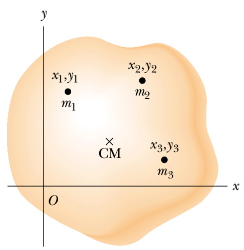


Figure 12.5 An object can be divided into many small particles each having a specific mass and specific coordinates. These particles can be used to locate the center of mass.

Equation 9.28 we defined the x coordinate of the center of mass of such an object to be

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_ix_i}{\sum_i m_i}$$

We use a similar equation to define the y coordinate of the center of mass, replacing each x with its y counterpart.

Let us now examine the situation from another point of view by considering the force of gravity exerted on each particle, as shown in Figure 12.6. Each particle contributes a torque about the origin equal in magnitude to the particle's weight mg multiplied by its moment arm. For example, the torque due to the force $m_1\mathbf{g}_1$ is $m_1g_1x_1$, where g_1 is the magnitude of the gravitational field at the position of the particle of mass m_1. We wish to locate the center of gravity, the point at which application of the single gravitational force $M\mathbf{g}$ (where $M = m_1 + m_2 + m_3 + \cdots$ is the total mass of the object) has the same effect on rotation as does the combined effect of all the individual gravitational forces $m_i\mathbf{g}_i$. Equating the torque resulting from $M\mathbf{g}$ acting at the center of gravity to the sum of the torques acting on the individual particles gives

$$(m_1g_1 + m_2g_2 + m_3g_3 + \cdots)x_{CG} = m_1g_1x_1 + m_2g_2x_2 + m_3g_3x_3 + \cdots$$

This expression accounts for the fact that the gravitational field strength g can in general vary over the object. If we assume uniform g over the object (as is usually the case), then the g terms cancel and we obtain

$$x_{CG} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad \text{(12.4)}$$

Comparing this result with Equation 9.28, we see that **the center of gravity is located at the center of mass as long as the object is in a uniform gravitational field.**

In several examples presented in the next section, we are concerned with homogeneous, symmetric objects. The center of gravity for any such object coincides with its geometric center.

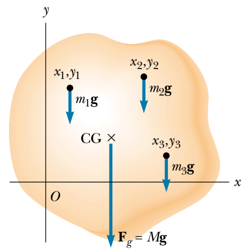
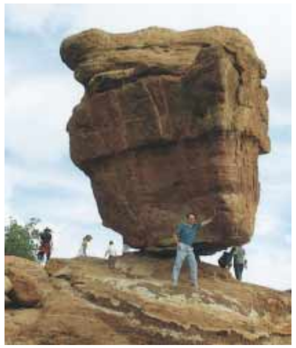


Figure 12.6 The center of gravity of an object is located at the center of mass if **g** is constant over the object.

12.3 EXAMPLES OF RIGID OBJECTS IN STATIC EQUILIBRIUM

The photograph of the one-bottle wine holder on the first page of this chapter shows one example of a balanced mechanical system that seems to defy gravity. For the system (wine holder plus bottle) to be in equilibrium, the net external force must be zero (see Eq. 12.1) and the net external torque must be zero (see Eq. 12.2). The second condition can be satisfied only when the center of gravity of the system is directly over the support point.

In working static equilibrium problems, it is important to recognize all the external forces acting on the object. Failure to do so results in an incorrect analysis. When analyzing an object in equilibrium under the action of several external forces, use the following procedure.



A large balanced rock at the Garden of the Gods in Colorado Springs, Colorado—an example of stable equilibrium.

Problem-Solving Hints

Objects in Static Equilibrium

- Draw a simple, neat diagram of the system.
- Isolate the object being analyzed. Draw a free-body diagram and then show and label all external forces acting on the object, indicating where those forces are applied. Do not include forces exerted by the object on its surroundings. (For systems that contain more than one object, draw a *separate* free-body diagram for each one.) Try to guess the correct direction for each force. If the direction you select leads to a negative force, do not be alarmed; this merely means that the direction of the force is the opposite of what you guessed.
- Establish a convenient coordinate system for the object and find the components of the forces along the two axes. Then apply the first condition for equilibrium. Remember to keep track of the signs of all force components.
- Choose a convenient axis for calculating the net torque on the object. Remember that the choice of origin for the torque equation is arbitrary; therefore, choose an origin that simplifies your calculation as much as possible. Note that a force that acts along a line passing through the point chosen as the origin gives zero contribution to the torque and thus can be ignored.

The first and second conditions for equilibrium give a set of linear equations containing several unknowns, and these equations can be solved simultaneously.

EXAMPLE 12.1 The Seesaw

A uniform 40.0-N board supports a father and daughter weighing 800 N and 350 N, respectively, as shown in Figure 12.7. If the support (called the *fulcrum*) is under the center of gravity of the board and if the father is 1.00 m from the center, (a) determine the magnitude of the upward force **n** exerted on the board by the support.

Solution First note that, in addition to **n**, the external forces acting on the board are the downward forces exerted by each person and the force of gravity acting on the board. We know that the board's center of gravity is at its geometric center because we were told the board is uniform. Because the system is in static equilibrium, the upward force **n** must balance all the downward forces. From $\Sigma F_y = 0$, we have, once we define upward as the positive y direction,

$$n - 800 \text{ N} - 350 \text{ N} - 40.0 \text{ N} = 0$$

$$n = \boxed{1\,190 \text{ N}}$$

(The equation $\Sigma F_x = 0$ also applies, but we do not need to consider it because no forces act horizontally on the board.)

(b) Determine where the child should sit to balance the system.

Solution To find this position, we must invoke the second condition for equilibrium. Taking an axis perpendicular to the page through the center of gravity of the board as the axis for our torque equation (this means that the torques

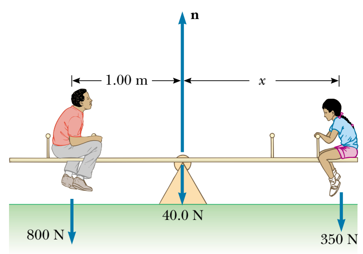


Figure 12.7 A balanced system.

produced by **n** and the force of gravity acting on the board about this axis are zero), we see from $\sum\tau = 0$ that

$$(800\text{ N})(1.00\text{ m}) - (350\text{ N})x = 0$$

$$x = \boxed{2.29\text{ m}}$$

(c) Repeat part (b) for another axis.

Solution To illustrate that the choice of axis is arbitrary, let us choose an axis perpendicular to the page and passing through the location of the father. Recall that the sign of the torque associated with a force is positive if that force tends to rotate the system counterclockwise, while the sign of the torque is negative if the force tends to rotate the system clockwise. In this case, $\sum\tau = 0$ yields

$$n(1.00\text{ m}) - (40.0\text{ N})(1.00\text{ m}) - (350\text{ N})(1.00\text{ m} + x) = 0$$

From part (a) we know that n = 1 190 N. Thus, we can solve for x to find $\boxed{x = 2.29\text{ m.}}$ This result is in agreement with the one we obtained in part (b).

Quick Quiz 12.2

In Example 12.1, if the fulcrum did not lie under the board's center of gravity, what other information would you need to solve the problem?

EXAMPLE 12.2 A Weighted Hand

A person holds a 50.0-N sphere in his hand. The forearm is horizontal, as shown in Figure 12.8a. The biceps muscle is attached 3.00 cm from the joint, and the sphere is 35.0 cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of the forearm.

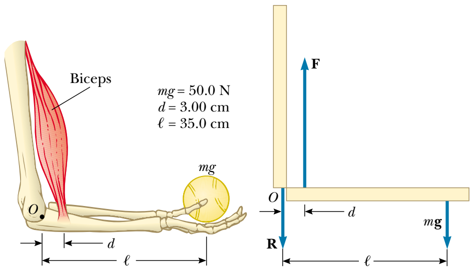


Figure 12.8 (a) The biceps muscle pulls upward with a force **F** that is essentially at right angles to the forearm. (b) The mechanical model for the system described in part (a).

Solution We simplify the situation by modeling the forearm as a bar as shown in Figure 12.8b, where **F** is the upward force exerted by the biceps and **R** is the downward force exerted by the upper arm at the joint. From the first condition for equilibrium, we have, with upward as the positive y direction,

$$(1) \qquad \sum F_y = F - R - 50.0\text{ N} = 0$$

From the second condition for equilibrium, we know that the sum of the torques about any point must be zero. With the joint O as the axis, we have

$$Fd - mg\ell = 0$$

$$F(3.00\text{ cm}) - (50.0\text{ N})(35.0\text{ cm}) = 0$$

$$F = \boxed{583\text{ N}}$$

This value for F can be substituted into Equation (1) to give R = 533 N. As this example shows, the forces at joints and in muscles can be extremely large.

Exercise In reality, the biceps makes an angle of 15.0° with the vertical; thus, **F** has both a vertical and a horizontal component. Find the magnitude of **F** and the components of **R** when you include this fact in your analysis.

Answer F = 604 N, R_x = 156 N, R_y = 533 N.

EXAMPLE 12.3 Standing on a Horizontal Beam

A uniform horizontal beam with a length of 8.00 m and a weight of 200 N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0° with the horizontal (Fig. 12.9a). If a 600-N person stands 2.00 m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.

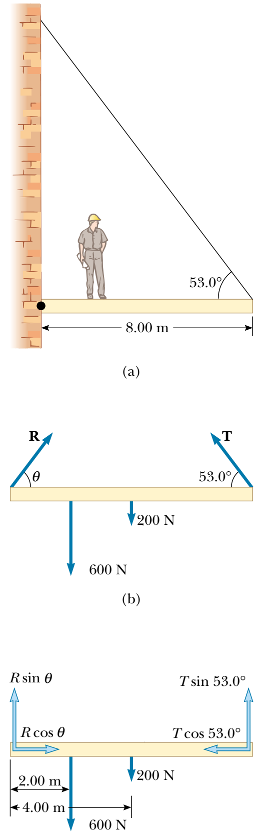


Figure 12.9 (a) A uniform beam supported by a cable. (b) The free-body diagram for the beam. (c) The free-body diagram for the beam showing the components of **R** and **T**.

Solution First we must identify all the external forces acting on the beam: They are the 200-N force of gravity, the force **T** exerted by the cable, the force **R** exerted by the wall at the pivot, and the 600-N force that the person exerts on the beam. These forces are all indicated in the free-body diagram for the beam shown in Figure 12.9b. When we consider directions for forces, it sometimes is helpful if we imagine what would happen if a force were suddenly removed. For example, if the wall were to vanish suddenly, the left end of the beam would probably move to the left as it begins to fall. This tells us that the wall is not only holding the beam up but is also pressing outward against it. Thus, we draw the vector **R** as shown in Figure 12.9b. If we resolve **T** and **R** into horizontal and vertical components, as shown in Figure 12.9c, and apply the first condition for equilibrium, we obtain

$$(1) \qquad \sum F_x = R\cos\theta - T\cos 53.0° = 0$$

$$(2) \qquad \sum F_y = R\sin\theta + T\sin 53.0° - 600\text{ N} - 200\text{ N} = 0$$

where we have chosen rightward and upward as our positive directions. Because R, T, and θ are all unknown, we cannot obtain a solution from these expressions alone. (The number of simultaneous equations must equal the number of unknowns for us to be able to solve for the unknowns.)

Now let us invoke the condition for rotational equilibrium. A convenient axis to choose for our torque equation is the one that passes through the pin connection. The feature that makes this point so convenient is that the force **R** and the horizontal component of **T** both have a moment arm of zero; hence, these forces provide no torque about this point. Recalling our counterclockwise-equals-positive convention for the sign of the torque about an axis and noting that the moment arms of the 600-N, 200-N, and $T\sin 53.0°$ forces are 2.00 m, 4.00 m, and 8.00 m, respectively, we obtain

$$\sum\tau = (T\sin 53.0°)(8.00\text{ m}) - (600\text{ N})(2.00\text{ m}) - (200\text{ N})(4.00\text{ m}) = 0$$

$$T = \boxed{313\text{ N}}$$

Thus, the torque equation with this axis gives us one of the unknowns directly! We now substitute this value into Equations (1) and (2) and find that

$$R\cos\theta = 188\text{ N}$$

$$R\sin\theta = 550\text{ N}$$

We divide the second equation by the first and, recalling the trigonometric identity $\sin\theta/\cos\theta = \tan\theta$, we obtain

$$\tan\theta = \frac{550\text{ N}}{188\text{ N}} = 2.93$$

$$\theta = \boxed{71.1°}$$

This positive value indicates that our estimate of the direction of **R** was accurate.

Finally,

$$R = \frac{188\text{ N}}{\cos\theta} = \frac{188\text{ N}}{\cos 71.1°} = \boxed{580\text{ N}}$$

If we had selected some other axis for the torque equation, the solution would have been the same. For example, if we had chosen an axis through the center of gravity of the beam, the torque equation would involve both T and R. However, this equation, coupled with Equations (1) and (2), could still be solved for the unknowns. Try it!

When many forces are involved in a problem of this nature, it is convenient to set up a table. For instance, for the example just given, we could construct the following table. Setting the sum of the terms in the last column equal to zero represents the condition of rotational equilibrium.

Force Component	Moment Arm Relative to O (m)	Torque About O (N·m)
$T\sin 53.0°$	8.00	$(8.00)\,T\sin 53.0°$
$T\cos 53.0°$	0	0
200 N	4.00	$-(4.00)(200)$
600 N	2.00	$-(2.00)(600)$
$R\sin\theta$	0	0
$R\cos\theta$	0	0

EXAMPLE 12.4 The Leaning Ladder

A uniform ladder of length ℓ and weight $mg = 50$ N rests against a smooth, vertical wall (Fig. 12.10a). If the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$, find the minimum angle $\theta_{\min}$ at which the ladder does not slip.

Solution The free-body diagram showing all the external forces acting on the ladder is illustrated in Figure 12.10b. The reaction force $\mathbf{R}$ exerted by the ground on the ladder is the vector sum of a normal force $\mathbf{n}$ and the force of static friction $\mathbf{f}_s$. The reaction force $\mathbf{P}$ exerted by the wall on the ladder is horizontal because the wall is frictionless. Notice how we have included only forces that act on the ladder. For example, the forces exerted by the ladder on the ground and on the wall are not part of the problem and thus do not appear in the free-body diagram. Applying the first condition for equilibrium to the ladder, we have

$$\sum F_x = f - P = 0$$

$$\sum F_y = n - mg = 0$$

From the second equation we see that $n = mg = 50$ N. Furthermore, when the ladder is on the verge of slipping, the force of friction must be a maximum, which is given by $f_{s,\max} = \mu_s n = 0.40(50\text{ N}) = 20$ N. (Recall Eq. 5.8: $f_s \le \mu_s n$.) Thus, at this angle, $P = 20$ N.

To find $\theta_{\min}$, we must use the second condition for equilibrium. When we take the torques about an axis through the origin O at the bottom of the ladder, we have

$$\sum \tau_O = P\ell\sin\theta - mg\frac{\ell}{2}\cos\theta = 0$$

Because $P = 20$ N when the ladder is about to slip, and because $mg = 50$ N, this expression gives

$$\tan\theta_{\min} = \frac{mg}{2P} = \frac{50\text{ N}}{40\text{ N}} = 1.25$$

$$\theta_{\min} = \boxed{51°}$$

An alternative approach is to consider the intersection O' of the lines of action of forces $m\mathbf{g}$ and $\mathbf{P}$. Because the torque about any origin must be zero, the torque about O' must be zero. This requires that the line of action of $\mathbf{R}$ (the resultant of $\mathbf{n}$ and $\mathbf{f}$) pass through O'. In other words, because the ladder is stationary, the three forces acting on it must all pass through some common point. (We say that such forces are *concurrent.*) With this condition, you could then obtain the angle ϕ that $\mathbf{R}$ makes with the horizontal (where ϕ is greater than θ). Because this approach depends on the length of the ladder, you would have to know the value of ℓ to obtain a value for $\theta_{\min}$.

Exercise For the angles labeled in Figure 12.10, show that $\tan\phi = 2\tan\theta$.

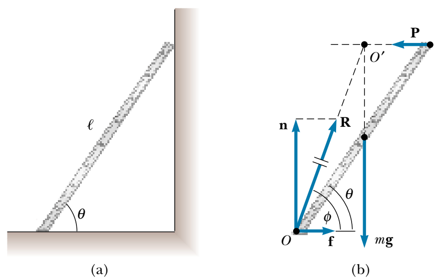


Figure 12.10 (a) A uniform ladder at rest, leaning against a smooth wall. The ground is rough. (b) The free-body diagram for the ladder. Note that the forces $\mathbf{R}$, $m\mathbf{g}$, and $\mathbf{P}$ pass through a common point O'.

EXAMPLE 12.5 Negotiating a Curb

(a) Estimate the magnitude of the force $\mathbf{F}$ a person must apply to a wheelchair's main wheel to roll up over a sidewalk curb (Fig. 12.11a). This main wheel, which is the one that comes in contact with the curb, has a radius r, and the height of the curb is h.

Solution Normally, the person's hands supply the required force to a slightly smaller wheel that is concentric with the main wheel. We assume that the radius of the smaller wheel is the same as the radius of the main wheel, and so we can use r for our radius. Let us estimate a combined weight of $mg = 1\,400$ N for the person and the wheelchair and choose a wheel radius of $r = 30$ cm, as shown in Figure 12.11b. We also pick a curb height of $h = 10$ cm. We assume that the wheelchair and occupant are symmetric, and that each wheel supports a weight of 700 N. We then proceed to analyze only one of the wheels.

When the wheel is just about to be raised from the street, the reaction force exerted by the ground on the wheel at point Q goes to zero. Hence, at this time only three forces act on the wheel, as shown in Figure 12.11c. However, the force $\mathbf{R}$, which is the force exerted on the wheel by the curb, acts at point P, and so if we choose to have our axis of rotation pass through point P, we do not need to include $\mathbf{R}$ in our torque equation. From the triangle OPQ shown in Figure 12.11b, we see that the moment arm d of the gravitational force $m\mathbf{g}$ acting on the wheel relative to point P is

$$d = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$$

The moment arm of $\mathbf{F}$ relative to point P is $2r - h$. Therefore, the net torque acting on the wheel about point P is

$$mgd - F(2r-h) = 0$$

$$mg\sqrt{2rh - h^2} - F(2r-h) = 0$$

$$F = \frac{mg\sqrt{2rh-h^2}}{2r-h}$$

$$F = \frac{(700\text{ N})\sqrt{2(0.3\text{ m})(0.1\text{ m}) - (0.1\text{ m})^2}}{2(0.3\text{ m}) - 0.1\text{ m}} = \boxed{300\text{ N}}$$

(Notice that we have kept only one digit as significant.) This result indicates that the force that must be applied to each wheel is substantial. You may want to estimate the force required to roll a wheelchair up a typical sidewalk accessibility ramp for comparison.

(b) Determine the magnitude and direction of $\mathbf{R}$.

Solution We use the first condition for equilibrium to determine the direction:

$$\sum F_x = F - R\cos\theta = 0$$

$$\sum F_y = R\sin\theta - mg = 0$$

Dividing the second equation by the first gives

$$\tan\theta = \frac{mg}{F} = \frac{700\text{ N}}{300\text{ N}};\quad \theta = \boxed{70°}$$

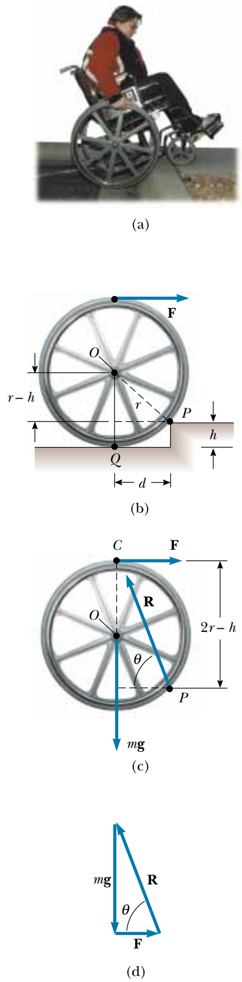


Figure 12.11 (a) A wheelchair and person of total weight mg being raised over a curb by a force $\mathbf{F}$. (b) Details of the wheel and curb. (c) The free-body diagram for the wheel when it is just about to be raised. Three forces act on the wheel at this instant: $\mathbf{F}$, which is exerted by the hand; $\mathbf{R}$, which is exerted by the curb; and the gravitational force $m\mathbf{g}$. (d) The vector sum of the three external forces acting on the wheel is zero.

We can use the right triangle shown in Figure 12.11d to obtain R:

$$R = \sqrt{(mg)^2 + F^2} = \sqrt{(700\ \text{N})^2 + (300\ \text{N})^2} = \boxed{800\ \text{N}}$$

Exercise Solve this problem by noting that the three forces acting on the wheel are concurrent (that is, that all three pass through the point C). The three forces form the sides of the triangle shown in Figure 12.11d.

APPLICATION Analysis of a Truss

Roofs, bridges, and other structures that must be both strong and lightweight often are made of trusses similar to the one shown in Figure 12.12a. Imagine that this truss structure represents part of a bridge. To approach this problem, we assume that the structural components are connected by pin joints. We also assume that the entire structure is free to slide horizontally because it sits on "rockers" on each end, which allow it to move back and forth as it undergoes thermal expansion and contraction. Assuming the mass of the bridge structure is negligible compared with the load, let us calculate the forces of tension or compression in all the structural components when it is supporting a 7 200-N load at the center (see Problem 58).

The force notation that we use here is not of our usual format. Until now, we have used the notation F_{AB} to mean "the force exerted by A on B." For this application, however, all double-letter subscripts on F indicate only the body exerting the force. The body on which a given force acts is not named in the subscript. For example, in Figure 12.12, F_{AB} is the force exerted by strut AB on the pin at A.

First, we apply Newton's second law to the truss as a whole in the vertical direction. Internal forces do not enter into this accounting. We balance the weight of the load with the normal forces exerted at the two ends by the supports on which the bridge rests:

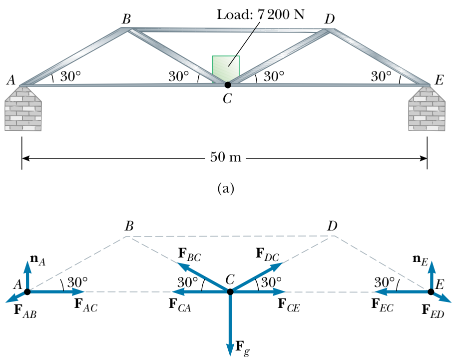


Figure 12.12 (a) Truss structure for a bridge. (b) The forces acting on the pins at points A, C, and E. As an exercise, you should diagram the forces acting on the pin at point B.

$$\sum F_y = n_A + n_E - F_g = 0$$
$$n_A + n_E = 7\,200\ \text{N}$$

Next, we calculate the torque about A, noting that the overall length of the bridge structure is $L = 50$ m:

$$\sum \tau = Ln_E - (L/2)F_g = 0$$
$$n_E = F_g/2 = 3\,600\ \text{N}$$

Although we could repeat the torque calculation for the right end (point E), it should be clear from symmetry arguments that $n_A = 3\,600$ N.

Now let us balance the vertical forces acting on the pin at point A. If we assume that strut AB is in compression, then the force F_{AB} that the strut exerts on the pin at point A has a negative y component. (If the strut is actually in tension, our calculations will result in a negative value for the magnitude of the force, still of the correct size):

$$\sum F_y = n_A - F_{AB}\sin 30° = 0$$
$$F_{AB} = 7\,200\ \text{N}$$

The positive result shows that our assumption of compression was correct.

We can now find the forces acting in the strut between A and C by considering the horizontal forces acting on the pin at point A. Because point A is not accelerating, we can safely assume that F_{AC} must point toward the right (Fig. 12.12b); this indicates that the bar between points A and C is under tension:

$$\sum F_x = F_{AC} - F_{AB}\cos 30° = 0$$
$$F_{AC} = (7\,200\ \text{N})\cos 30° = 6\,200\ \text{N}$$

Now let us consider the vertical forces acting on the pin at point C. We shall assume that strut BC is in tension. (Imagine the subsequent motion of the pin at point C if strut BC were to break suddenly.) On the basis of symmetry, we assert that $F_{BC} = F_{DC}$ and that $F_{AC} = F_{EC}$:

$$\sum F_y = 2\,F_{BC}\sin 30° - 7\,200\ \text{N} = 0$$
$$F_{BC} = 7\,200\ \text{N}$$

Finally, we balance the horizontal forces on B, assuming that strut BD is in compression:

$$\sum F_x = F_{AB}\cos 30° + F_{BC}\cos 30° - F_{BD} = 0$$
$$(7\,200\ \text{N})\cos 30° + (7\,200\ \text{N})\cos 30° - F_{BD} = 0$$
$$F_{BD} = 12\,000\ \text{N}$$

Thus, the top bar in a bridge of this design must be very strong.

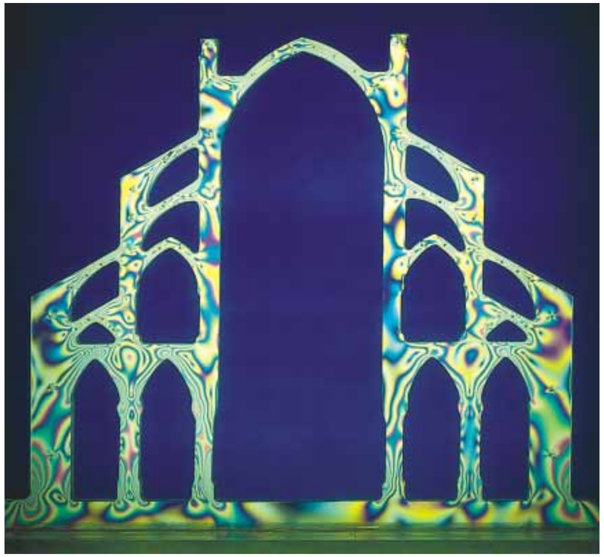
12.4 ELASTIC PROPERTIES OF SOLIDS

In our study of mechanics thus far, we have assumed that objects remain undeformed when external forces act on them. In reality, all objects are deformable. That is, it is possible to change the shape or the size of an object (or both) by applying external forces. As these changes take place, however, internal forces in the object resist the deformation.

We shall discuss the deformation of solids in terms of the concepts of stress and strain. **Stress** is a quantity that is proportional to the force causing a deformation; more specifically, stress is the external force acting on an object per unit cross-sectional area. **Strain** is a measure of the degree of deformation. It is found that, for sufficiently small stresses, **strain is proportional to stress;** the constant of proportionality depends on the material being deformed and on the nature of the deformation. We call this proportionality constant the **elastic modulus.** The elastic modulus is therefore the ratio of the stress to the resulting strain:

$$\text{Elastic modulus} \equiv \frac{\text{stress}}{\text{strain}} \qquad \textbf{(12.5)}$$

In a very real sense it is a comparison of what is done to a solid object (a force is applied) and how that object responds (it deforms to some extent).



A plastic model of an arch structure under load conditions. The wavy lines indicate regions where the stresses are greatest. Such models are useful in designing architectural components.

We consider three types of deformation and define an elastic modulus for each:

1. **Young's modulus,** which measures the resistance of a solid to a change in its length
2. **Shear modulus,** which measures the resistance to motion of the planes of a solid sliding past each other
3. **Bulk modulus,** which measures the resistance of solids or liquids to changes in their volume

Young's Modulus: Elasticity in Length

Consider a long bar of cross-sectional area A and initial length L_i that is clamped at one end, as in Figure 12.13. When an external force is applied perpendicular to the cross section, internal forces in the bar resist distortion ("stretching"), but the bar attains an equilibrium in which its length L_f is greater than L_i and in which the external force is exactly balanced by internal forces. In such a situation, the bar is said to be stressed. We define the **tensile stress** as the ratio of the magnitude of the external force F to the cross-sectional area A. The **tensile strain** in this case is defined as the ratio of the change in length ΔL to the original length L_i. We define **Young's modulus** by a combination of these two ratios:

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{\Delta L/L_i} \qquad \textbf{(12.6)}$$

Young's modulus

Young's modulus is typically used to characterize a rod or wire stressed under either tension or compression. Note that because strain is a dimensionless quantity, Y has units of force per unit area. Typical values are given in Table 12.1. Experiments show (a) that for a fixed applied force, the change in length is proportional to the original length and (b) that the force necessary to produce a given strain is proportional to the cross-sectional area. Both of these observations are in accord with Equation 12.6.

The **elastic limit** of a substance is defined as the maximum stress that can be applied to the substance before it becomes permanently deformed. It is possible to exceed the elastic limit of a substance by applying a sufficiently large stress, as seen in Figure 12.14. Initially, a stress–strain curve is a straight line. As the stress increases, however, the curve is no longer straight. When the stress exceeds the elas-

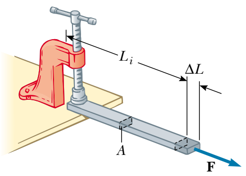


Figure 12.13 A long bar clamped at one end is stretched by an amount ΔL under the action of a force **F**.

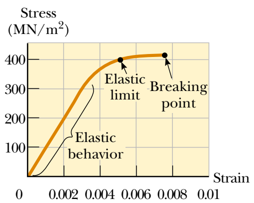


Figure 12.14 Stress-versus-strain curve for an elastic solid.

TABLE 12.1 Typical Values for Elastic Modulus

Substance	Young's Modulus (N/m²)	Shear Modulus (N/m²)	Bulk Modulus (N/m²)
Tungsten	35×10^{10}	14×10^{10}	20×10^{10}
Steel	20×10^{10}	8.4×10^{10}	6×10^{10}
Copper	11×10^{10}	4.2×10^{10}	14×10^{10}
Brass	9.1×10^{10}	3.5×10^{10}	6.1×10^{10}
Aluminum	7.0×10^{10}	2.5×10^{10}	7.0×10^{10}
Glass	$6.5–7.8 \times 10^{10}$	$2.6–3.2 \times 10^{10}$	$5.0–5.5 \times 10^{10}$
Quartz	5.6×10^{10}	2.6×10^{10}	2.7×10^{10}
Water	—	—	0.21×10^{10}
Mercury	—	—	2.8×10^{10}

tic limit, the object is permanently distorted and does not return to its original shape after the stress is removed. Hence, the shape of the object is permanently changed. As the stress is increased even further, the material ultimately breaks.

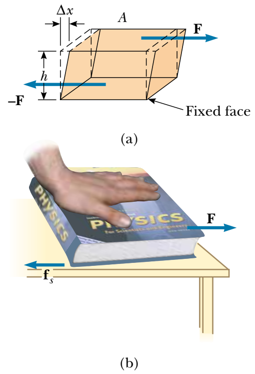


Figure 12.15 (a) A shear deformation in which a rectangular block is distorted by two forces of equal magnitude but opposite directions applied to two parallel faces. (b) A book under shear stress.

Quick Quiz 12.3

What is Young's modulus for the elastic solid whose stress–strain curve is depicted in Figure 12.14?

Quick Quiz 12.4

A material is said to be *ductile* if it can be stressed well beyond its elastic limit without breaking. A *brittle* material is one that breaks soon after the elastic limit is reached. How would you classify the material in Figure 12.14?

Shear Modulus: Elasticity of Shape

Another type of deformation occurs when an object is subjected to a force tangential to one of its faces while the opposite face is held fixed by another force (Fig. 12.15a). The stress in this case is called a shear stress. If the object is originally a rectangular block, a shear stress results in a shape whose cross-section is a parallelogram. A book pushed sideways, as shown in Figure 12.15b, is an example of an object subjected to a shear stress. To a first approximation (for small distortions), no change in volume occurs with this deformation.

We define the **shear stress** as F/A, the ratio of the tangential force to the area A of the face being sheared. The **shear strain** is defined as the ratio $\Delta x/h$, where Δx is the horizontal distance that the sheared face moves and h is the height of the object. In terms of these quantities, the **shear modulus** is

Shear modulus

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h} \qquad \textbf{(12.7)}$$

Values of the shear modulus for some representative materials are given in Table 12.1. The unit of shear modulus is force per unit area.

QuickLab

Estimate the shear modulus for the pages of your textbook. Does the thickness of the book have any effect on the modulus value?

Bulk Modulus: Volume Elasticity

Bulk modulus characterizes the response of a substance to uniform squeezing or to a reduction in pressure when the object is placed in a partial vacuum. Suppose that the external forces acting on an object are at right angles to all its faces, as shown in Figure 12.16, and that they are distributed uniformly over all the faces. As we shall see in Chapter 15, such a uniform distribution of forces occurs when an object is immersed in a fluid. An object subject to this type of deformation undergoes a change in volume but no change in shape. The **volume stress** is defined as the ratio of the magnitude of the normal force F to the area A. The quantity $P = F/A$ is called the **pressure.** If the pressure on an object changes by an amount $\Delta P = \Delta F/A$, then the object will experience a volume change ΔV. The **volume strain** is equal to the change in volume ΔV divided by the initial volume V_i. Thus, from Equation 12.5, we can characterize a volume ("bulk") compression in terms of the **bulk modulus,** which is defined as

Bulk modulus

$$B \equiv \frac{\text{volume stress}}{\text{volume strain}} = -\frac{\Delta F/A}{\Delta V/V_i} = -\frac{\Delta P}{\Delta V/V_i} \qquad \textbf{(12.8)}$$

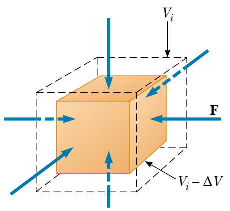


Figure 12.16 When a solid is under uniform pressure, it undergoes a change in volume but no change in shape. This cube is compressed on all sides by forces normal to its six faces.

A negative sign is inserted in this defining equation so that B is a positive number. This maneuver is necessary because an increase in pressure (positive ΔP) causes a decrease in volume (negative ΔV) and vice versa.

Table 12.1 lists bulk moduli for some materials. If you look up such values in a different source, you often find that the reciprocal of the bulk modulus is listed. The reciprocal of the bulk modulus is called the **compressibility** of the material.

Note from Table 12.1 that both solids and liquids have a bulk modulus. However, no shear modulus and no Young's modulus are given for liquids because a liquid does not sustain a shearing stress or a tensile stress (it flows instead).

Prestressed Concrete

If the stress on a solid object exceeds a certain value, the object fractures. The maximum stress that can be applied before fracture occurs depends on the nature of the material and on the type of applied stress. For example, concrete has a tensile strength of about 2×10^6 N/m^2, a compressive strength of 20×10^6 N/m^2, and a shear strength of 2×10^6 N/m^2. If the applied stress exceeds these values, the concrete fractures. It is common practice to use large safety factors to prevent failure in concrete structures.

Concrete is normally very brittle when it is cast in thin sections. Thus, concrete slabs tend to sag and crack at unsupported areas, as shown in Figure 12.17a. The slab can be strengthened by the use of steel rods to reinforce the concrete, as illustrated in Figure 12.17b. Because concrete is much stronger under compression (squeezing) than under tension (stretching) or shear, vertical columns of concrete can support very heavy loads, whereas horizontal beams of concrete tend to sag and crack. However, a significant increase in shear strength is achieved if the reinforced concrete is prestressed, as shown in Figure 12.17c. As the concrete is being poured, the steel rods are held under tension by external forces. The external

QuickLab

Support a new flat eraser (art gum or Pink Pearl will do) on two parallel pencils at least 3 cm apart. Press down on the middle of the top surface just enough to make the top face of the eraser curve a bit. Is the top face under tension or compression? How about the bottom? Why does a flat slab of concrete supported at the ends tend to crack on the bottom face and not the top?

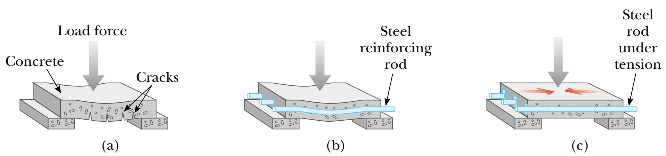


Figure 12.17 (a) A concrete slab with no reinforcement tends to crack under a heavy load. (b) The strength of the concrete is increased by using steel reinforcement rods. (c) The concrete is further strengthened by prestressing it with steel rods under tension.

forces are released after the concrete cures; this results in a permanent tension in the steel and hence a compressive stress on the concrete. This enables the concrete slab to support a much heavier load.

EXAMPLE 12.6 Stage Design

Recall Example 8.10, in which we analyzed a cable used to support an actor as he swung onto the stage. The tension in the cable was 940 N. What diameter should a 10-m-long steel wire have if we do not want it to stretch more than 0.5 cm under these conditions?

Solution From the definition of Young's modulus, we can solve for the required cross-sectional area. Assuming that the cross section is circular, we can determine the diameter of the wire. From Equation 12.6, we have

$$Y = \frac{F/A}{\Delta L/L_i}$$

$$A = \frac{FL_i}{Y\,\Delta L} = \frac{(940\ \text{N})(10\ \text{m})}{(20 \times 10^{10}\ \text{N/m}^2)(0.005\ \text{m})} = 9.4 \times 10^{-6}\ \text{m}^2$$

The radius of the wire can be found from $A = \pi r^2$:

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{9.4 \times 10^{-6}\ \text{m}^2}{\pi}} = 1.7 \times 10^{-3}\ \text{m} = 1.7\ \text{mm}$$

$$d = 2r = 2(1.7\ \text{mm}) = \boxed{3.4\ \text{mm}}$$

To provide a large margin of safety, we would probably use a flexible cable made up of many smaller wires having a total cross-sectional area substantially greater than our calculated value.

EXAMPLE 12.7 Squeezing a Brass Sphere

A solid brass sphere is initially surrounded by air, and the air pressure exerted on it is 1.0×10^5 N/m^2 (normal atmospheric pressure). The sphere is lowered into the ocean to a depth at which the pressure is 2.0×10^7 N/m^2. The volume of the sphere in air is 0.50 m^3. By how much does this volume change once the sphere is submerged?

Solution From the definition of bulk modulus, we have

$$B = -\frac{\Delta P}{\Delta V/V_i}$$

$$\Delta V = -\frac{V_i \Delta P}{B}$$

Because the final pressure is so much greater than the initial pressure, we can neglect the initial pressure and state that $\Delta P = P_f - P_i \approx P_f = 2.0 \times 10^7$ N/m^2. Therefore,

$$\Delta V = -\frac{(0.50\ \text{m}^3)(2.0 \times 10^7\ \text{N/m}^2)}{6.1 \times 10^{10}\ \text{N/m}^2} = \boxed{-1.6 \times 10^{-4}\ \text{m}^3}$$

The negative sign indicates a decrease in volume.

SUMMARY

A rigid object is in **equilibrium** if and only if **the resultant external force acting on it is zero and the resultant external torque on it is zero about any axis:**

$$\sum \mathbf{F} = 0 \qquad \textbf{(12.1)}$$

$$\sum \boldsymbol{\tau} = 0 \qquad \textbf{(12.2)}$$

The first condition is the condition for translational equilibrium, and the second is the condition for rotational equilibrium. These two equations allow you to analyze a great variety of problems. Make sure you can identify forces unambiguously, create a free-body diagram, and then apply Equations 12.1 and 12.2 and solve for the unknowns.

The force of gravity exerted on an object can be considered as acting at a single point called the **center of gravity.** The center of gravity of an object coincides with its center of mass if the object is in a uniform gravitational field.

We can describe the elastic properties of a substance using the concepts of stress and strain. **Stress** is a quantity proportional to the force producing a deformation; **strain** is a measure of the degree of deformation. Strain is proportional to stress, and the constant of proportionality is the **elastic modulus:**

$$\text{Elastic modulus} \equiv \frac{\text{stress}}{\text{strain}} \quad \textbf{(12.5)}$$

Three common types of deformation are (1) the resistance of a solid to elongation under a load, characterized by **Young's modulus** Y; (2) the resistance of a solid to the motion of internal planes sliding past each other, characterized by the **shear modulus** S; and (3) the resistance of a solid or fluid to a volume change, characterized by the **bulk modulus** B.

QUESTIONS

1. Can a body be in equilibrium if only one external force acts on it? Explain.
2. Can a body be in equilibrium if it is in motion? Explain.
3. Locate the center of gravity for the following uniform objects: (a) sphere, (b) cube, (c) right circular cylinder.
4. The center of gravity of an object may be located outside the object. Give a few examples for which this is the case.
5. You are given an arbitrarily shaped piece of plywood, together with a hammer, nail, and plumb bob. How could you use these items to locate the center of gravity of the plywood? (*Hint:* Use the nail to suspend the plywood.)
6. For a chair to be balanced on one leg, where must the center of gravity of the chair be located?
7. Can an object be in equilibrium if the only torques acting on it produce clockwise rotation?
8. A tall crate and a short crate of equal mass are placed side by side on an incline (without touching each other). As the incline angle is increased, which crate will topple first? Explain.
9. When lifting a heavy object, why is it recommended to keep the back as vertical as possible, lifting from the knees, rather than bending over and lifting from the waist?
10. Give a few examples in which several forces are acting on a system in such a way that their sum is zero but the system is not in equilibrium.
11. If you measure the net torque and the net force on a system to be zero, (a) could the system still be rotating with respect to you? (b) Could it be translating with respect to you?
12. A ladder is resting inclined against a wall. Would you feel safer climbing up the ladder if you were told that the ground is frictionless but the wall is rough or that the wall is frictionless but the ground is rough? Justify your answer.
13. What kind of deformation does a cube of Jell-O exhibit when it "jiggles"?
14. Ruins of ancient Greek temples often have intact vertical columns, but few horizontal slabs of stone are still in place. Can you think of a reason why this is so?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 12.1 The Conditions for Equilibrium

1. A baseball player holds a 36-oz bat (weight = 10.0 N) with one hand at the point O (Fig. P12.1). The bat is in equilibrium. The weight of the bat acts along a line 60.0 cm to the right of O. Determine the force and the torque exerted on the bat by the player.

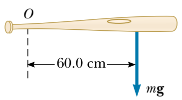


Figure P12.1

2. Write the necessary conditions of equilibrium for the body shown in Figure P12.2. Take the origin of the torque equation at the point O.

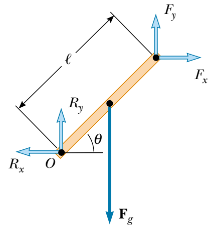


Figure P12.2

WEB 3. A uniform beam of mass m_b and length ℓ supports blocks of masses m_1 and m_2 at two positions, as shown in Figure P12.3. The beam rests on two points. For what value of x will the beam be balanced at P such that the normal force at O is zero?

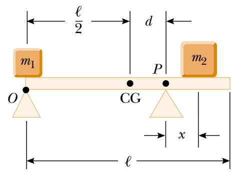


Figure P12.3

4. A student gets his car stuck in a snow drift. Not at a loss, having studied physics, he attaches one end of a stout rope to the vehicle and the other end to the trunk of a nearby tree, allowing for a very small amount of slack. The student then exerts a force **F** on the center of the rope in the direction perpendicular to the car–tree line, as shown in Figure P12.4. If the rope is inextensible and if the magnitude of the applied force is 500 N, what is the force on the car? (Assume equilibrium conditions.)

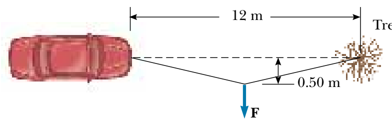


Figure P12.4

Section 12.2 More on the Center of Gravity

5. A 3.00-kg particle is located on the x axis at $x = -5.00$ m, and a 4.00-kg particle is located on the x axis at $x = 3.00$ m. Find the center of gravity of this two-particle system.
6. A circular pizza of radius R has a circular piece of radius $R/2$ removed from one side, as shown in Figure P12.6. Clearly, the center of gravity has moved from C to C' along the x axis. Show that the distance from C to C' is $R/6$. (Assume that the thickness and density of the pizza are uniform throughout.)

Figure P12.6

7. A carpenter's square has the shape of an L, as shown in Figure P12.7. Locate its center of gravity.

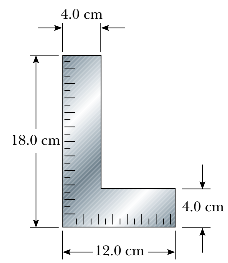


Figure P12.7

8. Pat builds a track for his model car out of wood, as illustrated in Figure P12.8. The track is 5.00 cm wide, 1.00 m high, and 3.00 m long, and it is solid. The runway is cut so that it forms a parabola described by the equation $y = (x - 3)^2/9$. Locate the horizontal position of the center of gravity of this track.

WEB 9. Consider the following mass distribution: 5.00 kg at (0, 0) m, 3.00 kg at (0, 4.00) m, and 4.00 kg at (3.00, 0) m. Where should a fourth mass of 8.00 kg be placed so that the center of gravity of the four-mass arrangement will be at (0, 0)?

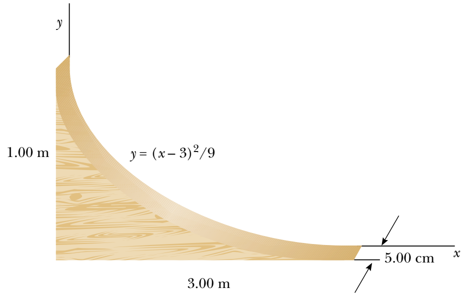


Figure P12.8

10. Figure P12.10 shows three uniform objects: a rod, a right triangle, and a square. Their masses in kilograms and their coordinates in meters are given. Determine the center of gravity for the three-object system.

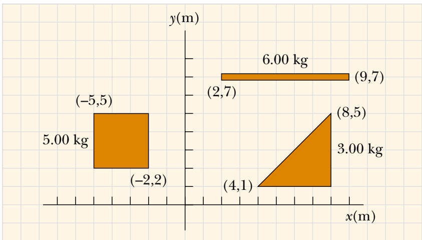


Figure P12.10

Section 12.3 Examples of Rigid Objects in Static Equilibrium

11. Stephen is pushing his sister Joyce in a wheelbarrow when it is stopped by a brick 8.00 cm high (Fig. P12.11). The handles make an angle of 15.0° from the horizontal. A downward force of 400 N is exerted on the wheel, which has a radius of 20.0 cm. (a) What force must Stephen apply along the handles to just start the wheel over the brick? (b) What is the force (magnitude and direction) that the brick exerts on the wheel just as the wheel begins to lift over the brick? Assume in both parts (a) and (b) that the brick remains fixed and does not slide along the ground.

12. Two pans of a balance are 50.0 cm apart. The fulcrum of the balance has been shifted 1.00 cm away from the center by a dishonest shopkeeper. By what percentage is the true weight of the goods being marked up by the shopkeeper? (Assume that the balance has negligible mass.)

Figure P12.11

13. A 15.0-m uniform ladder weighing 500 N rests against a frictionless wall. The ladder makes a 60.0° angle with the horizontal. (a) Find the horizontal and vertical forces that the ground exerts on the base of the ladder when an 800-N firefighter is 4.00 m from the bottom. (b) If the ladder is just on the verge of slipping when the firefighter is 9.00 m up, what is the coefficient of static friction between the ladder and the ground?

14. A uniform ladder of length L and mass m_1 rests against a frictionless wall. The ladder makes an angle θ with the horizontal. (a) Find the horizontal and vertical forces that the ground exerts on the base of the ladder when a firefighter of mass m_2 is a distance x from the bottom. (b) If the ladder is just on the verge of slipping when the firefighter is a distance d from the bottom, what is the coefficient of static friction between the ladder and the ground?

15. Figure P12.15 shows a claw hammer as it is being used to pull a nail out of a horizontal surface. If a force of magnitude 150 N is exerted horizontally as shown, find

Figure P12.15

(a) the force exerted by the hammer claws on the nail and (b) the force exerted by the surface on the point of contact with the hammer head. Assume that the force the hammer exerts on the nail is parallel to the nail.

16. A uniform plank with a length of 6.00 m and a mass of 30.0 kg rests horizontally across two horizontal bars of a scaffold. The bars are 4.50 m apart, and 1.50 m of the plank hangs over one side of the scaffold. Draw a free-body diagram for the plank. How far can a painter with a mass of 70.0 kg walk on the overhanging part of the plank before it tips?

17. A 1 500-kg automobile has a wheel base (the distance between the axles) of 3.00 m. The center of mass of the automobile is on the center line at a point 1.20 m behind the front axle. Find the force exerted by the ground on each wheel.

18. A vertical post with a square cross section is 10.0 m tall. Its bottom end is encased in a base 1.50 m tall that is precisely square but slightly loose. A force of 5.50 N to the right acts on the top of the post. The base maintains the post in equilibrium. Find the force that the top of the right sidewall of the base exerts on the post. Find the force that the bottom of the left sidewall of the base exerts on the post.

19. A flexible chain weighing 40.0 N hangs between two hooks located at the same height (Fig. P12.19). At each hook, the tangent to the chain makes an angle θ = 42.0° with the horizontal. Find (a) the magnitude of the force each hook exerts on the chain and (b) the tension in the chain at its midpoint. (*Hint:* For part (b), make a free-body diagram for half the chain.)

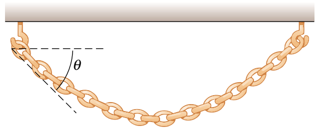


Figure P12.19

20. A hemispherical sign 1.00 m in diameter and of uniform mass density is supported by two strings, as shown in Figure P12.20. What fraction of the sign's weight is supported by each string?

21. Sir Lost-a-Lot dons his armor and sets out from the castle on his trusty steed in his quest to improve communication between damsels and dragons (Fig. P12.21). Unfortunately, his squire lowered the draw bridge too far and finally stopped lowering it when it was 20.0° below the horizontal. Lost-a-Lot and his horse stop when their combined center of mass is 1.00 m from the end of the bridge. The bridge is 8.00 m long and has a mass of 2 000 kg. The lift cable is attached to the bridge 5.00 m from the hinge at the castle end and to a point on the castle wall 12.0 m above the bridge. Lost-a-Lot's mass combined with that of his armor and steed is 1 000 kg. Determine (a) the tension in the cable, as well as (b) the horizontal and (c) the vertical force components acting on the bridge at the hinge.

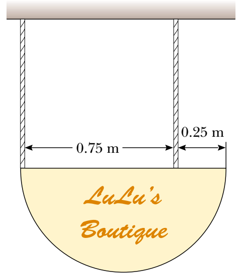


Figure P12.20

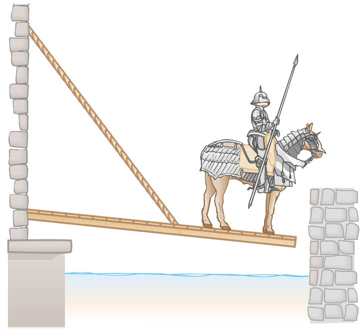


Figure P12.21

22. Two identical, uniform bricks of length L are placed in a stack over the edge of a horizontal surface such that the maximum possible overhang without falling is achieved, as shown in Figure P12.22. Find the distance x.

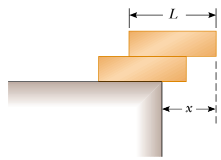


Figure P12.22

23. A vaulter holds a 29.4-N pole in equilibrium by exerting an upward force **U** with her leading hand and a downward force **D** with her trailing hand, as shown in Figure P12.23. Point C is the center of gravity of the pole. What are the magnitudes of **U** and **D**?

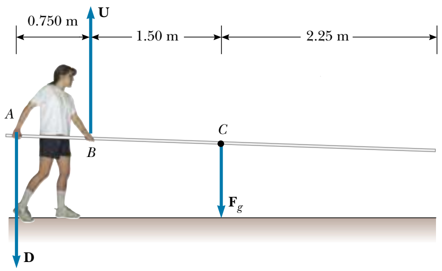


Figure P12.23

Section 12.4 Elastic Properties of Solids

24. Assume that Young's modulus for bone is 1.50×10^{10} N/m^2 and that a bone will fracture if more than 1.50×10^8 N/m^2 is exerted. (a) What is the maximum force that can be exerted on the femur bone in the leg if it has a minimum effective diameter of 2.50 cm? (b) If a force of this magnitude is applied compressively, by how much does the 25.0-cm-long bone shorten?

25. A 200-kg load is hung on a wire with a length of 4.00 m, a cross-sectional area of 0.200×10^{-4} m^2, and a Young's modulus of 8.00×10^{10} N/m^2. What is its increase in length?

26. A steel wire 1 mm in diameter can support a tension of 0.2 kN. Suppose you need a cable made of these wires to support a tension of 20 kN. The cable's diameter should be of what order of magnitude?

27. A child slides across a floor in a pair of rubber-soled shoes. The frictional force acting on each foot is 20.0 N. The footprint area of each shoe's sole is 14.0 cm^2, and the thickness of each sole is 5.00 mm. Find the horizontal distance by which the upper and lower surfaces of each sole are offset. The shear modulus of the rubber is 3.00×10^6 N/m^2.

28. **Review Problem.** A 30.0-kg hammer strikes a steel spike 2.30 cm in diameter while moving with a speed of 20.0 m/s. The hammer rebounds with a speed of 10.0 m/s after 0.110 s. What is the average strain in the spike during the impact?

29. If the elastic limit of copper is 1.50×10^8 N/m^2, determine the minimum diameter a copper wire can have under a load of 10.0 kg if its elastic limit is not to be exceeded.

30. **Review Problem.** A 2.00-m-long cylindrical steel wire with a cross-sectional diameter of 4.00 mm is placed over a light frictionless pulley, with one end of the wire connected to a 5.00-kg mass and the other end connected to a 3.00-kg mass. By how much does the wire stretch while the masses are in motion?

31. **Review Problem.** A cylindrical steel wire of length L_i with a cross-sectional diameter d is placed over a light frictionless pulley, with one end of the wire connected to a mass m_1 and the other end connected to a mass m_2. By how much does the wire stretch while the masses are in motion?

32. Calculate the density of sea water at a depth of 1 000 m, where the water pressure is about 1.00×10^7 N/m^2. (The density of sea water is 1.030×10^3 kg/m^3 at the surface.)

WEB 33. If the shear stress exceeds about 4.00×10^8 N/m^2, steel ruptures. Determine the shearing force necessary (a) to shear a steel bolt 1.00 cm in diameter and (b) to punch a 1.00-cm-diameter hole in a steel plate 0.500 cm thick.

34. (a) Find the minimum diameter of a steel wire 18.0 m long that elongates no more than 9.00 mm when a load of 380 kg is hung on its lower end. (b) If the elastic limit for this steel is 3.00×10^8 N/m^2, does permanent deformation occur with this load?

35. When water freezes, it expands by about 9.00%. What would be the pressure increase inside your automobile's engine block if the water in it froze? (The bulk modulus of ice is 2.00×10^9 N/m^2.)

36. For safety in climbing, a mountaineer uses a 50.0-m nylon rope that is 10.0 mm in diameter. When supporting the 90.0-kg climber on one end, the rope elongates by 1.60 m. Find Young's modulus for the rope material.

ADDITIONAL PROBLEMS

37. A bridge with a length of 50.0 m and a mass of 8.00×10^4 kg is supported on a smooth pier at each end, as illustrated in Figure P12.37. A truck of mass 3.00×10^4 kg is located 15.0 m from one end. What are the forces on the bridge at the points of support?

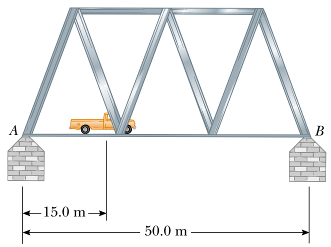


Figure P12.37

38. A frame in the shape of the letter A is formed from two uniform pieces of metal, each of weight 26.0 N and length 1.00 m. They are hinged at the top and held together by a horizontal wire 1.20 m in length (Fig. P12.38). The structure rests on a frictionless surface. If the wire is connected at points a distance of 0.650 m from the top of the frame, determine the tension in the wire.

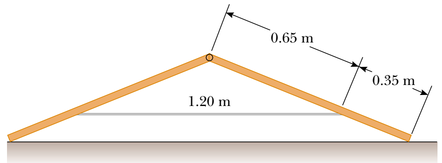


Figure P12.38

39. Refer to Figure 12.17c. A lintel of prestressed reinforced concrete is 1.50 m long. The cross-sectional area of the concrete is 50.0 cm^2. The concrete encloses one steel reinforcing rod with a cross-sectional area of 1.50 cm^2. The rod joins two strong end plates. Young's modulus for the concrete is 30.0×10^9 N/m^2. After the concrete cures and the original tension T_1 in the rod is released, the concrete will be under a compressive stress of 8.00×10^6 N/m^2. (a) By what distance will the rod compress the concrete when the original tension in the rod is released? (b) Under what tension T_2 will the rod still be? (c) How much longer than its unstressed length will the rod then be? (d) When the concrete was poured, the rod should have been stretched by what extension distance from its unstressed length? (e) Find the required original tension T_1 in the rod.

40. A solid sphere of radius R and mass M is placed in a trough, as shown in Figure P12.40. The inner surfaces of the trough are frictionless. Determine the forces exerted by the trough on the sphere at the two contact points.

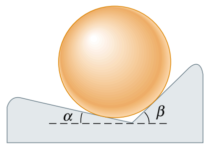


Figure P12.40

41. A 10.0-kg monkey climbs up a 120-N uniform ladder of length L, as shown in Figure P12.41. The upper and lower ends of the ladder rest on frictionless surfaces. The lower end is fastened to the wall by a horizontal rope that can support a maximum tension of 110 N. (a) Draw a free-body diagram for the ladder. (b) Find the tension in the rope when the monkey is one third the way up the ladder. (c) Find the maximum distance d that the monkey can climb up the ladder before the rope breaks. Express your answer as a fraction of L.

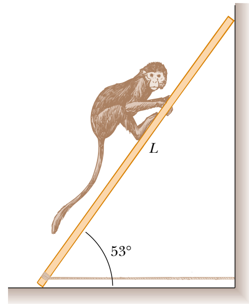


Figure P12.41

42. A hungry bear weighing 700 N walks out on a beam in an attempt to retrieve a basket of food hanging at the end of the beam (Fig. P12.42). The beam is uniform, weighs 200 N, and is 6.00 m long; the basket weighs 80.0 N. (a) Draw a free-body diagram for the beam. (b) When the bear is at $x = 1.00$ m, find the tension in the wire and the components of the force exerted by the wall on the left end of the beam. (c) If the wire can withstand a maximum tension of 900 N, what is the maximum distance that the bear can walk before the wire breaks?

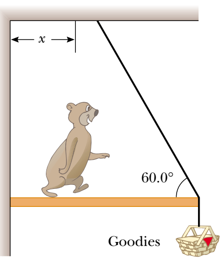


Figure P12.42

43. Old MacDonald had a farm, and on that farm he had a gate (Fig. P12.43). The gate is 3.00 m wide and 1.80 m

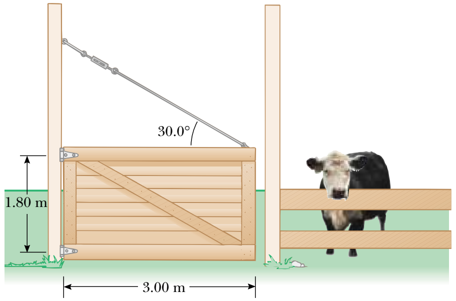


Figure P12.43

high, with hinges attached to the top and bottom. The guy wire makes an angle of 30.0° with the top of the gate and is tightened by a turn buckle to a tension of 200 N. The mass of the gate is 40.0 kg. (a) Determine the horizontal force exerted on the gate by the bottom hinge. (b) Find the horizontal force exerted by the upper hinge. (c) Determine the combined vertical force exerted by both hinges. (d) What must the tension in the guy wire be so that the horizontal force exerted by the upper hinge is zero?

44. A 1 200-N uniform boom is supported by a cable, as illustrated in Figure P12.44. The boom is pivoted at the bottom, and a 2 000-N object hangs from its top. Find the tension in the cable and the components of the reaction force exerted on the boom by the floor.

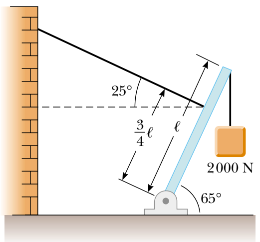


Figure P12.44

WEB **45.** A uniform sign of weight F_g and width $2L$ hangs from a light, horizontal beam hinged at the wall and supported by a cable (Fig. P12.45). Determine (a) the tension in the cable and (b) the components of the reaction force exerted by the wall on the beam in terms of F_g, d, L, and θ.

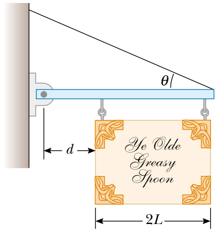


Figure P12.45

46. A crane of mass 3 000 kg supports a load of 10 000 kg as illustrated in Figure P12.46. The crane is pivoted with a frictionless pin at A and rests against a smooth support at B. Find the reaction forces at A and B.

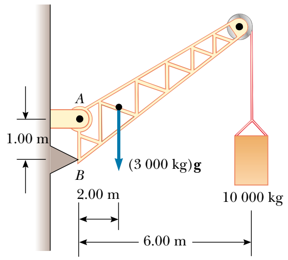


Figure P12.46

47. A ladder having a uniform density and a mass m rests against a frictionless vertical wall, making an angle 60.0° with the horizontal. The lower end rests on a flat surface, where the coefficient of static friction is $\mu_s = 0.400$. A window cleaner having a mass $M = 2m$ attempts to climb the ladder. What fraction of the length L of the ladder will the worker have reached when the ladder begins to slip?

48. A uniform ladder weighing 200 N is leaning against a wall (see Fig. 12.10). The ladder slips when $\theta = 60.0°$. Assuming that the coefficients of static friction at the wall and the ground are the same, obtain a value for μ_s.

49. A 10 000-N shark is supported by a cable attached to a 4.00-m rod that can pivot at its base. Calculate the tension in the tie-rope between the wall and the rod if it is holding the system in the position shown in Figure P12.49. Find the horizontal and vertical forces exerted on the base of the rod. (Neglect the weight of the rod.)

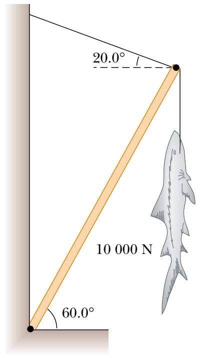


Figure P12.49

50. When a person stands on tiptoe (a strenuous position), the position of the foot is as shown in Figure P12.50a. The total weight of the body $\mathbf{F}_g$ is supported by the force $\mathbf{n}$ exerted by the floor on the toe. A mechanical model for the situation is shown in Figure P12.50b,

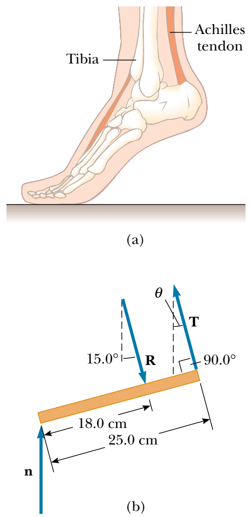


Figure P12.50

where **T** is the force exerted by the Achilles tendon on the foot and **R** is the force exerted by the tibia on the foot. Find the values of T, R, and θ when $F_g = 700$ N.

51. A person bends over and lifts a 200-N object as shown in Figure P12.51a, with his back in a horizontal position (a terrible way to lift an object). The back muscle attached at a point two thirds the way up the spine maintains the position of the back, and the angle between the spine and this muscle is 12.0°. Using the mechanical model shown in Figure P12.51b and taking the weight of the upper body to be 350 N, find the tension in the back muscle and the compressional force in the spine.

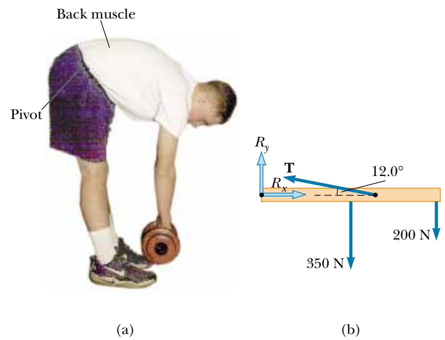


Figure P12.51

52. Two 200-N traffic lights are suspended from a single cable, as shown in Figure 12.52. Neglecting the cable's weight, (a) prove that if $\theta_1 = \theta_2$, then $T_1 = T_2$. (b) Determine the three tensions T_1, T_2, and T_3 if $\theta_1 = \theta_2 = 8.00°$.

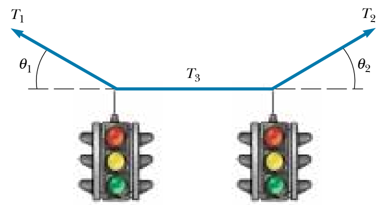


Figure P12.52

53. A force acts on a rectangular cabinet weighing 400 N, as illustrated in Figure P12.53. (a) If the cabinet slides with constant speed when $F = 200$ N and $h = 0.400$ m, find the coefficient of kinetic friction and the position of the resultant normal force. (b) If $F = 300$ N, find the value of h for which the cabinet just begins to tip.

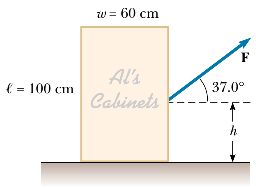


Figure P12.53 Problems 53 and 54.

54. Consider the rectangular cabinet of Problem 53, but with a force **F** applied horizontally at its upper edge. (a) What is the minimum force that must be applied for the cabinet to start tipping? (b) What is the minimum coefficient of static friction required to prevent the cabinet from sliding with the application of a force of this magnitude? (c) Find the magnitude and direction of the minimum force required to tip the cabinet if the point of application can be chosen anywhere on it.

55. A uniform rod of weight F_g and length L is supported at its ends by a frictionless trough, as shown in Figure P12.55. (a) Show that the center of gravity of the rod is directly over point O when the rod is in equilibrium. (b) Determine the equilibrium value of the angle θ.

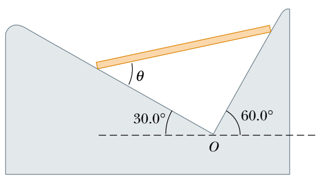


Figure P12.55

56. Review Problem. A cue stick strikes a cue ball and delivers a horizontal impulse in such a way that the ball rolls without slipping as it starts to move. At what height above the ball's center (in terms of the radius of the ball) was the blow struck?

57. A uniform beam of mass m is inclined at an angle θ to the horizontal. Its upper end produces a 90° bend in a very rough rope tied to a wall, and its lower end rests on a rough floor (Fig. P12.57). (a) If the coefficient of static friction between the beam and the floor is μ_s, determine an expression for the maximum mass M that can be suspended from the top before the beam slips. (b) Determine the magnitude of the reaction force at the floor and the magnitude of the force exerted by the beam on the rope at P in terms of m, M, and μ_s.

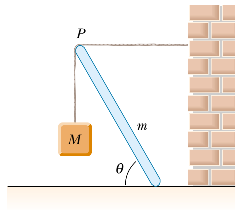


Figure P12.57

58. Figure P12.58 shows a truss that supports a downward force of 1 000 N applied at the point B. The truss has negligible weight. The piers at A and C are smooth. (a) Apply the conditions of equilibrium to prove that $n_A = 366$ N and that $n_C = 634$ N. (b) Show that, because forces act on the light truss only at the hinge joints, each bar of the truss must exert on each hinge pin only a force along the length of that bar—a force of tension or compression. (c) Find the force of tension or compression in each of the three bars.

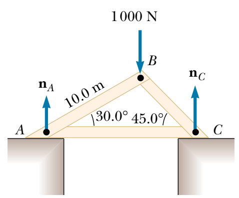


Figure P12.58

59. A stepladder of negligible weight is constructed as shown in Figure P12.59. A painter with a mass of 70.0 kg stands on the ladder 3.00 m from the bottom. Assuming that the floor is frictionless, find (a) the tension in the horizontal bar connecting the two halves of the ladder, (b) the normal forces at A and B, and (c) the components of the reaction force at the single hinge C that the left half of the ladder exerts on the right half. (*Hint:* Treat each half of the ladder separately.)

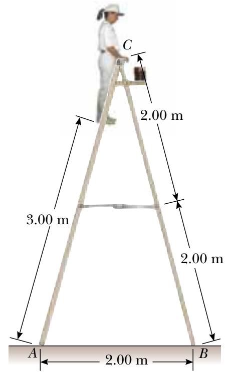


Figure P12.59

60. A flat dance floor of dimensions 20.0 m by 20.0 m has a mass of 1 000 kg. Three dance couples, each of mass 125 kg, start in the top left, top right, and bottom left corners. (a) Where is the initial center of gravity? (b) The couple in the bottom left corner moves 10.0 m to the right. Where is the new center of gravity? (c) What was the average velocity of the center of gravity if it took that couple 8.00 s to change position?

61. A shelf bracket is mounted on a vertical wall by a single screw, as shown in Figure P12.61. Neglecting the weight of the bracket, find the horizontal component of the force that the screw exerts on the bracket when an 80.0-N vertical force is applied as shown. (*Hint:* Imagine that the bracket is slightly loose.)

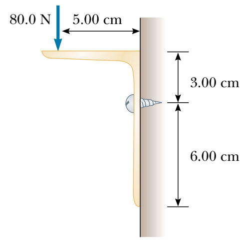


Figure P12.61

62. Figure P12.62 shows a vertical force applied tangentially to a uniform cylinder of weight F_g. The coefficient of

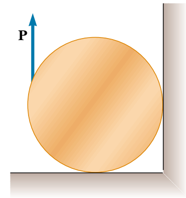


Figure P12.62

static friction between the cylinder and all surfaces is 0.500. In terms of F_g, find the maximum force **P** that can be applied that does not cause the cylinder to rotate. (*Hint:* When the cylinder is on the verge of slipping, both friction forces are at their maximum values. Why?)

WEB **63. Review Problem.** A wire of length L_i, Young's modulus Y, and cross-sectional area A is stretched elastically by an amount ΔL. According to Hooke's law, the restoring force is $-k\,\Delta L$. (a) Show that $k = YA/L_i$. (b) Show that the work done in stretching the wire by an amount ΔL is $W = YA(\Delta L)^2/2L_i$.

64. Two racquetballs are placed in a glass jar, as shown in Figure P12.64. Their centers and the point A lie on a straight line. (a) Assuming that the walls are frictionless, determine P_1, P_2, and P_3. (b) Determine the magnitude of the force exerted on the right ball by the left ball. Assume each ball has a mass of 170 g.

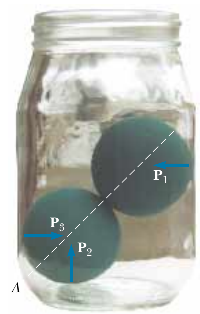


Figure P12.64

65. In Figure P12.65, the scales read $F_{g1} = 380$ N and $F_{g2} = 320$ N. Neglecting the weight of the supporting plank,

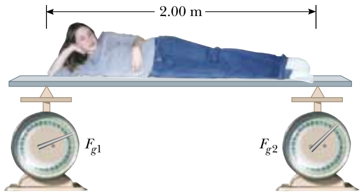


Figure P12.65

how far from the woman's feet is her center of mass, given that her height is 2.00 m?

66. A steel cable 3.00 cm^2 in cross-sectional area has a mass of 2.40 kg per meter of length. If 500 m of the cable is hung over a vertical cliff, how much does the cable stretch under its own weight? (For Young's modulus for steel, refer to Table 12.1.)

67. (a) Estimate the force with which a karate master strikes a board if the hand's speed at time of impact is 10.0 m/s and decreases to 1.00 m/s during a 0.002 00-s time-of-contact with the board. The mass of coordinated hand-and-arm is 1.00 kg. (b) Estimate the shear stress if this force is exerted on a 1.00-cm-thick pine board that is 10.0 cm wide. (c) If the maximum shear stress a pine board can receive before breaking is 3.60×10^6 N/m^2, will the board break?

68. A bucket is made from thin sheet metal. The bottom and top of the bucket have radii of 25.0 cm and 35.0 cm, respectively. The bucket is 30.0 cm high and filled with water. Where is the center of gravity? (Ignore the weight of the bucket itself.)

69. Review Problem. A trailer with a loaded weight of F_g is being pulled by a vehicle with a force **P**, as illustrated in Figure P12.69. The trailer is loaded such that its center of mass is located as shown. Neglect the force of rolling friction and let a represent the x component of the acceleration of the trailer. (a) Find the vertical component of **P** in terms of the given parameters. (b) If a = 2.00 m/s^2 and h = 1.50 m, what must be the value of d so that $P_y = 0$ (that is, no vertical load on the vehicle)? (c) Find the values of P_x and P_y given that F_g = 1 500 N, d = 0.800 m, L = 3.00 m, h = 1.50 m, and a = − 2.00 m/s^2.

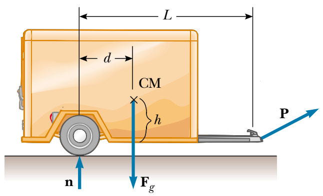


Figure P12.69

70. Review Problem. An aluminum wire is 0.850 m long and has a circular cross section of diameter 0.780 mm. Fixed at the top end, the wire supports a 1.20-kg mass that swings in a horizontal circle. Determine the angular velocity required to produce strain 1.00×10^{-3}.

71. A 200-m-long bridge truss extends across a river (Fig. P12.71). Calculate the force of tension or compression in each structural component when a 1 360-kg car is at the center of the bridge. Assume that the structure is free to slide horizontally to permit thermal expansion and contraction, that the structural components are connected by pin joints, and that the masses of the structural components are small compared with the mass of the car.

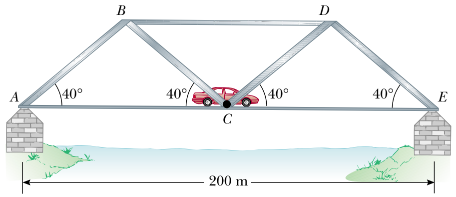


Figure P12.71

72. A 100-m-long bridge truss is supported at its ends so that it can slide freely (Fig. P12.72). A 1 500-kg car is halfway between points A and C. Show that the weight of the car is evenly distributed between points A and C, and calculate the force in each structural component. Specify whether each structural component is under tension or compression. Assume that the structural components are connected by pin joints and that the masses of the components are small compared with the mass of the car.

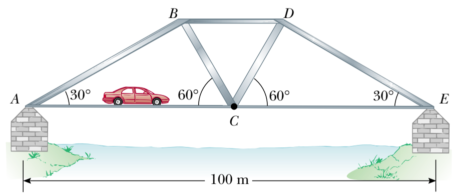


Figure P12.72

Answers to Quick Quizzes

12.1 (a) Yes, as Figure 12.3 shows. The unbalanced torques cause an angular acceleration even though the linear acceleration is zero. (b) Yes, again. This happens when the lines of action of all the forces intersect at a common point. If a net force acts on the object, then the object has a translational acceleration. However, because there is no net torque on the object, the object has no angular acceleration. There are other instances in which torques cancel but the forces do not. You should be able to draw at least two.

12.2 The location of the board's center of gravity relative to the fulcrum.

12.3 Young's modulus is given by the ratio of stress to strain, which is the slope of the elastic behavior section of the graph in Figure 12.14. Reading from the graph, we note that a stress of approximately 3×10^8 N/m^2 results in a strain of 0.003. The slope, and hence Young's modulus, are therefore 10×10^{10} N/m^2.

12.4 A substantial part of the graph extends beyond the elastic limit, indicating permanent deformation. Thus, the material is ductile.

PUZZLER

Inside the pocket watch is a small disk (called a torsional pendulum) that oscillates back and forth at a very precise rate and controls the watch gears. A grandfather clock keeps accurate time because of its pendulum. The tall wooden case provides the space needed by the long pendulum as it advances the clock gears with each swing. In both of these timepieces, the vibration of a carefully shaped component is critical to accurate operation. What properties of oscillating objects make them so useful in timing devices? *(Photograph of pocket watch, George Semple; photograph of grandfather clock, Charles D. Winters)*

chapter

13

Oscillatory Motion

Chapter Outline

A very special kind of motion occurs when the force acting on a body is proportional to the displacement of the body from some equilibrium position. If this force is always directed toward the equilibrium position, repetitive back-and-forth motion occurs about this position. Such motion is called *periodic motion, harmonic motion, oscillation,* or *vibration* (the four terms are completely equivalent).

You are most likely familiar with several examples of periodic motion, such as the oscillations of a block attached to a spring, the swinging of a child on a playground swing, the motion of a pendulum, and the vibrations of a stringed musical instrument. In addition to these everyday examples, numerous other systems exhibit periodic motion. For example, the molecules in a solid oscillate about their equilibrium positions; electromagnetic waves, such as light waves, radar, and radio waves, are characterized by oscillating electric and magnetic field vectors; and in alternating-current electrical circuits, voltage, current, and electrical charge vary periodically with time.

Most of the material in this chapter deals with *simple harmonic motion,* in which an object oscillates such that its position is specified by a sinusoidal function of time with no loss in mechanical energy. In real mechanical systems, damping (frictional) forces are often present. These forces are considered in optional Section 13.6 at the end of this chapter.

13.1 SIMPLE HARMONIC MOTION

8.10 Consider a physical system that consists of a block of mass m attached to the end of a spring, with the block free to move on a horizontal, frictionless surface (Fig. 13.1). When the spring is neither stretched nor compressed, the block is at the position $x = 0$, called the *equilibrium position* of the system. We know from experience that such a system oscillates back and forth if disturbed from its equilibrium position.

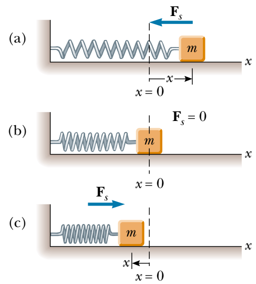


Figure 13.1 A block attached to a spring moving on a frictionless surface. (a) When the block is displaced to the right of equilibrium ($x > 0$), the force exerted by the spring acts to the left. (b) When the block is at its equilibrium position ($x = 0$), the force exerted by the spring is zero. (c) When the block is displaced to the left of equilibrium ($x < 0$), the force exerted by the spring acts to the right.

We can understand the motion in Figure 13.1 qualitatively by first recalling that when the block is displaced a small distance x from equilibrium, the spring exerts on the block a force that is proportional to the displacement and given by Hooke's law (see Section 7.3):

$$F_s = -kx \tag{13.1}$$

We call this a **restoring force** because it is is always directed toward the equilibrium position and therefore *opposite* the displacement. That is, when the block is displaced to the right of $x = 0$ in Figure 13.1, then the displacement is positive and the restoring force is directed to the left. When the block is displaced to the left of $x = 0$, then the displacement is negative and the restoring force is directed to the right.

Applying Newton's second law to the motion of the block, together with Equation 13.1, we obtain

$$F_s = -kx = ma$$

$$a = -\frac{k}{m}x \tag{13.2}$$

That is, the acceleration is proportional to the displacement of the block, and its direction is opposite the direction of the displacement. Systems that behave in this way are said to exhibit **simple harmonic motion. An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed.**

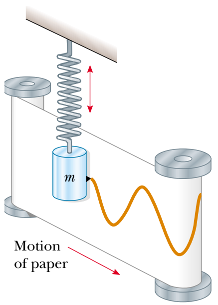


Figure 13.2 An experimental apparatus for demonstrating simple harmonic motion. A pen attached to the oscillating mass traces out a wavelike pattern on the moving chart paper.

An experimental arrangement that exhibits simple harmonic motion is illustrated in Figure 13.2. A mass oscillating vertically on a spring has a pen attached to it. While the mass is oscillating, a sheet of paper is moved perpendicular to the direction of motion of the spring, and the pen traces out a wavelike pattern.

8.2 & 8.3

In general, a particle moving along the x axis exhibits simple harmonic motion when x, the particle's displacement from equilibrium, varies in time according to the relationship

Displacement versus time for simple harmonic motion

$$x = A\cos(\omega t + \phi) \quad \textbf{(13.3)}$$

where A, ω, and ϕ are constants. To give physical significance to these constants, we have labeled a plot of x as a function of t in Figure 13.3a. This is just the pattern that is observed with the experimental apparatus shown in Figure 13.2. The **amplitude** A of the motion is the maximum displacement of the particle in either the positive or negative x direction. The constant ω is called the **angular frequency** of the motion and has units of radians per second. (We shall discuss the geometric significance of ω in Section 13.2.) The constant angle ϕ, called the **phase constant** (or phase angle), is determined by the initial displacement and velocity of the particle. If the particle is at its maximum position $x = A$ at $t = 0$, then $\phi = 0$ and the curve of x versus t is as shown in Figure 13.3b. If the particle is at some other position at $t = 0$, the constants ϕ and A tell us what the position was at time $t = 0$. The quantity $(\omega t + \phi)$ is called the **phase** of the motion and is useful in comparing the motions of two oscillators.

Note from Equation 13.3 that the trigonometric function x is *periodic* and repeats itself every time ωt increases by 2π rad. **The period T of the motion is the time it takes for the particle to go through one full cycle.** We say that the particle has made *one oscillation.* This definition of T tells us that the value of x at time t equals the value of x at time $t + T$. We can show that $T = 2\pi/\omega$ by using the preceding observation that the phase $(\omega t + \phi)$ increases by 2π rad in a time T:

$$\omega t + \phi + 2\pi = \omega(t + T) + \phi$$

Hence, $\omega T = 2\pi$, or

$$T = \frac{2\pi}{\omega} \quad \textbf{(13.4)}$$

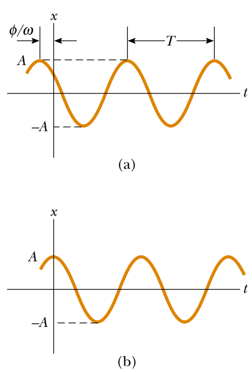


Figure 13.3 (a) An x–t curve for a particle undergoing simple harmonic motion. The amplitude of the motion is A, the period is T, and the phase constant is ϕ. (b) The x–t curve in the special case in which $x = A$ at $t = 0$ and hence $\phi = 0$.

The inverse of the period is called the **frequency** f of the motion. **The frequency represents the number of oscillations that the particle makes per unit time:**

Frequency

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \textbf{(13.5)}$$

The units of f are cycles per second $= s^{-1}$, or **hertz** (Hz).

Rearranging Equation 13.5, we obtain the angular frequency:

Angular frequency

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \textbf{(13.6)}$$

Quick Quiz 13.1

What would the phase constant ϕ have to be in Equation 13.3 if we were describing an oscillating object that happened to be at the origin at $t = 0$?

Quick Quiz 13.2

An object undergoes simple harmonic motion of amplitude A. Through what total distance does the object move during one complete cycle of its motion? (a) $A/2$. (b) A. (c) $2A$. (d) $4A$.

We can obtain the linear velocity of a particle undergoing simple harmonic motion by differentiating Equation 13.3 with respect to time:

Velocity in simple harmonic motion

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \quad \textbf{(13.7)}$$

The acceleration of the particle is

Acceleration in simple harmonic motion

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) \quad \textbf{(13.8)}$$

Because $x = A\cos(\omega t + \phi)$, we can express Equation 13.8 in the form

$$a = -\omega^2 x \quad \textbf{(13.9)}$$

From Equation 13.7 we see that, because the sine function oscillates between ± 1, the extreme values of v are $\pm\omega A$. Because the cosine function also oscillates between ± 1, Equation 13.8 tells us that the extreme values of a are $\pm\omega^2 A$. Therefore, the maximum speed and the magnitude of the maximum acceleration of a particle moving in simple harmonic motion are

Maximum values of speed and acceleration in simple harmonic motion

$$v_{max} = \omega A \quad \textbf{(13.10)}$$

$$a_{max} = \omega^2 A \quad \textbf{(13.11)}$$

Figure 13.4a represents the displacement versus time for an arbitrary value of the phase constant. The velocity and acceleration curves are illustrated in Figure 13.4b and c. These curves show that the phase of the velocity differs from the phase of the displacement by $\pi/2$ rad, or 90°. That is, when x is a maximum or a minimum, the velocity is zero. Likewise, when x is zero, the speed is a maximum.

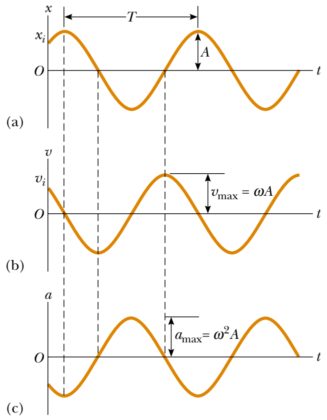


Figure 13.4 Graphical representation of simple harmonic motion. (a) Displacement versus time. (b) Velocity versus time. (c) Acceleration versus time. Note that at any specified time the velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement.

Furthermore, note that the phase of the acceleration differs from the phase of the displacement by π rad, or 180°. That is, when x is a maximum, a is a maximum in the opposite direction.

The phase constant ϕ is important when we compare the motion of two or more oscillating objects. Imagine two identical pendulum bobs swinging side by side in simple harmonic motion, with one having been released later than the other. The pendulum bobs have different phase constants. Let us show how the phase constant and the amplitude of any particle moving in simple harmonic motion can be determined if we know the particle's initial speed and position and the angular frequency of its motion.

Suppose that at $t = 0$ the initial position of a single oscillator is $x = x_i$ and its initial speed is $v = v_i$. Under these conditions, Equations 13.3 and 13.7 give

$$x_i = A\cos\phi \qquad \textbf{(13.12)}$$

$$v_i = -\omega A \sin\phi \qquad \textbf{(13.13)}$$

Dividing Equation 13.13 by Equation 13.12 eliminates A, giving $v_i/x_i = -\omega \tan\phi$, or

$$\tan\phi = -\frac{v_i}{\omega x_i} \qquad \textbf{(13.14)}$$

Furthermore, if we square Equations 13.12 and 13.13, divide the velocity equation by ω^2, and then add terms, we obtain

$$x_i^2 + \left(\frac{v_i}{\omega}\right)^2 = A^2\cos^2\phi + A^2\sin^2\phi$$

Using the identity $\sin^2\phi + \cos^2\phi = 1$, we can solve for A:

$$A = \sqrt{x_i^2 + \left(\frac{v_i}{\omega}\right)^2} \qquad \textbf{(13.15)}$$

The following properties of a particle moving in simple harmonic motion are important:

Properties of simple harmonic motion

- The acceleration of the particle is proportional to the displacement but is in the opposite direction. This is the *necessary and sufficient condition for simple harmonic motion,* as opposed to all other kinds of vibration.
- The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time but are not in phase, as shown in Figure 13.4.
- The frequency and the period of the motion are independent of the amplitude. (We show this explicitly in the next section.)

Quick Quiz 13.3

Can we use Equations 2.8, 2.10, 2.11, and 2.12 (see pages 35 and 36) to describe the motion of a simple harmonic oscillator?

EXAMPLE 13.1 An Oscillating Object

An object oscillates with simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation

$$x = (4.00\text{ m})\cos\left(\pi t + \frac{\pi}{4}\right)$$

where t is in seconds and the angles in the parentheses are in radians. (a) Determine the amplitude, frequency, and period of the motion.

Solution By comparing this equation with Equation 13.3, the general equation for simple harmonic motion—$x = A\cos(\omega t + \phi)$—we see that $A = 4.00$ m and $\omega = \pi$ rad/s. Therefore, $f = \omega/2\pi = \pi/2\pi = 0.500$ Hz and $T = 1/f = 2.00$ s.

(b) Calculate the velocity and acceleration of the object at any time t.

Solution

$$v = \frac{dx}{dt} = -(4.00\text{ m})\sin\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$= -(4.00\pi\text{ m/s})\sin\left(\pi t + \frac{\pi}{4}\right)$$

$$a = \frac{dv}{dt} = -(4.00\pi\text{ m/s})\cos\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$= -(4.00\pi^2\text{ m/s}^2)\cos\left(\pi t + \frac{\pi}{4}\right)$$

(c) Using the results of part (b), determine the position, velocity, and acceleration of the object at $t = 1.00$ s.

Solution Noting that the angles in the trigonometric functions are in radians, we obtain, at $t = 1.00$ s,

$$x = (4.00\text{ m})\cos\left(\pi + \frac{\pi}{4}\right) = (4.00\text{ m})\cos\left(\frac{5\pi}{4}\right)$$

$$= (4.00\text{ m})(-0.707) = -2.83\text{ m}$$

$$v = -(4.00\pi\text{ m/s})\sin\left(\frac{5\pi}{4}\right) = -(4.00\pi\text{ m/s})(-0.707)$$

$$= 8.89\text{ m/s}$$

$$a = -(4.00\pi^2\text{ m/s}^2)\cos\left(\frac{5\pi}{4}\right)$$

$$= -(4.00\pi^2\text{ m/s}^2)(-0.707) = 27.9\text{ m/s}^2$$

(d) Determine the maximum speed and maximum acceleration of the object.

Solution In the general expressions for v and a found in part (b), we use the fact that the maximum values of the sine and cosine functions are unity. Therefore, v varies between $\pm 4.00\pi$ m/s, and a varies between $\pm 4.00\pi^2$ m/s². Thus,

$$v_{max} = 4.00\pi\text{ m/s} = 12.6\text{ m/s}$$

$$a_{max} = 4.00\pi^2\text{ m/s}^2 = 39.5\text{ m/s}^2$$

We obtain the same results using $v_{max} = \omega A$ and $a_{max} = \omega^2 A$, where $A = 4.00$ m and $\omega = \pi$ rad/s.

(e) Find the displacement of the object between $t = 0$ and $t = 1.00$ s.

Solution The x coordinate at $t = 0$ is

$$x_i = (4.00 \text{ m}) \cos\left(0 + \frac{\pi}{4}\right) = (4.00 \text{ m})(0.707) = 2.83 \text{ m}$$

In part (c), we found that the x coordinate at $t = 1.00$ s is -2.83 m; therefore, the displacement between $t = 0$ and $t = 1.00$ s is

$$\Delta x = x_f - x_i = -2.83 \text{ m} - 2.83 \text{ m} = \boxed{-5.66 \text{ m}}$$

Because the object's velocity changes sign during the first second, the magnitude of Δx is not the same as the distance traveled in the first second. (By the time the first second is over, the object has been through the point $x = -2.83$ m once, traveled to $x = -4.00$ m, and come back to $x = -2.83$ m.)

Exercise What is the phase of the motion at $t = 2.00$ s?

Answer $9\pi/4$ rad.

13.2 THE BLOCK–SPRING SYSTEM REVISITED

Let us return to the block–spring system (Fig. 13.5). Again we assume that the surface is frictionless; hence, when the block is displaced from equilibrium, the only force acting on it is the restoring force of the spring. As we saw in Equation 13.2, when the block is displaced a distance x from equilibrium, it experiences an acceleration $a = -(k/m)x$. If the block is displaced a maximum distance $x = A$ at some initial time and then released from rest, its initial acceleration at that instant is $-kA/m$ (its extreme negative value). When the block passes through the equilibrium position $x = 0$, its acceleration is zero. At this instant, its speed is a maximum. The block then continues to travel to the left of equilibrium and finally reaches $x = -A$, at which time its acceleration is kA/m (maximum positive) and its speed is again zero. Thus, we see that the block oscillates between the turning points $x = \pm A$.

Let us now describe the oscillating motion in a quantitative fashion. Recall that $a = dv/dt = d^2x/dt^2$, and so we can express Equation 13.2 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \textbf{(13.16)}$$

If we denote the ratio k/m with the symbol ω^2, this equation becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \textbf{(13.17)}$$

Now we require a solution to Equation 13.17—that is, a function $x(t)$ that satisfies this second-order differential equation. Because Equations 13.17 and 13.9 are equivalent, each solution must be that of simple harmonic motion:

$$x = A\cos(\omega t + \phi)$$

To see this explicitly, assume that $x = A\cos(\omega t + \phi)$. Then

$$\frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \phi) = -\omega A \sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}\sin(\omega t + \phi) = -\omega^2 A\cos(\omega t + \phi)$$

Comparing the expressions for x and d^2x/dt^2, we see that $d^2x/dt^2 = -\omega^2 x$, and Equation 13.17 is satisfied. We conclude that **whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium**

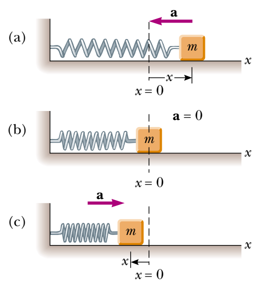


Figure 13.5 A block of mass m attached to a spring on a frictionless surface undergoes simple harmonic motion. (a) When the block is displaced to the right of equilibrium, the displacement is positive and the acceleration is negative. (b) At the equilibrium position, $x = 0$, the acceleration is zero and the speed is a maximum. (c) When the block is displaced to the left of equilibrium, the displacement is negative and the acceleration is positive.

position and in the opposite direction ($F = -kx$), the particle moves in simple harmonic motion.

Recall that the period of any simple harmonic oscillator is $T = 2\pi/\omega$ (Eq. 13.4) and that the frequency is the inverse of the period. We know from Equations 13.16 and 13.17 that $\omega = \sqrt{k/m}$, so we can express the period and frequency of the block–spring system as

Period and frequency for a block–spring system

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \textbf{(13.18)}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \textbf{(13.19)}$$

That is, **the frequency and period depend only on the mass of the block and on the force constant of the spring.** Furthermore, the frequency and period are independent of the amplitude of the motion. As we might expect, the frequency is greater for a stiffer spring (the stiffer the spring, the greater the value of k) and decreases with increasing mass.

Special Case 1. Let us consider a special case to better understand the physical significance of Equation 13.3, the defining expression for simple harmonic motion. We shall use this equation to describe the motion of an oscillating block–spring system. Suppose we pull the block a distance A from equilibrium and then release it from rest at this stretched position, as shown in Figure 13.6. Our solution for x must obey the initial conditions that $x_i = A$ and $v_i = 0$ at $t = 0$. It does if we choose $\phi = 0$, which gives $x = A\cos\omega t$ as the solution. To check this solution, we note that it satisfies the condition that $x_i = A$ at $t = 0$ because $\cos 0 = 1$. Thus, we see that A and ϕ contain the information on initial conditions.

Now let us investigate the behavior of the velocity and acceleration for this special case. Because $x = A\cos\omega t$,

$$v = \frac{dx}{dt} = -\omega A\sin\omega t$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos\omega t$$

From the velocity expression we see that, because $\sin 0 = 0$, $v_i = 0$ at $t = 0$, as we require. The expression for the acceleration tells us that $a = -\omega^2 A$ at $t = 0$. Physically, this negative acceleration makes sense because the force acting on the block is directed to the left when the displacement is positive. In fact, at the extreme po-

QuickLab

Hang an object from a rubber band and start it oscillating. Measure T. Now tie four identical rubber bands together, end to end. How should k for this longer band compare with k for the single band? Again, time the oscillations with the same object. Can you verify Equation 13.19?

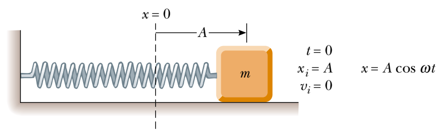


Figure 13.6 A block–spring system that starts from rest at $x_i = A$. In this case, $\phi = 0$ and thus $x = A\cos\omega t$.

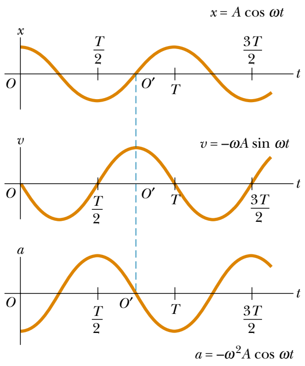


Figure 13.7 Displacement, velocity, and acceleration versus time for a block–spring system like the one shown in Figure 13.6, undergoing simple harmonic motion under the initial conditions that at $t = 0$, $x_i = A$ and $v_i = 0$ (Special Case 1). The origins at O' correspond to Special Case 2, the block–spring system under the initial conditions shown in Figure 13.8.

sition shown in Figure 13.6, $F_s = -kA$ (to the left) and the initial acceleration is $-\omega^2 A = -kA/m$.

Another approach to showing that $x = A\cos\omega t$ is the correct solution involves using the relationship $\tan\phi = -v_i/\omega x_i$ (Eq. 13.14). Because $v_i = 0$ at $t = 0$, $\tan\phi = 0$ and thus $\phi = 0$. (The tangent of π also equals zero, but $\phi = \pi$ gives the wrong value for x_i.)

Figure 13.7 is a plot of displacement, velocity, and acceleration versus time for this special case. Note that the acceleration reaches extreme values of $\pm\omega^2 A$ while the displacement has extreme values of $\pm A$ because the force is maximal at those positions. Furthermore, the velocity has extreme values of $\pm\omega A$, which both occur at $x = 0$. Hence, the quantitative solution agrees with our qualitative description of this system.

Special Case 2. Now suppose that the block is given an initial velocity $\mathbf{v}_i$ to the right at the instant it is at the equilibrium position, so that $x_i = 0$ and $v = v_i$ at $t = 0$ (Fig. 13.8). The expression for x must now satisfy these initial conditions. Because the block is moving in the positive x direction at $t = 0$ and because $x_i = 0$ at $t = 0$, the expression for x must have the form $x = A\sin\omega t$.

Applying Equation 13.14 and the initial condition that $x_i = 0$ at $t = 0$, we find that $\tan\phi = -\infty$ and $\phi = -\pi/2$. Hence, Equation 13.3 becomes $x = A\cos(\omega t - \pi/2)$, which can be written $x = A\sin\omega t$. Furthermore, from Equation 13.15 we see that $A = v_i/\omega$; therefore, we can express x as

$$x = \frac{v_i}{\omega}\sin\omega t$$

The velocity and acceleration in this case are

$$v = \frac{dx}{dt} = v_i\cos\omega t$$

$$a = \frac{dv}{dt} = -\omega v_i\sin\omega t$$

These results are consistent with the facts that (1) the block always has a maximum

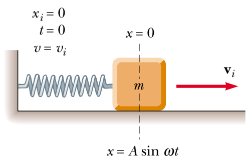


Figure 13.8 The block–spring system starts its motion at the equilibrium position at $t = 0$. If its initial velocity is v_i to the right, the block's x coordinate varies as $x = (v_i/\omega)\sin\omega t$.

speed at $x = 0$ and (2) the force and acceleration are zero at this position. The graphs of these functions versus time in Figure 13.7 correspond to the origin at O'.

Quick Quiz 13.4

What is the solution for x if the block is initially moving to the left in Figure 13.8?

Example 13.2 Watch Out for Potholes!

A car with a mass of 1 300 kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20 000 N/m. If two people riding in the car have a combined mass of 160 kg, find the frequency of vibration of the car after it is driven over a pothole in the road.

Solution We assume that the mass is evenly distributed. Thus, each spring supports one fourth of the load. The total mass is 1 460 kg, and therefore each spring supports 365 kg. Hence, the frequency of vibration is, from Equation 13.19,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20\,000\ \text{N/m}}{365\ \text{kg}}} = 1.18\ \text{Hz}$$

Exercise How long does it take the car to execute two complete vibrations?

Answer 1.70 s.

Example 13.3 A Block–Spring System

A block with a mass of 200 g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from rest, as shown in Figure 13.6. (a) Find the period of its motion.

Solution From Equations 13.16 and 13.17, we know that the angular frequency of any block–spring system is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00\ \text{N/m}}{200\times10^{-3}\ \text{kg}}} = 5.00\ \text{rad/s}$$

and the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00\ \text{rad/s}} = 1.26\ \text{s}$$

(b) Determine the maximum speed of the block.

Solution We use Equation 13.10:

$$v_{max} = \omega A = (5.00\ \text{rad/s})(5.00\times10^{-2}\ \text{m}) = 0.250\ \text{m/s}$$

(c) What is the maximum acceleration of the block?

Solution We use Equation 13.11:

$$a_{max} = \omega^2 A = (5.00\ \text{rad/s})^2(5.00\times10^{-2}\ \text{m}) = 1.25\ \text{m/s}^2$$

(d) Express the displacement, speed, and acceleration as functions of time.

Solution This situation corresponds to Special Case 1, where our solution is $x = A\cos\omega t$. Using this expression and the results from (a), (b), and (c), we find that

$$x = A\cos\omega t = (0.050\ \text{m})\cos 5.00t$$

$$v = \omega A\sin\omega t = -(0.250\ \text{m/s})\sin 5.00t$$

$$a = \omega^2 A\cos\omega t = -(1.25\ \text{m/s}^2)\cos 5.00t$$

13.3 ENERGY OF THE SIMPLE HARMONIC OSCILLATOR

Let us examine the mechanical energy of the block–spring system illustrated in Figure 13.6. Because the surface is frictionless, we expect the total mechanical energy to be constant, as was shown in Chapter 8. We can use Equation 13.7 to express the kinetic energy as

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi) \quad \textbf{(13.20)}$$

Kinetic energy of a simple harmonic oscillator

The elastic potential energy stored in the spring for any elongation x is given by $\frac{1}{2}kx^2$ (see Eq. 8.4). Using Equation 13.3, we obtain

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t + \phi) \quad \textbf{(13.21)}$$

Potential energy of a simple harmonic oscillator

We see that K and U are *always* positive quantities. Because $\omega^2 = k/m$, we can express the total mechanical energy of the simple harmonic oscillator as

$$E = K + U = \tfrac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

From the identity $\sin^2\theta + \cos^2\theta = 1$, we see that the quantity in square brackets is unity. Therefore, this equation reduces to

$$E = \tfrac{1}{2}kA^2 \quad \textbf{(13.22)}$$

Total energy of a simple harmonic oscillator

That is, **the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.** Note that U is small when K is large, and vice versa, because the sum must be constant. In fact, the total mechanical energy is equal to the maximum potential energy stored in the spring when $x = \pm A$ because $v = 0$ at these points and thus there is no kinetic energy. At the equilibrium position, where $U = 0$ because $x = 0$, the total energy, all in the form of kinetic energy, is again $\frac{1}{2}kA^2$. That is,

$$E = \tfrac{1}{2}mv_{\text{max}}^2 = \tfrac{1}{2}m\omega^2A^2 = \tfrac{1}{2}m\frac{k}{m}A^2 = \tfrac{1}{2}kA^2 \qquad (\text{at } x = 0)$$

Plots of the kinetic and potential energies versus time appear in Figure 13.9a, where we have taken $\phi = 0$. As already mentioned, both K and U are always positive, and at all times their sum is a constant equal to $\frac{1}{2}kA^2$, the total energy of the system. The variations of K and U with the displacement x of the block are plotted

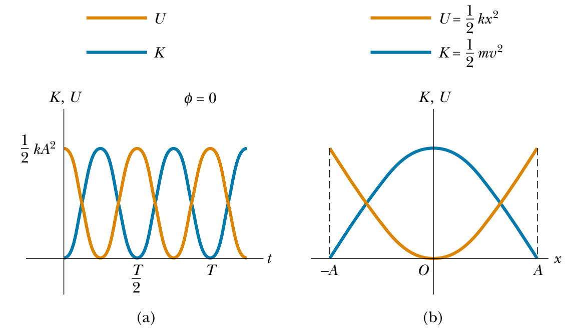


Figure 13.9 (a) Kinetic energy and potential energy versus time for a simple harmonic oscillator with $\phi = 0$. (b) Kinetic energy and potential energy versus displacement for a simple harmonic oscillator. In either plot, note that $K + U =$ constant.

in Figure 13.9b. Energy is continuously being transformed between potential energy stored in the spring and kinetic energy of the block.

Figure 13.10 illustrates the position, velocity, acceleration, kinetic energy, and potential energy of the block–spring system for one full period of the motion. Most of the ideas discussed so far are incorporated in this important figure. Study it carefully.

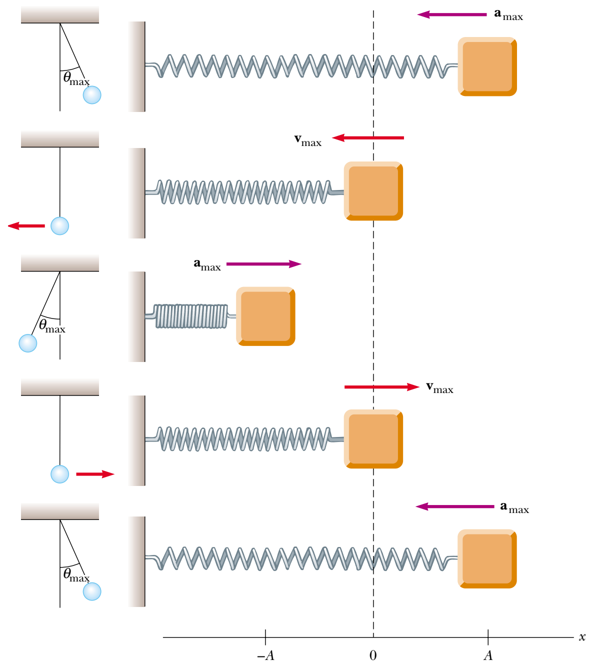
Finally, we can use the principle of conservation of energy to obtain the velocity for an arbitrary displacement by expressing the total energy at some arbitrary position x as

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\,\omega\sqrt{A^2 - x^2} \quad \textbf{(13.23)}$$

Velocity as a function of position for a simple harmonic oscillator

When we check Equation 13.23 to see whether it agrees with known cases, we find that it substantiates the fact that the speed is a maximum at $x = 0$ and is zero at the turning points $x = \pm A$.



t	x	v	a	K	U
0	A	0	$-\omega^2A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	ω^2A	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2A$	0	$\frac{1}{2}kA^2$

Figure 13.10 Simple harmonic motion for a block–spring system and its relationship to the motion of a simple pendulum. The parameters in the table refer to the block–spring system, assuming that $x = A$ at $t = 0$; thus, $x = A\cos\omega t$ (see Special Case 1).

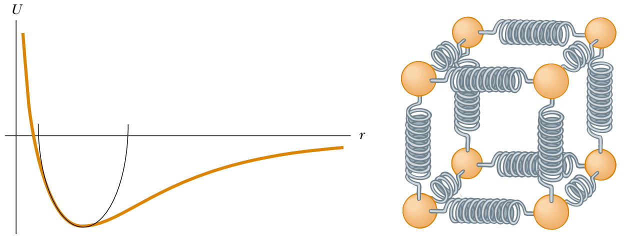


Figure 13.11 (a) If the atoms in a molecule do not move too far from their equilibrium positions, a graph of potential energy versus separation distance between atoms is similar to the graph of potential energy versus position for a simple harmonic oscillator. (b) Tiny springs approximate the forces holding atoms together.

You may wonder why we are spending so much time studying simple harmonic oscillators. We do so because they are good models of a wide variety of physical phenomena. For example, recall the Lennard–Jones potential discussed in Example 8.11. This complicated function describes the forces holding atoms together. Figure 13.11a shows that, for small displacements from the equilibrium position, the potential energy curve for this function approximates a parabola, which represents the potential energy function for a simple harmonic oscillator. Thus, we can approximate the complex atomic binding forces as tiny springs, as depicted in Figure 13.11b.

The ideas presented in this chapter apply not only to block–spring systems and atoms, but also to a wide range of situations that include bungee jumping, tuning in a television station, and viewing the light emitted by a laser. You will see more examples of simple harmonic oscillators as you work through this book.

EXAMPLE 13.4 Oscillations on a Horizontal Surface

A 0.500-kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. (a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

Solution Using Equation 13.22, we obtain

$$E = K + U = \tfrac{1}{2}kA^2 = \tfrac{1}{2}(20.0\text{ N/m})(3.00 \times 10^{-2}\text{ m})^2$$
$$= \boxed{9.00 \times 10^{-3}\text{ J}}$$

When the cube is at $x = 0$, we know that $U = 0$ and $E = \frac{1}{2}mv_{\text{max}}^2$; therefore,

$$\tfrac{1}{2}mv_{\text{max}}^2 = 9.00 \times 10^{-3}\text{ J}$$
$$v_{\text{max}} = \sqrt{\frac{18.0 \times 10^{-3}\text{ J}}{0.500\text{ kg}}} = \boxed{0.190\text{ m/s}}$$

(b) What is the velocity of the cube when the displacement is 2.00 cm?

Solution We can apply Equation 13.23 directly:

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$
$$= \pm\sqrt{\frac{20.0\text{ N/m}}{0.500\text{ kg}}[(0.030\,0\text{ m})^2 - (0.020\,0\text{ m})^2]}$$
$$= \boxed{\pm 0.141\text{ m/s}}$$

The positive and negative signs indicate that the cube could be moving to either the right or the left at this instant.

(c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

Solution Using the result of (b), we find that

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.500\text{ kg})(0.141\text{ m/s})^2 = \boxed{5.00 \times 10^{-3}\text{ J}}$$

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(20.0\text{ N/m})(0.020\,0\text{ m})^2 = \boxed{4.00 \times 10^{-3}\text{ J}}$$

Note that $K + U = E$.

Exercise For what values of x is the speed of the cube 0.100 m/s?

Answer ± 2.55 cm.

13.4 THE PENDULUM

8.11 & 8.12

The **simple pendulum** is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end, as shown in Figure 13.12. The motion occurs in the vertical plane and is driven by the force of gravity. We shall show that, provided the angle θ is small (less than about 10°), the motion is that of a simple harmonic oscillator.

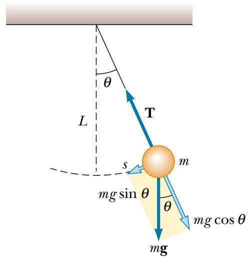


Figure 13.12 When θ is small, a simple pendulum oscillates in simple harmonic motion about the equilibrium position $\theta = 0$. The restoring force is $mg\sin\theta$, the component of the gravitational force tangent to the arc.

The forces acting on the bob are the force **T** exerted by the string and the gravitational force $m\mathbf{g}$. The tangential component of the gravitational force, $mg\sin\theta$, always acts toward $\theta = 0$, opposite the displacement. Therefore, the tangential force is a restoring force, and we can apply Newton's second law for motion in the tangential direction:

$$\sum F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$$

where s is the bob's displacement measured along the arc and the minus sign indicates that the tangential force acts toward the equilibrium (vertical) position. Because $s = L\theta$ (Eq. 10.1a) and L is constant, this equation reduces to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

The right side is proportional to $\sin\theta$ rather than to θ; hence, with $\sin\theta$ present, we would not expect simple harmonic motion because this expression is not of the form of Equation 13.17. However, if we assume that θ is small, we can use the approximation $\sin\theta \approx \theta$; thus the equation of motion for the simple pen-

The motion of a simple pendulum, captured with multiflash photography. Is the oscillating motion simple harmonic in this case?

dulum becomes

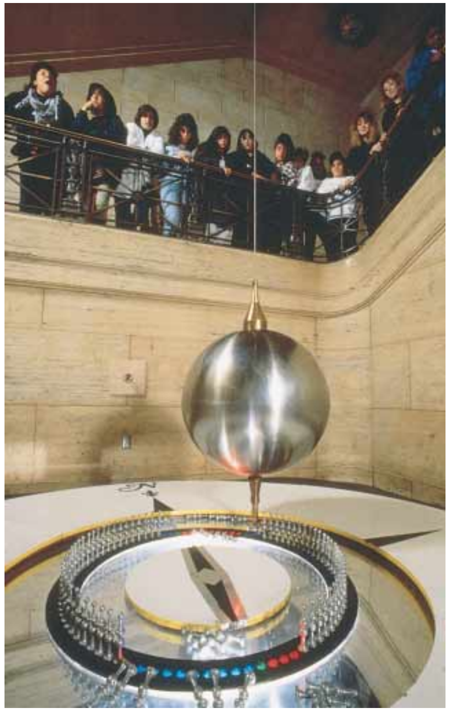
$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad \textbf{(13.24)}$$

Equation of motion for a simple pendulum (small θ)

Now we have an expression of the same form as Equation 13.17, and we conclude that the motion for small amplitudes of oscillation is simple harmonic motion. Therefore, θ can be written as $\theta = \theta_{max}\cos(\omega t + \phi)$, where θ_{max} is the *maximum angular displacement* and the angular frequency ω is

$$\omega = \sqrt{\frac{g}{L}} \quad \textbf{(13.25)}$$

Angular frequency of motion for a simple pendulum



The Foucault pendulum at the Franklin Institute in Philadelphia. This type of pendulum was first used by the French physicist Jean Foucault to verify the Earth's rotation experimentally. As the pendulum swings, the vertical plane in which it oscillates appears to rotate as the bob successively knocks over the indicators arranged in a circle on the floor. In reality, the plane of oscillation is fixed in space, and the Earth rotating beneath the swinging pendulum moves the indicators into position to be knocked down, one after the other.

The period of the motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \quad \textbf{(13.26)}$$

Period of motion for a simple pendulum

In other words, **the period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity.** Because the period is independent of the mass, we conclude that all simple pendulums that are of equal length and are at the same location (so that g is constant) oscillate with the same period. The analogy between the motion of a simple pendulum and that of a block–spring system is illustrated in Figure 13.10.

The simple pendulum can be used as a timekeeper because its period depends only on its length and the local value of g. It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important because variations in local values of g can provide information on the location of oil and of other valuable underground resources.

Quick Quiz 13.5

A block of mass m is first allowed to hang from a spring in static equilibrium. It stretches the spring a distance L beyond the spring's unstressed length. The block and spring are then set into oscillation. Is the period of this system less than, equal to, or greater than the period of a simple pendulum having a length L and a bob mass m?

EXAMPLE 13.5 A Connection Between Length and Time

Christian Huygens (1629–1695), the greatest clockmaker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1 s. How much shorter would our length unit be had his suggestion been followed?

Solution Solving Equation 13.26 for the length gives

$$L = \frac{T^2 g}{4\pi^2} = \frac{(1\ \text{s})^2(9.80\ \text{m/s}^2)}{4\pi^2} = 0.248\ \text{m}$$

Thus, the meter's length would be slightly less than one-fourth its current length. Note that the number of significant digits depends only on how precisely we know g because the time has been defined to be exactly 1 s.

QuickLab

Firmly hold a ruler so that about half of it is over the edge of your desk. With your other hand, pull down and then release the free end, watching how it vibrates. Now slide the ruler so that only about a quarter of it is free to vibrate. This time when you release it, how does the vibrational period compare with its earlier value? Why?

Physical Pendulum

Suppose you balance a wire coat hanger so that the hook is supported by your extended index finger. When you give the hanger a small displacement (with your other hand) and then release it, it oscillates. If a hanging object oscillates about a fixed axis that does not pass through its center of mass and the object cannot be approximated as a point mass, we cannot treat the system as a simple pendulum. In this case the system is called a **physical pendulum.**

Consider a rigid body pivoted at a point O that is a distance d from the center of mass (Fig. 13.13). The force of gravity provides a torque about an axis through O, and the magnitude of that torque is $mgd\sin\theta$, where θ is as shown in Figure 13.13. Using the law of motion $\Sigma\tau = I\alpha$, where I is the moment of inertia about the axis through O, we obtain

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

The minus sign indicates that the torque about O tends to decrease θ. That is, the force of gravity produces a restoring torque. Because this equation gives us the angular acceleration $d^2\theta/dt^2$ of the pivoted body, we can consider it the equation of motion for the system. If we again assume that θ is small, the approximation $\sin\theta \approx \theta$ is valid, and the equation of motion reduces to

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta \quad \textbf{(13.27)}$$

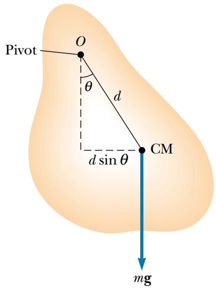


Figure 13.13 A physical pendulum.

Because this equation is of the same form as Equation 13.17, the motion is simple harmonic motion. That is, the solution of Equation 13.27 is $\theta = \theta_{max}\cos(\omega t + \phi)$, where θ_{max} is the maximum angular displacement and

$$\omega = \sqrt{\frac{mgd}{I}}$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} \quad \textbf{(13.28)}$$

Period of motion for a physical pendulum

One can use this result to measure the moment of inertia of a flat rigid body. If the location of the center of mass—and hence the value of d—are known, the moment of inertia can be obtained by measuring the period. Finally, note that Equation 13.28 reduces to the period of a simple pendulum (Eq. 13.26) when $I = md^2$—that is, when all the mass is concentrated at the center of mass.

EXAMPLE 13.6 A Swinging Rod

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane (Fig. 13.14). Find the period of oscillation if the amplitude of the motion is small.

Solution In Chapter 10 we found that the moment of inertia of a uniform rod about an axis through one end is $\frac{1}{3}ML^2$. The distance d from the pivot to the center of mass is $L/2$. Substituting these quantities into Equation 13.28 gives

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Comment In one of the Moon landings, an astronaut walking on the Moon's surface had a belt hanging from his space suit, and the belt oscillated as a physical pendulum. A scientist on the Earth observed this motion on television and used it to estimate the free-fall acceleration on the Moon. How did the scientist make this calculation?

Exercise Calculate the period of a meter stick that is pivoted about one end and is oscillating in a vertical plane.

Answer 1.64 s.

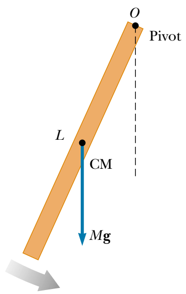


Figure 13.14 A rigid rod oscillating about a pivot through one end is a physical pendulum with $d = L/2$ and, from Table 10.2, $I = \frac{1}{3}ML^2$.

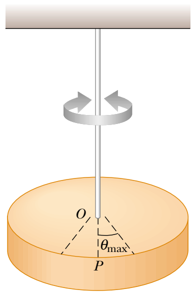


Figure 13.15 A torsional pendulum consists of a rigid body suspended by a wire attached to a rigid support. The body oscillates about the line OP with an amplitude θ_{max}.

Figure 13.16 The balance wheel of this antique pocket watch is a torsional pendulum and regulates the time-keeping mechanism.

Torsional Pendulum

Figure 13.15 shows a rigid body suspended by a wire attached at the top to a fixed support. When the body is twisted through some small angle θ, the twisted wire exerts on the body a restoring torque that is proportional to the angular displacement. That is,

$$\tau = -\kappa\theta$$

where κ (kappa) is called the *torsion constant* of the support wire. The value of κ can be obtained by applying a known torque to twist the wire through a measurable angle θ. Applying Newton's second law for rotational motion, we find

$$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \quad \textbf{(13.29)}$$

Again, this is the equation of motion for a simple harmonic oscillator, with $\omega = \sqrt{\kappa/I}$ and a period

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \quad \textbf{(13.30)}$$

Period of motion for a torsional pendulum

This system is called a *torsional pendulum.* There is no small-angle restriction in this situation as long as the elastic limit of the wire is not exceeded. Figure 13.16 shows the balance wheel of a watch oscillating as a torsional pendulum, energized by the mainspring.

13.5 COMPARING SIMPLE HARMONIC MOTION WITH UNIFORM CIRCULAR MOTION

8.8

We can better understand and visualize many aspects of simple harmonic motion by studying its relationship to uniform circular motion. Figure 13.17 is an overhead view of an experimental arrangement that shows this relationship. A ball is attached to the rim of a turntable of radius A, which is illuminated from the side by a lamp. The ball casts a shadow on a screen. We find that as the turntable rotates with constant angular speed, the shadow of the ball moves back and forth in simple harmonic motion.

Consider a particle located at point P on the circumference of a circle of radius A, as shown in Figure 13.18a, with the line OP making an angle ϕ with the x axis at $t = 0$. We call this circle a *reference circle* for comparing simple harmonic motion and uniform circular motion, and we take the position of P at $t = 0$ as our reference position. If the particle moves along the circle with constant angular speed ω until OP makes an angle θ with the x axis, as illustrated in Figure 13.18b, then at some time $t > 0$, the angle between OP and the x axis is $\theta = \omega t + \phi$. As the particle moves along the circle, the projection of P on the x axis, labeled point Q, moves back and forth along the x axis, between the limits $x = \pm A$.

Note that points P and Q always have the same x coordinate. From the right triangle OPQ, we see that this x coordinate is

$$x = A\cos(\omega t + \phi) \tag{13.31}$$

This expression shows that the point Q moves with simple harmonic motion along the x axis. Therefore, we conclude that

> simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.

We can make a similar argument by noting from Figure 13.18b that the projection of P along the y axis also exhibits simple harmonic motion. Therefore, **uniform circular motion can be considered a combination of two simple harmonic motions,** one along the x axis and one along the y axis, with the two differing in phase by 90°.

This geometric interpretation shows that the time for one complete revolution of the point P on the reference circle is equal to the period of motion T for simple harmonic motion between $x = \pm A$. That is, the angular speed ω of P is the same as the angular frequency ω of simple harmonic motion along the x axis (this is why we use the same symbol). The phase constant ϕ for simple harmonic motion corresponds to the initial angle that OP makes with the x axis. The radius A of the reference circle equals the amplitude of the simple harmonic motion.

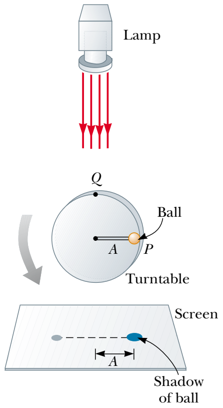


Figure 13.17 An experimental setup for demonstrating the connection between simple harmonic motion and uniform circular motion. As the ball rotates on the turntable with constant angular speed, its shadow on the screen moves back and forth in simple harmonic motion.

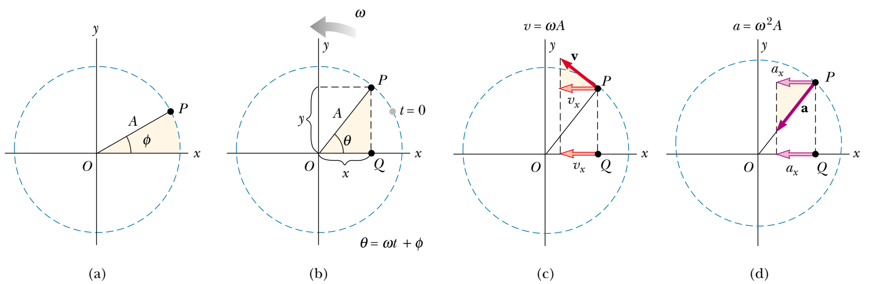


Figure 13.18 Relationship between the uniform circular motion of a point P and the simple harmonic motion of a point Q. A particle at P moves in a circle of radius A with constant angular speed ω. (a) A reference circle showing the position of P at $t = 0$. (b) The x coordinates of points P and Q are equal and vary in time as $x = A\cos(\omega t + \phi)$. (c) The x component of the velocity of P equals the velocity of Q. (d) The x component of the acceleration of P equals the acceleration of Q.

Because the relationship between linear and angular speed for circular motion is $v = r\omega$ (see Eq. 10.10), the particle moving on the reference circle of radius A has a velocity of magnitude ωA. From the geometry in Figure 13.18c, we see that the x component of this velocity is $-\omega A\sin(\omega t + \phi)$. By definition, the point Q has a velocity given by dx/dt. Differentiating Equation 13.31 with respect to time, we find that the velocity of Q is the same as the x component of the velocity of P.

The acceleration of P on the reference circle is directed radially inward toward O and has a magnitude $v^2/A = \omega^2 A$. From the geometry in Figure 13.18d, we see that the x component of this acceleration is $-\omega^2 A\cos(\omega t + \phi)$. This value is also the acceleration of the projected point Q along the x axis, as you can verify by taking the second derivative of Equation 13.31.

EXAMPLE 13.7 Circular Motion with Constant Angular Speed

A particle rotates counterclockwise in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. At $t = 0$, the particle has an x coordinate of 2.00 m and is moving to the right. (a) Determine the x coordinate as a function of time.

Solution Because the amplitude of the particle's motion equals the radius of the circle and $\omega = 8.00$ rad/s, we have

$$x = A\cos(\omega t + \phi) = (3.00\text{ m})\cos(8.00t + \phi)$$

We can evaluate ϕ by using the initial condition that $x = 2.00$ m at $t = 0$:

$$2.00\text{ m} = (3.00\text{ m})\cos(0 + \phi)$$

$$\phi = \cos^{-1}\left(\frac{2.00\text{ m}}{3.00\text{ m}}\right)$$

If we were to take our answer as $\phi = 48.2°$, then the coordinate $x = (3.00\text{ m})\cos(8.00t + 48.2°)$ would be decreasing at time $t = 0$ (that is, moving to the left). Because our particle is first moving to the right, we must choose $\phi = -48.2° = -0.841$ rad. The x coordinate as a function of time is then

$$x = (3.00\text{ m})\cos(8.00t - 0.841)$$

Note that ϕ in the cosine function must be in radians.

(b) Find the x components of the particle's velocity and acceleration at any time t.

Solution

$$v_x = \frac{dx}{dt} = (-3.00\text{ m})(8.00\text{ rad/s})\sin(8.00t - 0.841)$$

$$= -(24.0\text{ m/s})\sin(8.00t - 0.841)$$

$$a_x = \frac{dv_x}{dt} = (-24.0\text{ m/s})(8.00\text{ rad/s})\cos(8.00t - 0.841)$$

$$= -(192\text{ m/s}^2)\cos(8.00t - 0.841)$$

From these results, we conclude that $v_{max} = 24.0$ m/s and that $a_{max} = 192\text{ m/s}^2$. Note that these values also equal the tangential speed ωA and the centripetal acceleration $\omega^2 A$.

Optional Section

13.6 DAMPED OSCILLATIONS

The oscillatory motions we have considered so far have been for ideal systems—that is, systems that oscillate indefinitely under the action of a linear restoring force. In many real systems, dissipative forces, such as friction, retard the motion. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be *damped.*

One common type of retarding force is the one discussed in Section 6.4, where the force is proportional to the speed of the moving object and acts in the direction opposite the motion. This retarding force is often observed when an object moves through air, for instance. Because the retarding force can be expressed as $\mathbf{R} = -b\mathbf{v}$ (where b is a constant called the *damping coefficient*) and the restoring force of the system is $-kx$, we can write Newton's second law as

$$\sum F_x = -kx - bv = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \qquad \textbf{(13.32)}$$

The solution of this equation requires mathematics that may not be familiar to you yet; we simply state it here without proof. When the retarding force is small compared with the maximum restoring force—that is, when b is small—the solution to Equation 13.32 is

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi) \qquad \textbf{(13.33)}$$

where the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \qquad \textbf{(13.34)}$$

This result can be verified by substituting Equation 13.33 into Equation 13.32.

Figure 13.19a shows the displacement as a function of time for an object oscillating in the presence of a retarding force, and Figure 13.19b depicts one such system: a block attached to a spring and submersed in a viscous liquid. We see that **when the retarding force is much smaller than the restoring force, the oscillatory character of the motion is preserved but the amplitude decreases in time, with the result that the motion ultimately ceases.** Any system that behaves in this way is known as a **damped oscillator.** The dashed blue lines in Figure 13.19a, which define the *envelope* of the oscillatory curve, represent the exponential factor in Equation 13.33. This envelope shows that **the amplitude decays exponentially with time.** For motion with a given spring constant and block mass, the oscillations dampen more rapidly as the maximum value of the retarding force approaches the maximum value of the restoring force.

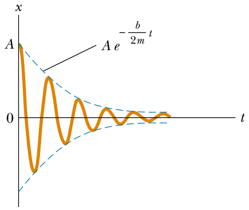
It is convenient to express the angular frequency of a damped oscillator in the form

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

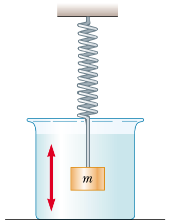
where $\omega_0 = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency** of the system. When the magnitude of the maximum retarding force $R_{max} = bv_{max} < kA$, the system is said to be **underdamped.** As the value of R approaches kA, the amplitudes of the oscillations decrease more and more rapidly. This motion is represented by the blue curve in Figure 13.20. When b reaches a critical value b_c such that $b_c/2m = \omega_0$, the system does not oscillate and is said to be **critically damped.** In this case the system, once released from rest at some nonequilibrium position, returns to equilibrium and then stays there. The graph of displacement versus time for this case is the red curve in Figure 13.20.

If the medium is so viscous that the retarding force is greater than the restoring force—that is, if $R_{max} = bv_{max} > kA$ and $b/2m > \omega_0$—the system is **overdamped.** Again, the displaced system, when free to move, does not oscillate but simply returns to its equilibrium position. As the damping increases, the time it takes the system to approach equilibrium also increases, as indicated by the black curve in Figure 13.20.

In any case in which friction is present, whether the system is overdamped or underdamped, the energy of the oscillator eventually falls to zero. The lost mechanical energy dissipates into internal energy in the retarding medium.



(a)



(b)

Figure 13.19 (a) Graph of displacement versus time for a damped oscillator. Note the decrease in amplitude with time. (b) One example of a damped oscillator is a mass attached to a spring and submersed in a viscous liquid.

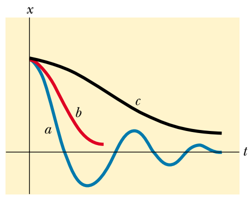
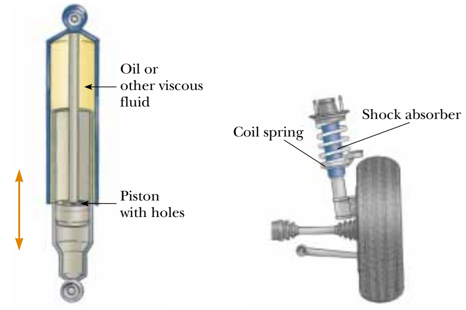


Figure 13.20 Graphs of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.



(a) (b)

Figure 13.21 (a) A shock absorber consists of a piston oscillating in a chamber filled with oil. As the piston oscillates, the oil is squeezed through holes between the piston and the chamber, causing a damping of the piston's oscillations. (b) One type of automotive suspension system, in which a shock absorber is placed inside a coil spring at each wheel.

web
To learn more about shock absorbers, visit
http://www.hdridecontrol.com

Quick Quiz 13.6

An automotive suspension system consists of a combination of springs and shock absorbers, as shown in Figure 13.21. If you were an automotive engineer, would you design a suspension system that was underdamped, critically damped, or overdamped? Discuss each case.

Optional Section

13.7 FORCED OSCILLATIONS

It is possible to compensate for energy loss in a damped system by applying an external force that does positive work on the system. At any instant, energy can be put into the system by an applied force that acts in the direction of motion of the oscillator. For example, a child on a swing can be kept in motion by appropriately timed pushes. The amplitude of motion remains constant if the energy input per cycle exactly equals the energy lost as a result of damping. Any motion of this type is called **forced oscillation.**

A common example of a forced oscillator is a damped oscillator driven by an external force that varies periodically, such as $F = F_{ext}\cos\omega t$, where ω is the angular frequency of the periodic force and F_{ext} is a constant. Adding this driving force to the left side of Equation 13.32 gives

$$F_{ext}\cos\omega t - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \qquad \textbf{(13.35)}$$

(As earlier, we present the solution of this equation without proof.) After a sufficiently long period of time, when the energy input per cycle equals the energy lost per cycle, a steady-state condition is reached in which the oscillations proceed with constant amplitude. At this time, when the system is in a steady state, the solution of Equation 13.35 is

$$x = A\cos(\omega t + \phi) \qquad \textbf{(13.36)}$$

where

$$A = \frac{F_{\text{ext}}/m}{\sqrt{(\omega^2 - \omega_0{}^2)^2 + \left(\dfrac{b\omega}{m}\right)^2}} \tag{13.37}$$

and where $\omega_0 = \sqrt{k/m}$ is the angular frequency of the undamped oscillator $(b = 0)$. One could argue that in steady state the oscillator must physically have the same frequency as the driving force, and thus the solution given by Equation 13.36 is expected. In fact, when this solution is substituted into Equation 13.35, one finds that it is indeed a solution, provided the amplitude is given by Equation 13.37.

Equation 13.37 shows that, because an external force is driving it, the motion of the forced oscillator is not damped. The external agent provides the necessary energy to overcome the losses due to the retarding force. Note that the system oscillates at the angular frequency ω of the driving force. For small damping, the amplitude becomes very large when the frequency of the driving force is near the natural frequency of oscillation. The dramatic increase in amplitude near the natural frequency ω_0 is called **resonance,** and for this reason ω_0 is sometimes called the **resonance frequency** of the system.

The reason for large-amplitude oscillations at the resonance frequency is that energy is being transferred to the system under the most favorable conditions. We can better understand this by taking the first time derivative of x in Equation 13.36, which gives an expression for the velocity of the oscillator. We find that v is proportional to $\sin(\omega t + \phi)$. When the applied force $\mathbf{F}$ is in phase with the velocity, the rate at which work is done on the oscillator by $\mathbf{F}$ equals the dot product $\mathbf{F} \cdot \mathbf{v}$. Remember that "rate at which work is done" is the definition of power. Because the product $\mathbf{F} \cdot \mathbf{v}$ is a maximum when $\mathbf{F}$ and $\mathbf{v}$ are in phase, we conclude that **at resonance the applied force is in phase with the velocity and that the power transferred to the oscillator is a maximum.**

Figure 13.22 is a graph of amplitude as a function of frequency for a forced oscillator with and without damping. Note that the amplitude increases with decreasing damping $(b \rightarrow 0)$ and that the resonance curve broadens as the damping increases. Under steady-state conditions and at any driving frequency, the energy transferred into the system equals the energy lost because of the damping force; hence, the average total energy of the oscillator remains constant. In the absence of a damping force $(b = 0)$, we see from Equation 13.37 that the steady-state amplitude approaches infinity as $\omega \rightarrow \omega_0$. In other words, if there are no losses in the system and if we continue to drive an initially motionless oscillator with a periodic force that is in phase with the velocity, the amplitude of motion builds without limit (see the red curve in Fig. 13.22). This limitless building does not occur in practice because some damping is always present.

The behavior of a driven oscillating system after the driving force is removed depends on b and on how close ω was to ω_0. This behavior is sometimes quantified by a parameter called the *quality factor* Q. The closer a system is to being undamped, the greater its Q. The amplitude of oscillation drops by a factor of e $(= 2.718\ .\ .\ .)$ in Q/π cycles.

Later in this book we shall see that resonance appears in other areas of physics. For example, certain electrical circuits have natural frequencies. A bridge has natural frequencies that can be set into resonance by an appropriate driving force. A dramatic example of such resonance occurred in 1940, when the Tacoma Narrows Bridge in the state of Washington was destroyed by resonant vibrations. Although the winds were not particularly strong on that occasion, the bridge ultimately collapsed (Fig. 13.23) because the bridge design had no built-in safety features.

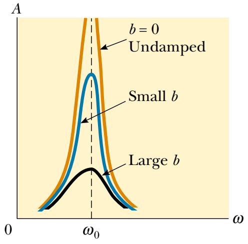
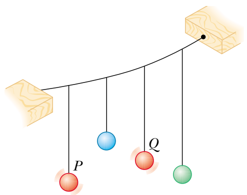


Figure 13.22 Graph of amplitude versus frequency for a damped oscillator when a periodic driving force is present. When the frequency of the driving force equals the natural frequency ω_0, resonance occurs. Note that the shape of the resonance curve depends on the size of the damping coefficient b.

QuickLab

Tie several objects to strings and suspend them from a horizontal string, as illustrated in the figure. Make two of the hanging strings approximately the same length. If one of this pair, such as P, is set into sideways motion, all the others begin to oscillate. But Q, whose length is the same as that of P, oscillates with the greatest amplitude. Must all the masses have the same value?



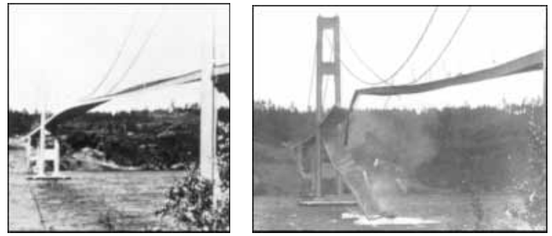


Figure 13.23 (a) In 1940 turbulent winds set up torsional vibrations in the Tacoma Narrows Bridge, causing it to oscillate at a frequency near one of the natural frequencies of the bridge structure. (b) Once established, this resonance condition led to the bridge's collapse.

Many other examples of resonant vibrations can be cited. A resonant vibration that you may have experienced is the "singing" of telephone wires in the wind. Machines often break if one vibrating part is at resonance with some other moving part. Soldiers marching in cadence across a bridge have been known to set up resonant vibrations in the structure and thereby cause it to collapse. Whenever any real physical system is driven near its resonance frequency, you can expect oscillations of very large amplitudes.

SUMMARY

When the acceleration of an object is proportional to its displacement from some equilibrium position and is in the direction opposite the displacement, the object moves with simple harmonic motion. The position x of a simple harmonic oscillator varies periodically in time according to the expression

$$x = A\cos(\omega t + \phi) \tag{13.3}$$

where A is the **amplitude** of the motion, ω is the **angular frequency,** and ϕ is the **phase constant.** The value of ϕ depends on the initial position and initial velocity of the oscillator. You should be able to use this formula to describe the motion of an object undergoing simple harmonic motion.

The time T needed for one complete oscillation is defined as the **period** of the motion:

$$T = \frac{2\pi}{\omega} \tag{13.4}$$

The inverse of the period is the **frequency** of the motion, which equals the number of oscillations per second.

The velocity and acceleration of a simple harmonic oscillator are

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \tag{13.7}$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) \tag{13.8}$$

$$v = \pm\omega\sqrt{A^2 - x^2} \tag{13.23}$$

Thus, the maximum speed is ωA, and the maximum acceleration is $\omega^2 A$. The speed is zero when the oscillator is at its turning points, $x = \pm A$, and is a maximum when the oscillator is at the equilibrium position $x = 0$. The magnitude of the acceleration is a maximum at the turning points and zero at the equilibrium position. You should be able to find the velocity and acceleration of an oscillating object at any time if you know the amplitude, angular frequency, and phase constant.

A block–spring system moves in simple harmonic motion on a frictionless surface, with a period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \textbf{(13.18)}$$

The kinetic energy and potential energy for a simple harmonic oscillator vary with time and are given by

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi) \qquad \textbf{(13.20)}$$

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t + \phi) \qquad \textbf{(13.21)}$$

These three formulas allow you to analyze a wide variety of situations involving oscillations. Be sure you recognize how the mass of the block and the spring constant of the spring enter into the calculations.

The total energy of a simple harmonic oscillator is a constant of the motion and is given by

$$E = \tfrac{1}{2}kA^2 \qquad \textbf{(13.22)}$$

The potential energy of the oscillator is a maximum when the oscillator is at its turning points and is zero when the oscillator is at the equilibrium position. The kinetic energy is zero at the turning points and a maximum at the equilibrium position. You should be able to determine the division of energy between potential and kinetic forms at any time t.

A **simple pendulum** of length L moves in simple harmonic motion. For small angular displacements from the vertical, its period is

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \textbf{(13.26)}$$

For small angular displacements from the vertical, a **physical pendulum** moves in simple harmonic motion about a pivot that does not go through the center of mass. The period of this motion is

$$T = 2\pi\sqrt{\frac{I}{mgd}} \qquad \textbf{(13.28)}$$

where I is the moment of inertia about an axis through the pivot and d is the distance from the pivot to the center of mass. You should be able to distinguish when to use the simple-pendulum formula and when the system must be considered a physical pendulum.

Uniform circular motion can be considered a combination of two simple harmonic motions, one along the x axis and the other along the y axis, with the two differing in phase by 90°.

QUESTIONS

1. Is a bouncing ball an example of simple harmonic motion? Is the daily movement of a student from home to school and back simple harmonic motion? Why or why not?

2. If the coordinate of a particle varies as $x = -A\cos\omega t$, what is the phase constant in Equation 13.3? At what position does the particle begin its motion?

3. Does the displacement of an oscillating particle between $t = 0$ and a later time t necessarily equal the position of the particle at time t? Explain.

4. Determine whether the following quantities can be in the same direction for a simple harmonic oscillator: (a) displacement and velocity, (b) velocity and acceleration, (c) displacement and acceleration.

5. Can the amplitude A and the phase constant ϕ be determined for an oscillator if only the position is specified at $t = 0$? Explain.

6. Describe qualitatively the motion of a mass–spring system when the mass of the spring is not neglected.

7. Make a graph showing the potential energy of a stationary block hanging from a spring, $U = \frac{1}{2}ky^2 + mgy$. Why is the lowest part of the graph offset from the origin?

8. A block–spring system undergoes simple harmonic motion with an amplitude A. Does the total energy change if the mass is doubled but the amplitude is not changed? Do the kinetic and potential energies depend on the mass? Explain.

9. What happens to the period of a simple pendulum if the pendulum's length is doubled? What happens to the period if the mass of the suspended bob is doubled?

10. A simple pendulum is suspended from the ceiling of a stationary elevator, and the period is determined. Describe the changes, if any, in the period when the elevator (a) accelerates upward, (b) accelerates downward, and (c) moves with constant velocity.

11. A simple pendulum undergoes simple harmonic motion when θ is small. Is the motion periodic when θ is large? How does the period of motion change as θ increases?

12. Will damped oscillations occur for any values of b and k? Explain.

13. As it possible to have damped oscillations when a system is at resonance? Explain.

14. At resonance, what does the phase constant ϕ equal in Equation 13.36? (*Hint:* Compare this equation with the expression for the driving force, which must be in phase with the velocity at resonance.)

15. Some parachutes have holes in them to allow air to move smoothly through them. Without such holes, sometimes the air that has gathered beneath the chute as a parachutist falls is released from under its edges alternately and periodically, at one side and then at the other. Why might this periodic release of air cause a problem?

16. If a grandfather clock were running slowly, how could we adjust the length of the pendulum to correct the time?

17. A pendulum bob is made from a sphere filled with water. What would happen to the frequency of vibration of this pendulum if the sphere had a hole in it that allowed the water to leak out slowly?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 13.1 Simple Harmonic Motion

1. The displacement of a particle at $t = 0.250$ s is given by the expression $x = (4.00\text{ m})\cos(3.00\pi t + \pi)$, where x is in meters and t is in seconds. Determine (a) the frequency and period of the motion, (b) the amplitude of the motion, (c) the phase constant, and (d) the displacement of the particle at $t = 0.250$ s.

2. A ball dropped from a height of 4.00 m makes a perfectly elastic collision with the ground. Assuming that no energy is lost due to air resistance, (a) show that the motion is periodic and (b) determine the period of the motion. (c) Is the motion simple harmonic? Explain.

3. A particle moves in simple harmonic motion with a frequency of 3.00 oscillations/s and an amplitude of 5.00 cm. (a) Through what total distance does the particle move during one cycle of its motion? (b) What is its maximum speed? Where does this occur? (c) Find the maximum acceleration of the particle. Where in the motion does the maximum acceleration occur?

4. In an engine, a piston oscillates with simple harmonic motion so that its displacement varies according to the expression

$$x = (5.00\text{ cm})\cos(2t + \pi/6)$$

where x is in centimeters and t is in seconds. At $t = 0$, find (a) the displacement of the particle, (b) its velocity, and (c) its acceleration. (d) Find the period and amplitude of the motion.

WEB **5.** A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is 2.00 cm, and the frequency is 1.50 Hz. (a) Show that the displacement of the particle is given by $x = (2.00\text{ cm})\sin(3.00\pi t)$. Determine (b) the maximum speed and the earliest time ($t > 0$) at which the particle has this speed, (c) the maximum acceleration and the earliest time ($t > 0$) at which the particle has this acceleration, and (d) the total distance traveled between $t = 0$ and $t = 1.00$ s.

6. The initial position and initial velocity of an object moving in simple harmonic motion are x_i and v_i; the angular frequency of oscillation is ω. (a) Show that the position and velocity of the object for all time can be written as

$$x(t) = x_i\cos\omega t + \left(\frac{v_i}{\omega}\right)\sin\omega t$$

$$v(t) = -x_i\omega\sin\omega t + v_i\cos\omega t$$

(b) If the amplitude of the motion is A, show that

$$v^2 - ax = v_i^2 - a_ix_i = \omega^2A^2$$

Section 13.2 The Block–Spring System Revisited

Note: Neglect the mass of the spring in all problems in this section.

7. A spring stretches by 3.90 cm when a 10.0-g mass is hung from it. If a 25.0-g mass attached to this spring oscillates in simple harmonic motion, calculate the period of the motion.

8. A simple harmonic oscillator takes 12.0 s to undergo five complete vibrations. Find (a) the period of its motion, (b) the frequency in hertz, and (c) the angular frequency in radians per second.

9. A 0.500-kg mass attached to a spring with a force constant of 8.00 N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm. Calculate (a) the maximum value of its speed and acceleration, (b) the speed and acceleration when the mass is 6.00 cm from the equilibrium position, and (c) the time it takes the mass to move from $x = 0$ to $x = 8.00$ cm.

10. A 1.00-kg mass attached to a spring with a force constant of 25.0 N/m oscillates on a horizontal, frictionless track. At $t = 0$, the mass is released from rest at $x = -3.00$ cm. (That is, the spring is compressed by 3.00 cm.) Find (a) the period of its motion; (b) the maximum values of its speed and acceleration; and (c) the displacement, velocity, and acceleration as functions of time.

11. A 7.00-kg mass is hung from the bottom end of a vertical spring fastened to an overhead beam. The mass is set into vertical oscillations with a period of 2.60 s. Find the force constant of the spring.

12. A block of unknown mass is attached to a spring with a spring constant of 6.50 N/m and undergoes simple harmonic motion with an amplitude of 10.0 cm. When the mass is halfway between its equilibrium position and the end point, its speed is measured to be +30.0 cm/s. Calculate (a) the mass of the block, (b) the period of the motion, and (c) the maximum acceleration of the block.

13. A particle that hangs from a spring oscillates with an angular frequency of 2.00 rad/s. The spring–particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed of 1.50 m/s. The car then stops suddenly. (a) With what amplitude does the particle oscillate? (b) What is the equation of motion for the particle? (Choose upward as the positive direction.)

14. A particle that hangs from a spring oscillates with an angular frequency ω. The spring–particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed v. The car then stops suddenly. (a) With what amplitude does the particle oscillate? (b) What is the equation of motion for the particle? (Choose upward as the positive direction.)

15. A 1.00-kg mass is attached to a horizontal spring. The spring is initially stretched by 0.100 m, and the mass is released from rest there. It proceeds to move without friction. After 0.500 s, the speed of the mass is zero. What is the maximum speed of the mass?

Section 13.3 Energy of the Simple Harmonic Oscillator

Note: Neglect the mass of the spring in all problems in this section.

16. A 200-g mass is attached to a spring and undergoes simple harmonic motion with a period of 0.250 s. If the total energy of the system is 2.00 J, find (a) the force constant of the spring and (b) the amplitude of the motion.

WEB **17.** An automobile having a mass of 1 000 kg is driven into a brick wall in a safety test. The bumper behaves as a spring of constant 5.00×10^6 N/m and compresses 3.16 cm as the car is brought to rest. What was the speed of the car before impact, assuming that no energy is lost during impact with the wall?

18. A mass–spring system oscillates with an amplitude of 3.50 cm. If the spring constant is 250 N/m and the mass is 0.500 kg, determine (a) the mechanical energy of the system, (b) the maximum speed of the mass, and (c) the maximum acceleration.

19. A 50.0-g mass connected to a spring with a force constant of 35.0 N/m oscillates on a horizontal, frictionless surface with an amplitude of 4.00 cm. Find (a) the total energy of the system and (b) the speed of the mass when the displacement is 1.00 cm. Find (c) the kinetic energy and (d) the potential energy when the displacement is 3.00 cm.

20. A 2.00-kg mass is attached to a spring and placed on a horizontal, smooth surface. A horizontal force of 20.0 N is required to hold the mass at rest when it is pulled 0.200 m from its equilibrium position (the origin of the x axis). The mass is now released from rest with an initial displacement of $x_i = 0.200$ m, and it subsequently undergoes simple harmonic oscillations. Find (a) the force constant of the spring, (b) the frequency of the oscillations, and (c) the maximum speed of the mass. Where does this maximum speed occur? (d) Find the maximum acceleration of the mass. Where does it occur? (e) Find the total energy of the oscillating system. Find (f) the speed and (g) the acceleration when the displacement equals one third of the maximum value.

21. A 1.50-kg block at rest on a tabletop is attached to a horizontal spring having force constant of 19.6 N/m. The spring is initially unstretched. A constant 20.0-N horizontal force is applied to the object, causing the spring to stretch. (a) Determine the speed of the block after it has moved 0.300 m from equilibrium, assuming that the surface between the block and the tabletop is frictionless. (b) Answer part (a) for a coefficient of kinetic friction of 0.200 between the block and the tabletop.

22. The amplitude of a system moving in simple harmonic motion is doubled. Determine the change in (a) the total energy, (b) the maximum speed, (c) the maximum acceleration, and (d) the period.

23. A particle executes simple harmonic motion with an amplitude of 3.00 cm. At what displacement from the midpoint of its motion does its speed equal one half of its maximum speed?

24. A mass on a spring with a constant of 3.24 N/m vibrates, with its position given by the equation $x = (5.00\text{ cm})\cos(3.60t\text{ rad/s})$. (a) During the first cycle, for $0 < t < 1.75$ s, when is the potential energy of the system changing most rapidly into kinetic energy? (b) What is the maximum rate of energy transformation?

Section 13.4 The Pendulum

25. A man enters a tall tower, needing to know its height. He notes that a long pendulum extends from the ceiling almost to the floor and that its period is 12.0 s. (a) How tall is the tower? (b) If this pendulum is taken to the Moon, where the free-fall acceleration is 1.67 m/s^2, what is its period there?

26. A "seconds" pendulum is one that moves through its equilibrium position once each second. (The period of the pendulum is 2.000 s.) The length of a seconds pendulum is 0.992 7 m at Tokyo and 0.994 2 m at Cambridge, England. What is the ratio of the free-fall accelerations at these two locations?

27. A rigid steel frame above a street intersection supports standard traffic lights, each of which is hinged to hang immediately below the frame. A gust of wind sets a light swinging in a vertical plane. Find the order of magnitude of its period. State the quantities you take as data and their values.

28. The angular displacement of a pendulum is represented by the equation $\theta = (0.320\text{ rad})\cos\omega t$, where θ is in radians and $\omega = 4.43$ rad/s. Determine the period and length of the pendulum.

WEB **29.** A simple pendulum has a mass of 0.250 kg and a length of 1.00 m. It is displaced through an angle of 15.0° and then released. What are (a) the maximum speed, (b) the maximum angular acceleration, and (c) the maximum restoring force?

30. A simple pendulum is 5.00 m long. (a) What is the period of simple harmonic motion for this pendulum if it is hanging in an elevator that is accelerating upward at 5.00 m/s^2? (b) What is its period if the elevator is accelerating downward at 5.00 m/s^2? (c) What is the period of simple harmonic motion for this pendulum if it is placed in a truck that is accelerating horizontally at 5.00 m/s^2?

31. A particle of mass m slides without friction inside a hemispherical bowl of radius R. Show that, if it starts from rest with a small displacement from equilibrium, the particle moves in simple harmonic motion with an angular frequency equal to that of a simple pendulum of length R. That is, $\omega = \sqrt{g/R}$.

32. A mass is attached to the end of a string to form a simple pendulum. The period of its harmonic motion is measured for small angular displacements and three lengths; in each case, the motion is clocked with a stopwatch for 50 oscillations. For lengths of 1.000 m, 0.750 m, and 0.500 m, total times of 99.8 s, 86.6 s, and 71.1 s, respectively, are measured for the 50 oscillations. (a) Determine the period of motion for each length. (b) Determine the mean value of g obtained from these three independent measurements, and compare it with the accepted value. (c) Plot T^2 versus L, and obtain a value for g from the slope of your best-fit straight-line graph. Compare this value with that obtained in part (b).

33. A physical pendulum in the form of a planar body moves in simple harmonic motion with a frequency of 0.450 Hz. If the pendulum has a mass of 2.20 kg and the pivot is located 0.350 m from the center of mass, determine the moment of inertia of the pendulum.

34. A very light, rigid rod with a length of 0.500 m extends straight out from one end of a meter stick. The stick is suspended from a pivot at the far end of the rod and is set into oscillation. (a) Determine the period of oscillation. (b) By what percentage does this differ from a 1.00-m-long simple pendulum?

35. Consider the physical pendulum of Figure 13.13. (a) If I_{CM} is its moment of inertia about an axis that passes through its center of mass and is parallel to the axis that passes through its pivot point, show that its period is

$$T = 2\pi\sqrt{\frac{I_{CM} + md^2}{mgd}}$$

where d is the distance between the pivot point and the center of mass. (b) Show that the period has a minimum value when d satisfies $md^2 = I_{CM}$.

36. A torsional pendulum is formed by attaching a wire to the center of a meter stick with a mass of 2.00 kg. If the resulting period is 3.00 min, what is the torsion constant for the wire?

37. A clock balance wheel has a period of oscillation of 0.250 s. The wheel is constructed so that 20.0 g of mass is concentrated around a rim of radius 0.500 cm. What are (a) the wheel's moment of inertia and (b) the torsion constant of the attached spring?

Section 13.5 Comparing Simple Harmonic Motion with Uniform Circular Motion

38. While riding behind a car that is traveling at 3.00 m/s, you notice that one of the car's tires has a small hemispherical boss on its rim, as shown in Figure P13.38. (a) Explain why the boss, from your viewpoint behind the car, executes simple harmonic motion. (b) If the radius of the car's tires is 0.300 m, what is the boss's period of oscillation?

39. Consider the simplified single-piston engine shown in Figure P13.39. If the wheel rotates with constant angular speed, explain why the piston rod oscillates in simple harmonic motion.

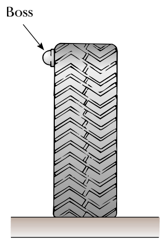


Figure P13.38

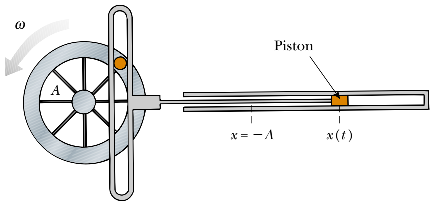


Figure P13.39

(Optional)
Section 13.6 Damped Oscillations

40. Show that the time rate of change of mechanical energy for a damped, undriven oscillator is given by $dE/dt = -bv^2$ and hence is always negative. (*Hint:* Differentiate the expression for the mechanical energy of an oscillator, $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, and use Eq. 13.32.)

41. A pendulum with a length of 1.00 m is released from an initial angle of 15.0°. After 1 000 s, its amplitude is reduced by friction to 5.50°. What is the value of $b/2m$?

42. Show that Equation 13.33 is a solution of Equation 13.32 provided that $b^2 < 4mk$.

(Optional)
Section 13.7 Forced Oscillations

43. A baby rejoices in the day by crowing and jumping up and down in her crib. Her mass is 12.5 kg, and the crib mattress can be modeled as a light spring with a force constant of 4.30 kN/m. (a) The baby soon learns to bounce with maximum amplitude and minimum effort by bending her knees at what frequency? (b) She learns to use the mattress as a trampoline—losing contact with it for part of each cycle—when her amplitude exceeds what value?

44. A 2.00-kg mass attached to a spring is driven by an external force $F = (3.00\text{ N})\cos(2\pi t)$. If the force constant of the spring is 20.0 N/m, determine (a) the period and (b) the amplitude of the motion. (*Hint:* Assume that there is no damping—that is, that $b = 0$—and use Eq. 13.37.)

45. Considering an *undamped,* forced oscillator ($b = 0$), show that Equation 13.36 is a solution of Equation 13.35, with an amplitude given by Equation 13.37.

46. A weight of 40.0 N is suspended from a spring that has a force constant of 200 N/m. The system is undamped and is subjected to a harmonic force with a frequency of 10.0 Hz, which results in a forced-motion amplitude of 2.00 cm. Determine the maximum value of the force.

47. Damping is negligible for a 0.150-kg mass hanging from a light 6.30-N/m spring. The system is driven by a force oscillating with an amplitude of 1.70 N. At what frequency will the force make the mass vibrate with an amplitude of 0.440 m?

48. You are a research biologist. Before dining at a fine restaurant, you set your pager to vibrate instead of beep, and you place it in the side pocket of your suit coat. The arm of your chair presses the light cloth against your body at one spot. Fabric with a length of 8.21 cm hangs freely below that spot, with the pager at the bottom. A co-worker telephones you. The motion of the vibrating pager makes the hanging part of your coat swing back and forth with remarkably large amplitude. The waiter, maître d', wine steward, and nearby diners notice immediately and fall silent. Your daughter pipes up and says, "Daddy, look! Your cockroaches must have gotten out again!" Find the frequency at which your pager vibrates.

ADDITIONAL PROBLEMS

49. A car with bad shock absorbers bounces up and down with a period of 1.50 s after hitting a bump. The car has a mass of 1 500 kg and is supported by four springs of equal force constant k. Determine the value of k.

50. A large passenger with a mass of 150 kg sits in the middle of the car described in Problem 49. What is the new period of oscillation?

51. A compact mass M is attached to the end of a uniform rod, of equal mass M and length L, that is pivoted at the top (Fig. P13.51). (a) Determine the tensions in the rod at the pivot and at the point P when the system is stationary. (b) Calculate the period of oscillation for small displacements from equilibrium, and determine this period for $L = 2.00$ m. (*Hint:* Assume that the mass at the end of the rod is a point mass, and use Eq. 13.28.)

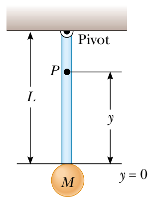


Figure P13.51

52. A mass, $m_1 = 9.00$ kg, is in equilibrium while connected to a light spring of constant $k = 100$ N/m that is fastened to a wall, as shown in Figure P13.52a. A second mass, $m_2 = 7.00$ kg, is slowly pushed up against mass m_1, compressing the spring by the amount $A = 0.200$ m (see Fig. P13.52b). The system is then released, and both masses start moving to the right on the frictionless surface. (a) When m_1 reaches the equilibrium point, m_2 loses contact with m_1 (see Fig. P13.52c) and moves to the right with speed v. Determine the value of v. (b) How far apart are the masses when the spring is fully stretched for the first time (D in Fig. P13.52d)? (*Hint:* First determine the period of oscillation and the amplitude of the m_1–spring system after m_2 loses contact with m_1.)

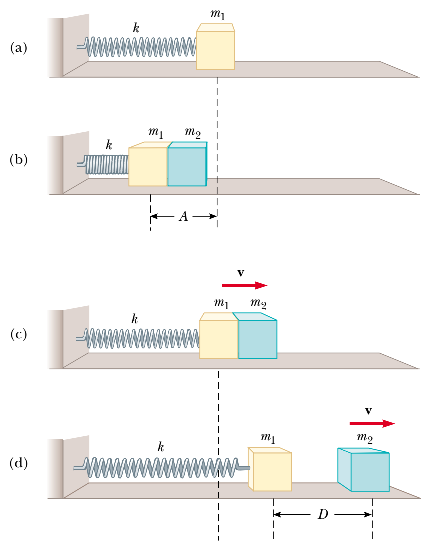


Figure P13.52

WEB **53.** A large block P executes horizontal simple harmonic motion as it slides across a frictionless surface with a frequency of $f = 1.50$ Hz. Block B rests on it, as shown in Figure P13.53, and the coefficient of static friction between the two is $\mu_s = 0.600$. What maximum amplitude of oscillation can the system have if block B is not to slip?

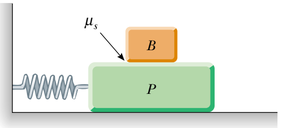


Figure P13.53 Problems 53 and 54.

54. A large block P executes horizontal simple harmonic motion as it slides across a frictionless surface with a frequency f. Block B rests on it, as shown in Figure P13.53, and the coefficient of static friction between the two is μ_s. What maximum amplitude of oscillation can the system have if the upper block is not to slip?

55. The mass of the deuterium molecule (D_2) is twice that of the hydrogen molecule (H_2). If the vibrational frequency of H_2 is 1.30×10^{14} Hz, what is the vibrational frequency of D_2? Assume that the "spring constant" of attracting forces is the same for the two molecules.

56. A solid sphere (radius = R) rolls without slipping in a cylindrical trough (radius = $5R$), as shown in Figure P13.56. Show that, for small displacements from equilibrium perpendicular to the length of the trough, the sphere executes simple harmonic motion with a period $T = 2\pi\sqrt{28R/5g}$.

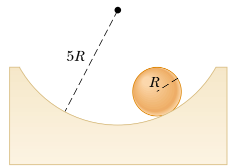
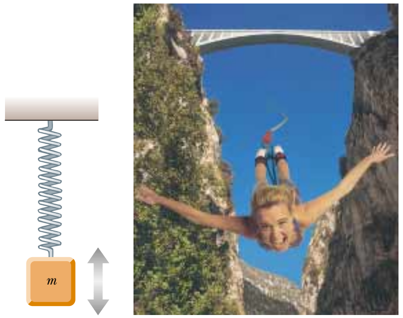


Figure P13.56

57. A light cubical container of volume a^3 is initially filled with a liquid of mass density ρ. The container is initially supported by a light string to form a pendulum of length L_i, measured from the center of mass of the filled container. The liquid is allowed to flow from the bottom of the container at a constant rate (dM/dt). At any time t, the level of the liquid in the container is h

and the length of the pendulum is L (measured relative to the instantaneous center of mass). (a) Sketch the apparatus and label the dimensions a, h, L_i, and L. (b) Find the time rate of change of the period as a function of time t. (c) Find the period as a function of time.

58. After a thrilling plunge, bungee-jumpers bounce freely on the bungee cords through many cycles. Your little brother can make a pest of himself by figuring out the mass of each person, using a proportion he set up by solving this problem: A mass m is oscillating freely on a vertical spring with a period T (Fig. P13.58a). An unknown mass m' on the same spring oscillates with a period T'. Determine (a) the spring constant k and (b) the unknown mass m'.



(a) (b)

Figure P13.58 (a) Mass–spring system for Problems 58 and 68. (b) Bungee-jumping from a bridge. *(Telegraph Colour Library/FPG International)*

59. A pendulum of length L and mass M has a spring of force constant k connected to it at a distance h below its point of suspension (Fig. P13.59). Find the frequency of vibration of the system for small values of the amplitude (small θ). (Assume that the vertical suspension of length L is rigid, but neglect its mass.)

60. A horizontal plank of mass m and length L is pivoted at one end. The plank's other end is supported by a spring of force constant k (Fig. P13.60). The moment of inertia of the plank about the pivot is $\frac{1}{3}mL^2$. (a) Show that the plank, after being displaced a small angle θ from its horizontal equilibrium position and released, moves with simple harmonic motion of angular frequency $\omega = \sqrt{3k/m}$. (b) Evaluate the frequency if the mass is 5.00 kg and the spring has a force constant of 100 N/m.

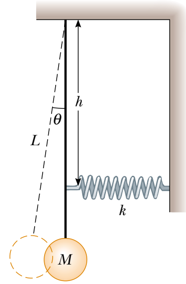


Figure P13.59

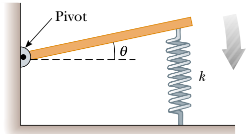


Figure P13.60

61. One end of a light spring with a force constant of 100 N/m is attached to a vertical wall. A light string is tied to the other end of the horizontal spring. The string changes from horizontal to vertical as it passes over a 4.00-cm-diameter solid pulley that is free to turn on a fixed smooth axle. The vertical section of the string supports a 200-g mass. The string does not slip at its contact with the pulley. Find the frequency of oscillation of the mass if the mass of the pulley is (a) negligible, (b) 250 g, and (c) 750 g.

62. A 2.00-kg block hangs without vibrating at the end of a spring ($k = 500$ N/m) that is attached to the ceiling of an elevator car. The car is rising with an upward acceleration of $g/3$ when the acceleration suddenly ceases (at $t = 0$). (a) What is the angular frequency of oscillation of the block after the acceleration ceases? (b) By what amount is the spring stretched during the acceleration of the elevator car? (c) What are the amplitude of the oscillation and the initial phase angle observed by a rider in the car? Take the upward direction to be positive.

63. A simple pendulum with a length of 2.23 m and a mass of 6.74 kg is given an initial speed of 2.06 m/s at its equilibrium position. Assume that it undergoes simple harmonic motion, and determine its (a) period, (b) total energy, and (c) maximum angular displacement.

64. People who ride motorcycles and bicycles learn to look out for bumps in the road and especially for *washboarding,* which is a condition of many equally spaced ridges worn into the road. What is so bad about washboarding? A motorcycle has several springs and shock absorbers in its suspension, but you can model it as a single spring supporting a mass. You can estimate the spring constant by thinking about how far the spring compresses when a big biker sits down on the seat. A motorcyclist traveling at highway speed must be particularly careful of washboard bumps that are a certain distance apart. What is the order of magnitude of their separation distance? State the quantities you take as data and the values you estimate or measure for them.

65. A wire is bent into the shape of one cycle of a cosine curve. It is held in a vertical plane so that the height y of the wire at any horizontal distance x from the center is given by $y = 20.0\text{ cm}[1 - \cos(0.160x \text{ rad/m})]$. A bead can slide without friction on the stationary wire. Show that if its excursion away from $x = 0$ is never large, the bead moves with simple harmonic motion. Determine its angular frequency. (*Hint:* $\cos\theta \cong 1 - \theta^2/2$ for small θ.)

66. A block of mass M is connected to a spring of mass m and oscillates in simple harmonic motion on a horizontal, frictionless track (Fig. P13.66). The force constant of the spring is k, and the equilibrium length is ℓ. Find (a) the kinetic energy of the system when the block has a speed v, and (b) the period of oscillation. (*Hint:* Assume that all portions of the spring oscillate in phase and that the velocity of a segment dx is proportional to the distance x from the fixed end; that is, $v_x = [x/\ell]v$. Also, note that the mass of a segment of the spring is $dm = [m/\ell]dx$.)

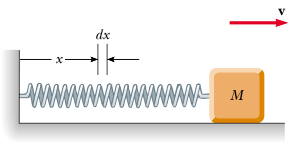


Figure P13.66

WEB **67.** A ball of mass m is connected to two rubber bands of length L, each under tension T, as in Figure P13.67. The ball is displaced by a small distance y perpendicular to the length of the rubber bands. Assuming that the tension does not change, show that (a) the restoring force is $-(2T/L)y$ and (b) the system exhibits simple harmonic motion with an angular frequency $\omega = \sqrt{2T/mL}$.

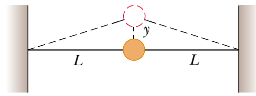


Figure P13.67

68. When a mass M, connected to the end of a spring of mass $m_s = 7.40$ g and force constant k, is set into simple harmonic motion, the period of its motion is

$$T = 2\pi\sqrt{\frac{M + (m_s/3)}{k}}$$

A two-part experiment is conducted with the use of various masses suspended vertically from the spring, as shown in Figure P13.58a. (a) Static extensions of 17.0, 29.3, 35.3, 41.3, 47.1, and 49.3 cm are measured for M values of 20.0, 40.0, 50.0, 60.0, 70.0, and 80.0 g, respectively. Construct a graph of Mg versus x, and perform a linear least-squares fit to the data. From the slope of your graph, determine a value for k for this spring. (b) The system is now set into simple harmonic motion, and periods are measured with a stopwatch. With $M =$ 80.0 g, the total time for 10 oscillations is measured to be 13.41 s. The experiment is repeated with M values of 70.0, 60.0, 50.0, 40.0, and 20.0 g, with corresponding times for 10 oscillations of 12.52, 11.67, 10.67, 9.62, and 7.03 s. Compute the experimental value for T for each of these measurements. Plot a graph of T^2 versus M, and determine a value for k from the slope of the linear least-squares fit through the data points. Compare this value of k with that obtained in part (a). (c) Obtain a value for m_s from your graph, and compare it with the given value of 7.40 g.

69. A small, thin disk of radius r and mass m is attached rigidly to the face of a second thin disk of radius R and mass M, as shown in Figure P13.69. The center of the small disk is located at the edge of the large disk. The large disk is mounted at its center on a frictionless axle. The assembly is rotated through a small angle θ from its equilibrium position and released. (a) Show that the

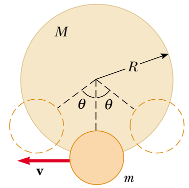


Figure P13.69

speed of the center of the small disk as it passes through the equilibrium position is

$$v = 2\left[\frac{Rg(1 - \cos\theta)}{(M/m) + (r/R)^2 + 2}\right]^{1/2}$$

(b) Show that the period of the motion is

$$T = 2\pi\left[\frac{(M + 2m)R^2 + mr^2}{2mgR}\right]^{1/2}$$

70. Consider the damped oscillator illustrated in Figure 13.19. Assume that the mass is 375 g, the spring constant is 100 N/m, and $b = 0.100$ kg/s. (a) How long does it takes for the amplitude to drop to half its initial value? (b) How long does it take for the mechanical energy to drop to half its initial value? (c) Show that, in general, the fractional rate at which the amplitude decreases in a damped harmonic oscillator is one-half the fractional rate at which the mechanical energy decreases.

71. A mass m is connected to two springs of force constants k_1 and k_2, as shown in Figure P13.71a and b. In each case, the mass moves on a frictionless table and is displaced from equilibrium and then released. Show that in the two cases the mass exhibits simple harmonic motion with periods

(a) $$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1k_2}}$$

(b) $$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

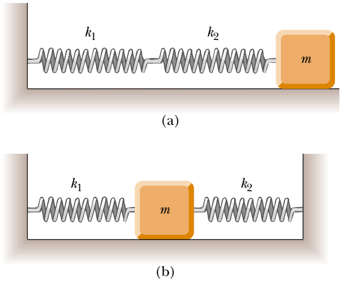


Figure P13.71

72. Consider a simple pendulum of length $L = 1.20$ m that is displaced from the vertical by an angle θ_{max} and then released. You are to predict the subsequent angular displacements when θ_{max} is small and also when it is large. Set up and carry out a numerical method to integrate the equation of motion for the simple pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Take the initial conditions to be $\theta = \theta_{max}$ and $d\theta/dt = 0$ at $t = 0$. On one trial choose $\theta_{max} = 5.00°$, and on another trial take $\theta_{max} = 100°$. In each case, find the displacement θ as a function of time. Using the same values for θ_{max}, compare your results for θ with those obtained from $\theta_{max}\cos\omega t$. How does the period for the large value of θ_{max} compare with that for the small value of θ_{max}? *Note:* Using the Euler method to solve this differential equation, you may find that the amplitude tends to increase with time. The fourth-order Runge–Kutta method would be a better choice to solve the differential equation. However, if you choose Δt small enough, the solution that you obtain using Euler's method can still be good.

ANSWERS TO QUICK QUIZZES

13.1 Because A can never be zero, ϕ must be any value that results in the cosine function's being zero at $t = 0$. In other words, $\phi = \cos^{-1}(0)$. This is true at $\phi = \pi/2$, $3\pi/2$ or, more generally, $\phi = \pm n\pi/2$, where n is any nonzero odd integer. If we want to restrict our choices of ϕ to values between 0 and 2π, we need to know whether the object was moving to the right or to the left at $t = 0$. If it was moving with a positive velocity, then $\phi = 3\pi/2$. If $v_i < 0$, then $\phi = \pi/2$.

13.2 (d) $4A$. From its maximum positive position to the equilibrium position, it travels a distance A, by definition of *amplitude.* It then goes an equal distance past the equilibrium position to its maximum negative position. It then repeats these two motions in the reverse direction to return to its original position and complete one cycle.

13.3 No, because in simple harmonic motion, the acceleration is not constant.

13.4 $x = -A\sin\omega t$, where $A = v_i/\omega$.

13.5 From Hooke's law, the spring constant must be $k = mg/L$. If we substitute this value for k into Equation 13.18, we find that

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

This is the same as Equation 13.26, which gives the period of a simple pendulum. Thus, when an object stretches a vertically hung spring, the period of the system is the same as that of a simple pendulum having a length equal to the amount of static extension of the spring.

13.6 If your goal is simply to stop the bounce from an absorbed shock as rapidly as possible, you should critically damp the suspension. Unfortunately, the stiffness of this design makes for an uncomfortable ride. If you underdamp the suspension, the ride is more comfortable but the car bounces. If you overdamp the suspension, the wheel is displaced from its equilibrium position longer than it should be. (For example, after hitting a bump, the spring stays compressed for a short time and the wheel does not quickly drop back down into contact with the road after the wheel is past the bump—a dangerous situation.) Because of all these considerations, automotive engineers usually design suspensions to be slightly underdamped. This allows the suspension to absorb a shock rapidly (minimizing the roughness of the ride) and then return to equilibrium after only one or two noticeable oscillations.

PUZZLER

More than 300 years ago, Isaac Newton realized that the same gravitational force that causes apples to fall to the Earth also holds the Moon in its orbit. In recent years, scientists have used the Hubble Space Telescope to collect evidence of the gravitational force acting even farther away, such as at this protoplanetary disk in the constellation Taurus. What properties of an object such as a protoplanet or the Moon determine the strength of its gravitational attraction to another object? *(Left, Larry West/FPG International; right, Courtesy of NASA)*

web

For more information about the Hubble, visit the Space Telescope Science Institute at **http://www.stsci.edu/**

chapter

14

The Law of Gravity

Chapter Outline

Before 1687, a large amount of data had been collected on the motions of the Moon and the planets, but a clear understanding of the forces causing these motions was not available. In that year, Isaac Newton provided the key that unlocked the secrets of the heavens. He knew, from his first law, that a net force had to be acting on the Moon because without such a force the Moon would move in a straight-line path rather than in its almost circular orbit. Newton reasoned that this force was the gravitational attraction exerted by the Earth on the Moon. He realized that the forces involved in the Earth–Moon attraction and in the Sun–planet attraction were not something special to those systems, but rather were particular cases of a general and universal attraction between objects. In other words, Newton saw that the same force of attraction that causes the Moon to follow its path around the Earth also causes an apple to fall from a tree. As he put it, "I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of their distances from the centers about which they revolve; and thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the Earth; and found them answer pretty nearly."

In this chapter we study the law of gravity. We place emphasis on describing the motion of the planets because astronomical data provide an important test of the validity of the law of gravity. We show that the laws of planetary motion developed by Johannes Kepler follow from the law of gravity and the concept of conservation of angular momentum. We then derive a general expression for gravitational potential energy and examine the energetics of planetary and satellite motion. We close by showing how the law of gravity is also used to determine the force between a particle and an extended object.

14.1 NEWTON'S LAW OF UNIVERSAL GRAVITATION

You may have heard the legend that Newton was struck on the head by a falling apple while napping under a tree. This alleged accident supposedly prompted him to imagine that perhaps all bodies in the Universe were attracted to each other in the same way the apple was attracted to the Earth. Newton analyzed astronomical data on the motion of the Moon around the Earth. From that analysis, he made the bold assertion that the force law governing the motion of planets was the *same* as the force law that attracted a falling apple to the Earth. This was the first time that "earthly" and "heavenly" motions were unified. We shall look at the mathematical details of Newton's analysis in Section 14.5.

In 1687 Newton published his work on the law of gravity in his treatise *Mathematical Principles of Natural Philosophy.* **Newton's law of universal gravitation** states that

> every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

If the particles have masses m_1 and m_2 and are separated by a distance r, the magnitude of this gravitational force is

The law of gravity

$$F_g = G\frac{m_1 m_2}{r^2} \qquad (14.1)$$

where G is a constant, called the *universal gravitational constant,* that has been measured experimentally. As noted in Example 6.6, its value in SI units is

$$G = 6.673 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2 \qquad \textbf{(14.2)}$$

The form of the force law given by Equation 14.1 is often referred to as an **inverse-square law** because the magnitude of the force varies as the inverse square of the separation of the particles.[1] We shall see other examples of this type of force law in subsequent chapters. We can express this force in vector form by defining a unit vector $\hat{\mathbf{r}}_{12}$ (Fig. 14.1). Because this unit vector is directed from particle 1 to particle 2, the force exerted by particle 1 on particle 2 is

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r^2}\,\hat{\mathbf{r}}_{12} \qquad \textbf{(14.3)}$$

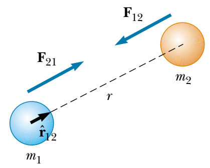


Figure 14.1 The gravitational force between two particles is attractive. The unit vector $\hat{\mathbf{r}}_{12}$ is directed from particle 1 to particle 2. Note that $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

where the minus sign indicates that particle 2 is attracted to particle 1, and hence the force must be directed toward particle 1. By Newton's third law, the force exerted by particle 2 on particle 1, designated $\mathbf{F}_{21}$, is equal in magnitude to $\mathbf{F}_{12}$ and in the opposite direction. That is, these forces form an action–reaction pair, and $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Several features of Equation 14.3 deserve mention. The gravitational force is a field force that always exists between two particles, regardless of the medium that separates them. Because the force varies as the inverse square of the distance between the particles, it decreases rapidly with increasing separation. We can relate this fact to the geometry of the situation by noting that the intensity of light emanating from a point source drops off in the same $1/r^2$ manner, as shown in Figure 14.2.

Properties of the gravitational force

Another important point about Equation 14.3 is that **the gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center.** For example, the force exerted by the

QuickLab

Inflate a balloon just enough to form a small sphere. Measure its diameter. Use a marker to color in a 1-cm square on its surface. Now continue inflating the balloon until it reaches twice the original diameter. Measure the size of the square you have drawn. Also note how the color of the marked area has changed. Have you verified what is shown in Figure 14.2?

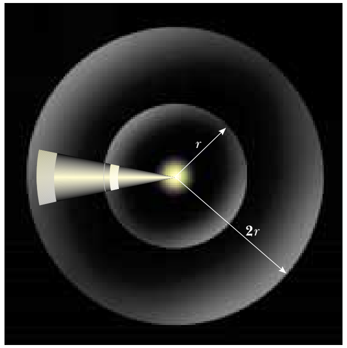


Figure 14.2 Light radiating from a point source drops off as $1/r^2$, a relationship that matches the way the gravitational force depends on distance. When the distance from the light source is doubled, the light has to cover four times the area and thus is one fourth as bright.

[1] An inverse relationship between two quantities x and y is one in which $y = k/x$, where k is a constant. A direct proportion between x and y exists when $y = kx$.

Earth on a particle of mass m near the Earth's surface has the magnitude

$$F_g = G\frac{M_E m}{R_E^2} \qquad \textbf{(14.4)}$$

where M_E is the Earth's mass and R_E its radius. This force is directed toward the center of the Earth.

We have evidence of the fact that the gravitational force acting on an object is directly proportional to its mass from our observations of falling objects, discussed in Chapter 2. All objects, regardless of mass, fall in the absence of air resistance at the same acceleration g near the surface of the Earth. According to Newton's second law, this acceleration is given by $g = F_g/m$, where m is the mass of the falling object. If this ratio is to be the same for all falling objects, then F_g must be directly proportional to m, so that the mass cancels in the ratio. If we consider the more general situation of a gravitational force between any two objects with mass, such as two planets, this same argument can be applied to show that the gravitational force is proportional to one of the masses. We can choose *either* of the masses in the argument, however; thus, the gravitational force must be directly proportional to *both* masses, as can be seen in Equation 14.3.

14.2 MEASURING THE GRAVITATIONAL CONSTANT

The universal gravitational constant G was measured in an important experiment by Henry Cavendish (1731–1810) in 1798. The Cavendish apparatus consists of two small spheres, each of mass m, fixed to the ends of a light horizontal rod suspended by a fine fiber or thin metal wire, as illustrated in Figure 14.3. When two large spheres, each of mass M, are placed near the smaller ones, the attractive force between smaller and larger spheres causes the rod to rotate and twist the wire suspension to a new equilibrium orientation. The angle of rotation is measured by the deflection of a light beam reflected from a mirror attached to the vertical suspension. The deflection of the light is an effective technique for amplifying the motion. The experiment is carefully repeated with different masses at various separations. In addition to providing a value for G, the results show experimentally that the force is attractive, proportional to the product mM, and inversely proportional to the square of the distance r.

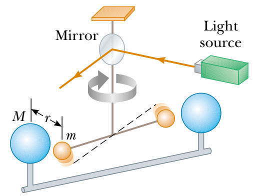


Figure 14.3 Schematic diagram of the Cavendish apparatus for measuring G. As the small spheres of mass m are attracted to the large spheres of mass M, the rod between the two small spheres rotates through a small angle. A light beam reflected from a mirror on the rotating apparatus measures the angle of rotation. The dashed line represents the original position of the rod.

EXAMPLE 14.1 Billiards, Anyone?

Three 0.300-kg billiard balls are placed on a table at the corners of a right triangle, as shown in Figure 14.4. Calculate the gravitational force on the cue ball (designated m_1) resulting from the other two balls.

Solution First we calculate separately the individual forces on the cue ball due to the other two balls, and then we find the vector sum to get the resultant force. We can see graphically that this force should point upward and toward the right. We locate our coordinate axes as shown in Figure 14.4, placing our origin at the position of the cue ball.

The force exerted by m_2 on the cue ball is directed upward and is given by

$$\mathbf{F}_{21} = G\frac{m_2 m_1}{{r_{21}}^2}\mathbf{j}$$

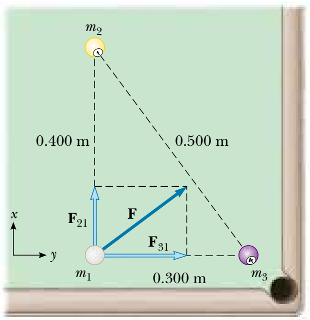


Figure 14.4 The resultant gravitational force acting on the cue ball is the vector sum $\mathbf{F}_{21} + \mathbf{F}_{31}$.

$$= \left(6.67 \times 10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\frac{(0.300\ \text{kg})(0.300\ \text{kg})}{(0.400\ \text{m})^2}\mathbf{j}$$

$$= 3.75 \times 10^{-11}\mathbf{j}\ \text{N}$$

This result shows that the gravitational forces between everyday objects have extremely small magnitudes. The force exerted by m_3 on the cue ball is directed to the right:

$$\mathbf{F}_{31} = G\frac{m_3 m_1}{{r_{31}}^2}\mathbf{i}$$

$$= \left(6.67 \times 10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\frac{(0.300\ \text{kg})(0.300\ \text{kg})}{(0.300\ \text{m})^2}\mathbf{i}$$

$$= 6.67 \times 10^{-11}\mathbf{i}\ \text{N}$$

Therefore, the resultant force on the cue ball is

$$\mathbf{F} = \mathbf{F}_{21} + \mathbf{F}_{31} = (3.75\mathbf{j} + 6.67\mathbf{i}) \times 10^{-11}\ \text{N}$$

and the magnitude of this force is

$$F = \sqrt{{F_{21}}^2 + {F_{31}}^2} = \sqrt{(3.75)^2 + (6.67)^2} \times 10^{-11}$$

$$= 7.65 \times 10^{-11}\ \text{N}$$

Exercise Find the direction of $\mathbf{F}$.

Answer 29.3° counterclockwise from the positive x axis.

14.3 FREE-FALL ACCELERATION AND THE GRAVITATIONAL FORCE

In Chapter 5, when defining mg as the weight of an object of mass m, we referred to g as the magnitude of the free-fall acceleration. Now we are in a position to obtain a more fundamental description of g. Because the force acting on a freely falling object of mass m near the Earth's surface is given by Equation 14.4, we can equate mg to this force to obtain

$$mg = G\frac{M_E m}{{R_E}^2}$$

$$g = G\frac{M_E}{{R_E}^2} \qquad \textbf{(14.5)}$$

Free-fall acceleration near the Earth's surface

Now consider an object of mass m located a distance h above the Earth's surface or a distance r from the Earth's center, where $r = R_E + h$. The magnitude of the gravitational force acting on this object is

$$F_g = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

The gravitational force acting on the object at this position is also $F_g = mg'$, where g' is the value of the free-fall acceleration at the altitude h. Substituting this expression for F_g into the last equation shows that g' is

Variation of g with altitude

$$g' = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \qquad \textbf{(14.6)}$$

Thus, it follows that g' *decreases* with *increasing altitude.* Because the weight of a body is mg', we see that as $r \rightarrow \infty$, its weight approaches zero.

EXAMPLE 14.2 Variation of g with Altitude h

The International Space Station is designed to operate at an altitude of 350 km. When completed, it will have a weight (measured at the Earth's surface) of 4.22×10^6 N. What is its weight when in orbit?

Solution Because the station is above the surface of the Earth, we expect its weight in orbit to be less than its weight on Earth, 4.22×10^6 N. Using Equation 14.6 with $h = 350$ km, we obtain

$$g' = \frac{GM_E}{(R_E + h)^2}$$

$$= \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{(6.37 \times 10^6\ \text{m} + 0.350 \times 10^6\ \text{m})^2}$$

$$= 8.83\ \text{m/s}^2$$

Because $g'/g = 8.83/9.80 = 0.901$, we conclude that the weight of the station at an altitude of 350 km is 90.1% of the value at the Earth's surface. So the station's weight in orbit is

$$(0.901)(4.22 \times 10^6\ \text{N}) = 3.80 \times 10^6\ \text{N}$$

Values of g' at other altitudes are listed in Table 14.1.

TABLE 14.1 Free-Fall Acceleration g' at Various Altitudes Above the Earth's Surface

Altitude h (km)	g' (m/s^2)
1 000	7.33
2 000	5.68
3 000	4.53
4 000	3.70
5 000	3.08
6 000	2.60
7 000	2.23
8 000	1.93
9 000	1.69
10 000	1.49
50 000	0.13
∞	0

web The official web site for the International Space Station is **www.station.nasa.gov**

EXAMPLE 14.3 The Density of the Earth

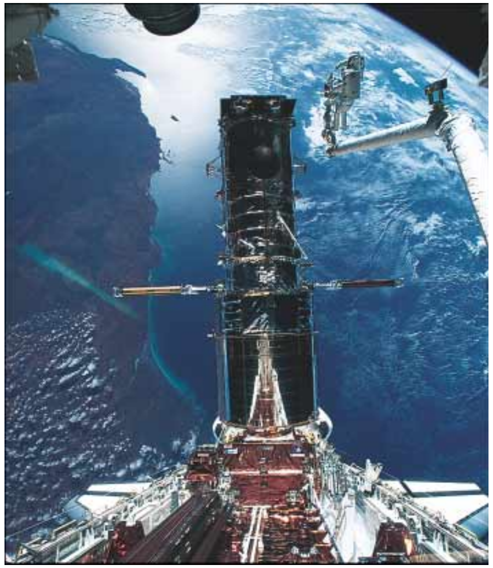
Using the fact that $g = 9.80\ \text{m/s}^2$ at the Earth's surface, find the average density of the Earth.

Solution Using $g = 9.80\ \text{m/s}^2$ and $R_E = 6.37 \times 10^6$ m, we find from Equation 14.5 that $M_E = 5.96 \times 10^{24}$ kg. From this result, and using the definition of density from Chapter 1, we obtain

$$\rho_E = \frac{M_E}{V_E} = \frac{M_E}{\frac{4}{3}\pi {R_E}^3} = \frac{5.96 \times 10^{24}\ \text{kg}}{\frac{4}{3}\pi(6.37 \times 10^6\ \text{m})^3}$$

$$= 5.50 \times 10^3\ \text{kg/m}^3$$

Because this value is about twice the density of most rocks at the Earth's surface, we conclude that the inner core of the Earth has a density much higher than the average value. It is most amazing that the Cavendish experiment, which determines G (and can be done on a tabletop), combined with simple free-fall measurements of g, provides information about the core of the Earth.



Astronauts F. Story Musgrave and Jeffrey A. Hoffman, along with the Hubble Space Telescope and the space shuttle *Endeavor,* are all falling around the Earth.

14.4 KEPLER'S LAWS

People have observed the movements of the planets, stars, and other celestial bodies for thousands of years. In early history, scientists regarded the Earth as the center of the Universe. This so-called geocentric model was elaborated and formalized by the Greek astronomer Claudius Ptolemy (c. 100–c. 170) in the second century A.D. and was accepted for the next 1 400 years. In 1543 the Polish astronomer Nicolaus Copernicus (1473–1543) suggested that the Earth and the other planets revolved in circular orbits around the Sun (the heliocentric model).

The Danish astronomer Tycho Brahe (1546–1601) wanted to determine how the heavens were constructed, and thus he developed a program to determine the positions of both stars and planets. It is interesting to note that those observations of the planets and 777 stars visible to the naked eye were carried out with only a large sextant and a compass. (The telescope had not yet been invented.)

The German astronomer Johannes Kepler was Brahe's assistant for a short while before Brahe's death, whereupon he acquired his mentor's astronomical data and spent 16 years trying to deduce a mathematical model for the motion of the planets. Such data are difficult to sort out because the Earth is also in motion around the Sun. After many laborious calculations, Kepler found that Brahe's data on the revolution of Mars around the Sun provided the answer.

Johannes Kepler **German astronomer (1571–1630)** The German astronomer Johannes Kepler is best known for developing the laws of planetary motion based on the careful observations of Tycho Brahe. *(Art Resource)*

For more information about Johannes Kepler, visit our Web site at **www.saunderscollege.com/physics/**

Kepler's analysis first showed that the concept of circular orbits around the Sun had to be abandoned. He eventually discovered that the orbit of Mars could be accurately described by an **ellipse.** Figure 14.5 shows the geometric description of an ellipse. The longest dimension is called the major axis and is of length $2a$, where a is the **semimajor axis.** The shortest dimension is the minor axis, of length $2b$, where b is the **semiminor axis.** On either side of the center is a **focal point,** a distance c from the center, where $a^2 = b^2 + c^2$. The Sun is located at one of the focal points of Mars's orbit. Kepler generalized his analysis to include the motions of all planets. The complete analysis is summarized in three statements known as **Kepler's laws:**

Kepler's laws

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

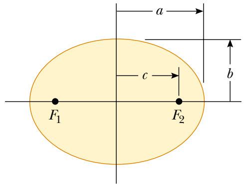


Figure 14.5 Plot of an ellipse. The semimajor axis has a length a, and the semiminor axis has a length b. The focal points are located at a distance c from the center, where $a^2 = b^2 + c^2$.

Most of the planetary orbits are close to circular in shape; for example, the semimajor and semiminor axes of the orbit of Mars differ by only 0.4%. Mercury and Pluto have the most elliptical orbits of the nine planets. In addition to the planets, there are many asteroids and comets orbiting the Sun that obey Kepler's laws. Comet Halley is such an object; it becomes visible when it is close to the Sun every 76 years. Its orbit is very elliptical, with a semiminor axis 76% smaller than its semimajor axis.

Although we do not prove it here, Kepler's first law is a direct consequence of the fact that the gravitational force varies as $1/r^2$. That is, under an inverse-square gravitational-force law, the orbit of a planet can be shown mathematically to be an ellipse with the Sun at one focal point. Indeed, half a century after Kepler developed his laws, Newton demonstrated that these laws are a consequence of the gravitational force that exists between any two masses. Newton's law of universal gravitation, together with his development of the laws of motion, provides the basis for a full mathematical solution to the motion of planets and satellites.

14.5 THE LAW OF GRAVITY AND THE MOTION OF PLANETS

In formulating his law of gravity, Newton used the following reasoning, which supports the assumption that the gravitational force is proportional to the inverse square of the separation between the two interacting bodies. He compared the acceleration of the Moon in its orbit with the acceleration of an object falling near the Earth's surface, such as the legendary apple (Fig. 14.6). Assuming that both accelerations had the same cause—namely, the gravitational attraction of the Earth—Newton used the inverse-square law to reason that the acceleration of the Moon toward the Earth (centripetal acceleration) should be proportional to $1/r_M^2$, where r_M is the distance between the centers of the Earth and the Moon. Furthermore, the acceleration of the apple toward the Earth should be proportional to $1/R_E^2$, where R_E is the radius of the Earth, or the distance between the centers of the Earth and the apple. Using the values $r_M = 3.84 \times 10^8$ m and

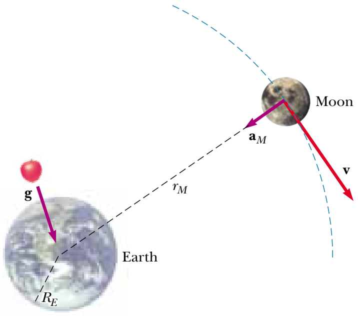


Figure 14.6 As it revolves around the Earth, the Moon experiences a centripetal acceleration $\mathbf{a}_M$ directed toward the Earth. An object near the Earth's surface, such as the apple shown here, experiences an acceleration **g**. (Dimensions are not to scale.)

$R_E = 6.37 \times 10^6$ m, Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6 \text{ m}}{3.84 \times 10^8 \text{ m}}\right)^2 = 2.75 \times 10^{-4}$$

Therefore, the centripetal acceleration of the Moon is

Acceleration of the Moon

$$a_M = (2.75 \times 10^{-4})(9.80 \text{ m/s}^2) = 2.70 \times 10^{-3} \text{ m/s}^2$$

Newton also calculated the centripetal acceleration of the Moon from a knowledge of its mean distance from the Earth and its orbital period, $T = 27.32$ days $= 2.36 \times 10^6$ s. In a time T, the Moon travels a distance $2\pi r_M$, which equals the circumference of its orbit. Therefore, its orbital speed is $2\pi r_M/T$ and its centripetal acceleration is

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2(3.84 \times 10^8 \text{ m})}{(2.36 \times 10^6 \text{ s})^2}$$

$$= 2.72 \times 10^{-3} \text{ m/s}^2 \approx \frac{9.80 \text{ m/s}^2}{60^2}$$

In other words, because the Moon is roughly 60 Earth radii away, the gravitational acceleration at that distance should be about $1/60^2$ of its value at the Earth's surface. This is just the acceleration needed to account for the circular motion of the Moon around the Earth. The nearly perfect agreement between this value and the value Newton obtained using g provides strong evidence of the inverse-square nature of the gravitational force law.

Although these results must have been very encouraging to Newton, he was deeply troubled by an assumption he made in the analysis. To evaluate the acceleration of an object at the Earth's surface, Newton treated the Earth as if its mass were all concentrated at its center. That is, he assumed that the Earth acted as a particle as far as its influence on an exterior object was concerned. Several years later, in 1687, on the basis of his pioneering work in the development of calculus, Newton proved that this assumption was valid and was a natural consequence of the law of universal gravitation.

Kepler's Third Law

It is informative to show that Kepler's third law can be predicted from the inverse-square law for circular orbits.[2] Consider a planet of mass M_p moving around the Sun of mass M_S in a circular orbit, as shown in Figure 14.7. Because the gravitational force exerted by the Sun on the planet is a radially directed force that keeps the planet moving in a circle, we can apply Newton's second law ($\Sigma F = ma$) to the planet:

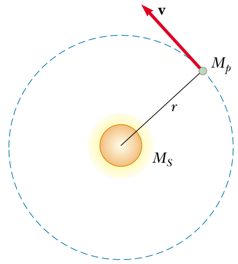


Figure 14.7 A planet of mass M_p moving in a circular orbit around the Sun. The orbits of all planets except Mercury and Pluto are nearly circular.

$$\frac{GM_S M_p}{r^2} = \frac{M_p v^2}{r}$$

Because the orbital speed v of the planet is simply $2\pi r/T$, where T is its period of revolution, the preceding expression becomes

$$\frac{GM_S}{r^2} = \frac{(2\pi r/T)^2}{r}$$

Kepler's third law

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) r^3 = K_S r^3 \qquad \textbf{(14.7)}$$

where K_S is a constant given by

$$K_S = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19} \text{ s}^2/\text{m}^3$$

Equation 14.7 is Kepler's third law. It can be shown that the law is also valid for elliptical orbits if we replace r with the length of the semimajor axis a. Note that the constant of proportionality K_S is independent of the mass of the planet. Therefore, Equation 14.7 is valid for *any* planet.[3] Table 14.2 contains a collection of useful planetary data. The last column verifies that T^2/r^3 is a constant. The small variations in the values in this column reflect uncertainties in the measured values of the periods and semimajor axes of the planets.

If we were to consider the orbit around the Earth of a satellite such as the Moon, then the proportionality constant would have a different value, with the Sun's mass replaced by the Earth's mass.

EXAMPLE 14.4 The Mass of the Sun

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.156×10^7 s and its distance from the Sun is 1.496×10^{11} m.

Solution Using Equation 14.7, we find that

$$M_S = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(1.496 \times 10^{11} \text{ m})^3}{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(3.156 \times 10^7 \text{ s})^2}$$

$$= 1.99 \times 10^{30} \text{ kg}$$

In Example 14.3, an understanding of gravitational forces enabled us to find out something about the density of the Earth's core, and now we have used this understanding to determine the mass of the Sun.

[2] The orbits of all planets except Mercury and Pluto are very close to being circular; hence, we do not introduce much error with this assumption. For example, the ratio of the semiminor axis to the semimajor axis for the Earth's orbit is $b/a = 0.999\,86$.

[3] Equation 14.7 is indeed a proportion because the ratio of the two quantities T^2 and r^3 is a constant. The variables in a proportion are not required to be limited to the first power only.

TABLE 14.2 Useful Planetary Data

Body	Mass (kg)	Mean Radius (m)	Period of Revolution (s)	Mean Distance from Sun (m)	$\frac{T^2}{r^3}$ (s^2/m^3)
Mercury	3.18×10^{23}	2.43×10^{6}	7.60×10^{6}	5.79×10^{10}	2.97×10^{-19}
Venus	4.88×10^{24}	6.06×10^{6}	1.94×10^{7}	1.08×10^{11}	2.99×10^{-19}
Earth	5.98×10^{24}	6.37×10^{6}	3.156×10^{7}	1.496×10^{11}	2.97×10^{-19}
Mars	6.42×10^{23}	3.37×10^{6}	5.94×10^{7}	2.28×10^{11}	2.98×10^{-19}
Jupiter	1.90×10^{27}	6.99×10^{7}	3.74×10^{8}	7.78×10^{11}	2.97×10^{-19}
Saturn	5.68×10^{26}	5.85×10^{7}	9.35×10^{8}	1.43×10^{12}	2.99×10^{-19}
Uranus	8.68×10^{25}	2.33×10^{7}	2.64×10^{9}	2.87×10^{12}	2.95×10^{-19}
Neptune	1.03×10^{26}	2.21×10^{7}	5.22×10^{9}	4.50×10^{12}	2.99×10^{-19}
Pluto	$\approx 1.4 \times 10^{22}$	$\approx 1.5 \times 10^{6}$	7.82×10^{9}	5.91×10^{12}	2.96×10^{-19}
Moon	7.36×10^{22}	1.74×10^{6}	—	—	—
Sun	1.991×10^{30}	6.96×10^{8}	—	—	—

Kepler's Second Law and Conservation of Angular Momentum

Consider a planet of mass M_p moving around the Sun in an elliptical orbit (Fig. 14.8). The gravitational force acting on the planet is always along the radius vector, directed toward the Sun, as shown in Figure 14.9a. When a force is directed toward or away from a fixed point and is a function of r only, it is called a **central force.** The torque acting on the planet due to this force is clearly zero; that is, because **F** is parallel to **r**,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F\hat{\mathbf{r}} = 0$$

(You may want to revisit Section 11.2 to refresh your memory on the vector product.) Recall from Equation 11.19, however, that torque equals the time rate of change of angular momentum: $\boldsymbol{\tau} = d\mathbf{L}/dt$. Therefore, **because the gravitational**

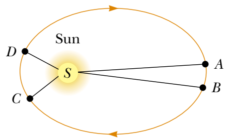


Figure 14.8 Kepler's second law is called the law of equal areas. When the time interval required for a planet to travel from A to B is equal to the time interval required for it to go from C to D, the two areas swept out by the planet's radius vector are equal. Note that in order for this to be true, the planet must be moving faster between C and D than between A and B.

Separate views of Jupiter and of Periodic Comet Shoemaker–Levy 9—both taken with the Hubble Space Telescope about two months before Jupiter and the comet collided in July 1994—were put together with the use of a computer. Their relative sizes and distances were altered. The black spot on Jupiter is the shadow of its moon Io.

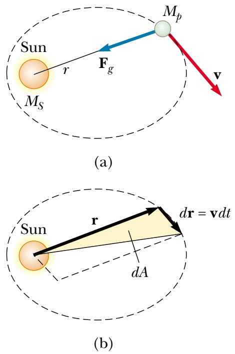


Figure 14.9 (a) The gravitational force acting on a planet is directed toward the Sun, along the radius vector. (b) As a planet orbits the Sun, the area swept out by the radius vector in a time dt is equal to one-half the area of the parallelogram formed by the vectors **r** and $d\mathbf{r} = \mathbf{v}dt$.

force exerted by the Sun on a planet results in no torque on the planet, the angular momentum L of the planet is constant:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times M_p\mathbf{v} = M_p\mathbf{r} \times \mathbf{v} = \text{constant} \quad \textbf{(14.8)}$$

Because **L** remains constant, the planet's motion at any instant is restricted to the plane formed by **r** and **v**.

We can relate this result to the following geometric consideration. The radius vector **r** in Figure 14.9b sweeps out an area dA in a time dt. This area equals one-half the area $|\mathbf{r} \times d\mathbf{r}|$ of the parallelogram formed by the vectors **r** and $d\mathbf{r}$ (see Section 11.2). Because the displacement of the planet in a time dt is $d\mathbf{r} = \mathbf{v}dt$, we can say that

$$dA = \tfrac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}\,dt| = \frac{L}{2M_p}\,dt$$

$$\frac{dA}{dt} = \frac{L}{2M_p} = \text{constant} \quad \textbf{(14.9)}$$

where L and M_p are both constants. Thus, we conclude that

the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

It is important to recognize that this result, which is Kepler's second law, is a consequence of the fact that the force of gravity is a central force, which in turn implies that angular momentum is constant. Therefore, Kepler's second law applies to *any* situation involving a central force, whether inverse-square or not.

EXAMPLE 14.5 Motion in an Elliptical Orbit

A satellite of mass m moves in an elliptical orbit around the Earth (Fig. 14.10). The minimum distance of the satellite from the Earth is called the *perigee* (indicated by p in Fig. 14.10), and the maximum distance is called the *apogee* (indicated by a). If the speed of the satellite at p is v_p, what is its speed at a?

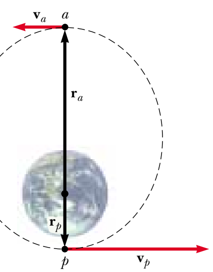


Figure 14.10 As a satellite moves around the Earth in an elliptical orbit, its angular momentum is constant. Therefore, $mv_ar_a = mv_pr_p$, where the subscripts a and p represent apogee and perigee, respectively.

Solution As the satellite moves from perigee toward apogee, it is moving farther from the Earth. Thus, a component of the gravitational force exerted by the Earth on the satellite is opposite the velocity vector. Negative work is done on the satellite, which causes it to slow down, according to the work–kinetic energy theorem. As a result, we expect the speed at apogee to be lower than the speed at perigee.

The angular momentum of the satellite relative to the Earth is $\mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times \mathbf{v}$. At the points a and p, **v** is perpendicular to **r**. Therefore, the magnitude of the angular momentum at these positions is $L_a = mv_ar_a$ and $L_p = mv_pr_p$. Because angular momentum is constant, we see that

$$mv_ar_a = mv_pr_p$$

$$v_a = \frac{r_p}{r_a}\,v_p$$

Quick Quiz 14.1

How would you explain the fact that Saturn and Jupiter have periods much greater than one year?

14.6 THE GRAVITATIONAL FIELD

When Newton published his theory of universal gravitation, it was considered a success because it satisfactorily explained the motion of the planets. Since 1687 the same theory has been used to account for the motions of comets, the deflection of a Cavendish balance, the orbits of binary stars, and the rotation of galaxies. Nevertheless, both Newton's contemporaries and his successors found it difficult to accept the concept of a force that acts through a distance, as mentioned in Section 5.1. They asked how it was possible for two objects to interact when they were not in contact with each other. Newton himself could not answer that question.

An approach to describing interactions between objects that are not in contact came well after Newton's death, and it enables us to look at the gravitational interaction in a different way. As described in Section 5.1, this alternative approach uses the concept of a **gravitational field** that exists at every point in space. When a particle of mass m is placed at a point where the gravitational field is $\mathbf{g}$, the particle experiences a force $\mathbf{F}_g = m\mathbf{g}$. In other words, the field exerts a force on the particle. Hence, the gravitational field $\mathbf{g}$ is defined as

Gravitational field

$$\mathbf{g} \equiv \frac{\mathbf{F}_g}{m} \tag{14.10}$$

That is, the gravitational field at a point in space equals the gravitational force experienced by a *test particle* placed at that point divided by the mass of the test particle. Notice that the presence of the test particle is not necessary for the field to exist—the Earth creates the gravitational field. We call the object creating the field the *source particle* (although the Earth is clearly not a particle; we shall discuss shortly the fact that we can approximate the Earth as a particle for the purpose of finding the gravitational field that it creates). We can detect the presence of the field and measure its strength by placing a test particle in the field and noting the force exerted on it.

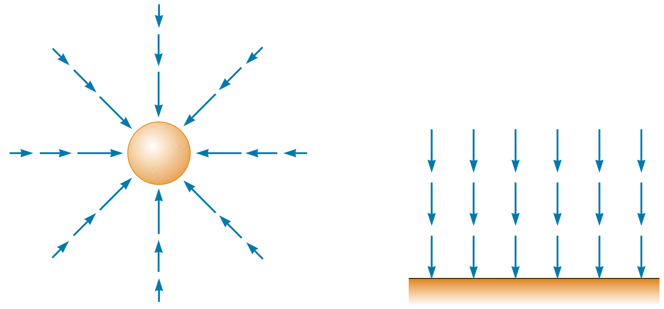
Although the gravitational force is inherently an interaction between two objects, the concept of a gravitational field allows us to "factor out" the mass of one of the objects. In essence, we are describing the "effect" that any object (in this case, the Earth) has on the empty space around itself in terms of the force that *would* be present *if* a second object were somewhere in that space.[4]

As an example of how the field concept works, consider an object of mass m near the Earth's surface. Because the gravitational force acting on the object has a magnitude GM_Em/r^2 (see Eq. 14.4), the field $\mathbf{g}$ at a distance r from the center of the Earth is

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} = -\frac{GM_E}{r^2}\hat{\mathbf{r}} \tag{14.11}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially outward from the Earth and the minus

[4] We shall return to this idea of mass affecting the space around it when we discuss Einstein's theory of gravitation in Chapter 39.



(a) (b)

Figure 14.11 (a) The gravitational field vectors in the vicinity of a uniform spherical mass such as the Earth vary in both direction and magnitude. The vectors point in the direction of the acceleration a particle would experience if it were placed in the field. The magnitude of the field vector at any location is the magnitude of the free-fall acceleration at that location. (b) The gravitational field vectors in a small region near the Earth's surface are uniform in both direction and magnitude.

sign indicates that the field points toward the center of the Earth, as illustrated in Figure 14.11a. Note that the field vectors at different points surrounding the Earth vary in both direction and magnitude. In a small region near the Earth's surface, the downward field $\mathbf{g}$ is approximately constant and uniform, as indicated in Figure 14.11b. Equation 14.11 is valid at all points *outside* the Earth's surface, assuming that the Earth is spherical. At the Earth's surface, where $r = R_E$, $\mathbf{g}$ has a magnitude of 9.80 N/kg.

14.7 GRAVITATIONAL POTENTIAL ENERGY

In Chapter 8 we introduced the concept of gravitational potential energy, which is the energy associated with the position of a particle. We emphasized that the gravitational potential energy function $U = mgy$ is valid only when the particle is near the Earth's surface, where the gravitational force is constant. Because the gravitational force between two particles varies as $1/r^2$, we expect that a more general potential energy function—one that is valid without the restriction of having to be near the Earth's surface—will be significantly different from $U = mgy$.

Before we calculate this general form for the gravitational potential energy function, let us first verify that *the gravitational force is conservative.* (Recall from Section 8.2 that a force is conservative if the work it does on an object moving between any two points is independent of the path taken by the object.) To do this, we first note that the gravitational force is a central force. By definition, a central force is any force that is directed along a radial line to a fixed center and has a magnitude that depends only on the radial coordinate r. Hence, a central force can be represented by $F(r)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector directed from the origin to the particle, as shown in Figure 14.12.

Consider a central force acting on a particle moving along the general path P to Q in Figure 14.12. The path from P to Q can be approximated by a series of steps according to the following procedure. In Figure 14.12, we draw several thin wedges, which are shown as dashed lines. The outer boundary of our set of wedges is a path consisting of short radial line segments and arcs (gray in the figure). We select the length of the radial dimension of each wedge such that the short arc at the wedge's wide end intersects the actual path of the particle. Then we can approximate the actual path with a series of zigzag movements that alternate between moving along an arc and moving along a radial line.

By definition, a central force is always directed along one of the radial segments; therefore, the work done by **F** along any radial segment is

$$dW = \mathbf{F} \cdot d\mathbf{r} = F(r)\ dr$$

You should recall that, by definition, the work done by a force that is perpendicular to the displacement is zero. Hence, the work done in moving along any arc is zero because **F** is perpendicular to the displacement along these segments. Therefore, the total work done by **F** is the sum of the contributions along the radial segments:

Work done by a central force

$$W = \int_{r_i}^{r_f} F(r)\ dr$$

where the subscripts i and f refer to the initial and final positions. Because the integrand is a function only of the radial position, this integral depends only on the initial and final values of r. Thus, the work done is the same over *any* path from P to Q. Because the work done is independent of the path and depends only on the end points, we conclude that *any central force is conservative.* We are now assured that a potential energy function can be obtained once the form of the central force is specified.

Recall from Equation 8.2 that the change in the gravitational potential energy associated with a given displacement is defined as the negative of the work done by the gravitational force during that displacement:

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)\ dr \qquad \textbf{(14.12)}$$

We can use this result to evaluate the gravitational potential energy function. Consider a particle of mass m moving between two points P and Q above the Earth's surface (Fig. 14.13). The particle is subject to the gravitational force given by Equation 14.1. We can express this force as

$$F(r) = -\frac{GM_E m}{r^2}$$

where the negative sign indicates that the force is attractive. Substituting this expression for $F(r)$ into Equation 14.12, we can compute the change in the gravita-

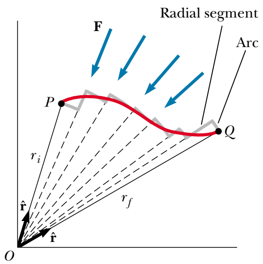


Figure 14.12 A particle moves from P to Q while acted on by a central force **F**, which is directed radially. The path is broken into a series of radial segments and arcs. Because the work done along the arcs is zero, the work done is independent of the path and depends only on r_f and r_i.

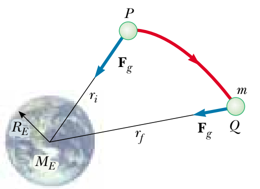


Figure 14.13 As a particle of mass m moves from P to Q above the Earth's surface, the gravitational potential energy changes according to Equation 14.12.

tional potential energy function:

$$U_f - U_i = GM_E m \int_{r_i}^{r_f} \frac{dr}{r^2} = GM_E m \left[-\frac{1}{r}\right]_{r_i}^{r_f}$$

Change in gravitational potential energy

$$U_f - U_i = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \qquad \textbf{(14.13)}$$

As always, the choice of a reference point for the potential energy is completely arbitrary. It is customary to choose the reference point where the force is zero. Taking $U_i = 0$ at $r_i = \infty$, we obtain the important result

Gravitational potential energy of the Earth–particle system for $r \geq R_E$

$$U = -\frac{GM_E m}{r} \qquad \textbf{(14.14)}$$

This expression applies to the Earth–particle system where the two masses are separated by a distance r, provided that $r \geq R_E$. The result is not valid for particles inside the Earth, where $r < R_E$. (The situation in which $r < R_E$ is treated in Section 14.10.) Because of our choice of U_i, the function U is always negative (Fig. 14.14).

Although Equation 14.14 was derived for the particle–Earth system, it can be applied to any two particles. That is, the gravitational potential energy associated with any pair of particles of masses m_1 and m_2 separated by a distance r is

$$U = -\frac{Gm_1 m_2}{r} \qquad \textbf{(14.15)}$$

This expression shows that the gravitational potential energy for any pair of particles varies as $1/r$, whereas the force between them varies as $1/r^2$. Furthermore, the potential energy is negative because the force is attractive and we have taken the potential energy as zero when the particle separation is infinite. Because the force between the particles is attractive, we know that an external agent must do positive work to increase the separation between them. The work done by the external agent produces an increase in the potential energy as the two particles are separated. That is, U becomes less negative as r increases.

When two particles are at rest and separated by a distance r, an external agent has to supply an energy at least equal to $+Gm_1 m_2/r$ in order to separate the particles to an infinite distance. It is therefore convenient to think of the absolute value of the potential energy as the *binding energy* of the system. If the external agent supplies an energy greater than the binding energy, the excess energy of the system will be in the form of kinetic energy when the particles are at an infinite separation.

We can extend this concept to three or more particles. In this case, the total potential energy of the system is the sum over all pairs of particles.[5] Each pair contributes a term of the form given by Equation 14.15. For example, if the system contains three particles, as in Figure 14.15, we find that

$$U_{\text{total}} = U_{12} + U_{13} + U_{23} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right) \qquad \textbf{(14.16)}$$

The absolute value of U_{total} represents the work needed to separate the particles by an infinite distance.

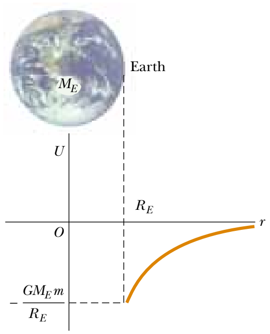


Figure 14.14 Graph of the gravitational potential energy U versus r for a particle above the Earth's surface. The potential energy goes to zero as r approaches infinity.

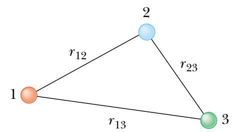


Figure 14.15 Three interacting particles.

[5] The fact that potential energy terms can be added for all pairs of particles stems from the experimental fact that gravitational forces obey the superposition principle.

EXAMPLE 14.6 The Change in Potential Energy

A particle of mass m is displaced through a small vertical distance Δy near the Earth's surface. Show that in this situation the general expression for the change in gravitational potential energy given by Equation 14.13 reduces to the familiar relationship $\Delta U = mg\,\Delta y$.

Solution We can express Equation 14.13 in the form

$$\Delta U = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = GM_E m\left(\frac{r_f - r_i}{r_i r_f}\right)$$

If both the initial and final positions of the particle are close to the Earth's surface, then $r_f - r_i = \Delta y$ and $r_i r_f \approx R_E^2$. (Recall that r is measured from the center of the Earth.) Therefore, the change in potential energy becomes

$$\Delta U \approx \frac{GM_E m}{R_E^2}\,\Delta y = mg\,\Delta y$$

where we have used the fact that $g = GM_E/R_E^2$ (Eq. 14.5). Keep in mind that the reference point is arbitrary because it is the *change* in potential energy that is meaningful.

14.8 ENERGY CONSIDERATIONS IN PLANETARY AND SATELLITE MOTION

Consider a body of mass m moving with a speed v in the vicinity of a massive body of mass M, where $M \gg m$. The system might be a planet moving around the Sun, a satellite in orbit around the Earth, or a comet making a one-time flyby of the Sun. If we assume that the body of mass M is at rest in an inertial reference frame, then the total mechanical energy E of the two-body system when the bodies are separated by a distance r is the sum of the kinetic energy of the body of mass m and the potential energy of the system, given by Equation 14.15:[6]

$$E = K + U$$

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \qquad (14.17)$$

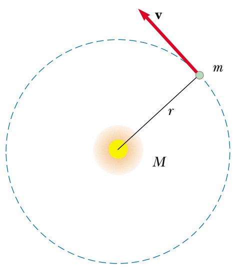


Figure 14.16 A body of mass m moving in a circular orbit about a much larger body of mass M.

This equation shows that E may be positive, negative, or zero, depending on the value of v. However, for a bound system,[7] such as the Earth–Sun system, E is necessarily *less than zero* because we have chosen the convention that $U \rightarrow 0$ as $r \rightarrow \infty$.

We can easily establish that $E < 0$ for the system consisting of a body of mass m moving in a circular orbit about a body of mass $M \gg m$ (Fig. 14.16). Newton's second law applied to the body of mass m gives

$$\frac{GMm}{r^2} = ma = \frac{mv^2}{r}$$

[6] You might recognize that we have ignored the acceleration and kinetic energy of the larger body. To see that this simplification is reasonable, consider an object of mass m falling toward the Earth. Because the center of mass of the object–Earth system is effectively stationary, it follows that $mv = M_E v_E$. Thus, the Earth acquires a kinetic energy equal to

$$\tfrac{1}{2}M_E v_E^2 = \tfrac{1}{2}\frac{m^2}{M_E}v^2 = \frac{m}{M_E}K$$

where K is the kinetic energy of the object. Because $M_E \gg m$, this result shows that the kinetic energy of the Earth is negligible.

[7] Of the three examples provided at the beginning of this section, the planet moving around the Sun and a satellite in orbit around the Earth are bound systems—the Earth will always stay near the Sun, and the satellite will always stay near the Earth. The one-time comet flyby represents an unbound system—the comet interacts once with the Sun but is not bound to it. Thus, in theory the comet can move infinitely far away from the Sun.

Multiplying both sides by r and dividing by 2 gives

$$\tfrac{1}{2}mv^2 = \frac{GMm}{2r} \qquad (14.18)$$

Substituting this into Equation 14.17, we obtain

$$E = \frac{GMm}{2r} - \frac{GMm}{r}$$

Total energy for circular orbits

$$E = -\frac{GMm}{2r} \qquad (14.19)$$

This result clearly shows that **the total mechanical energy is negative in the case of circular orbits.** Note that **the kinetic energy is positive and equal to one-half the absolute value of the potential energy.** The absolute value of E is also equal to the binding energy of the system, because this amount of energy must be provided to the system to move the two masses infinitely far apart.

The total mechanical energy is also negative in the case of elliptical orbits. The expression for E for elliptical orbits is the same as Equation 14.19 with r replaced by the semimajor axis length a. Furthermore, the total energy is constant if we assume that the system is isolated. Therefore, as the body of mass m moves from P to Q in Figure 14.13, the total energy remains constant and Equation 14.17 gives

$$E = \tfrac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \tfrac{1}{2}mv_f^2 - \frac{GMm}{r_f} \qquad (14.20)$$

Combining this statement of energy conservation with our earlier discussion of conservation of angular momentum, we see that **both the total energy and the total angular momentum of a gravitationally bound, two-body system are constants of the motion.**

EXAMPLE 14.7 Changing the Orbit of a Satellite

The space shuttle releases a 470-kg communications satellite while in an orbit that is 280 km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth. How much energy did the engine have to provide?

Solution First we must determine the radius of a geosynchronous orbit. Then we can calculate the change in energy needed to boost the satellite into orbit.

The period of the orbit T must be one day (86 400 s), so that the satellite travels once around the Earth in the same time that the Earth spins once on its axis. Knowing the period, we can then apply Kepler's third law (Eq. 14.7) to find the radius, once we replace K_S with $K_E = 4\pi^2/GM_E = 9.89 \times 10^{-14}\ \text{s}^2/\text{m}^3$:

$$T^2 = K_E r^3$$

$$r = \sqrt[3]{\frac{T^2}{K_E}} = \sqrt[3]{\frac{(86\,400\ \text{s})^2}{9.89 \times 10^{-14}\ \text{s}^2/\text{m}^3}} = 4.23 \times 10^7\ \text{m} = R_f$$

This is a little more than 26 000 mi above the Earth's surface.

We must also determine the initial radius (not the altitude above the Earth's surface) of the satellite's orbit when it was still in the shuttle's cargo bay. This is simply

$$R_E + 280\ \text{km} = 6.65 \times 10^6\ \text{m} = R_i$$

Now, applying Equation 14.19, we obtain, for the total initial and final energies,

$$E_i = -\frac{GM_E m}{2R_i} \qquad E_f = -\frac{GM_E m}{2R_f}$$

The energy required from the engine to boost the satellite is

$$E_{\text{engine}} = E_f - E_i = -\frac{GM_E m}{2}\left(\frac{1}{R_f} - \frac{1}{R_i}\right)$$

$$= -\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(470\ \text{kg})}{2}$$

$$\times\left(\frac{1}{4.23 \times 10^7\ \text{m}} - \frac{1}{6.65 \times 10^6\ \text{m}}\right)$$

$$= 1.19 \times 10^{10}\ \text{J}$$

This is the energy equivalent of 89 gal of gasoline. NASA engineers must account for the changing mass of the spacecraft as it ejects burned fuel, something we have not done here. Would you expect the calculation that includes the effect of this changing mass to yield a greater or lesser amount of energy required from the engine?

If we wish to determine how the energy is distributed after the engine is fired, we find from Equation 14.18 that the change in kinetic energy is $\Delta K = (GM_E m/2)(1/R_f - 1/R_i) = -1.19 \times 10^{10}$ J (a decrease), and the corresponding change in potential energy is $\Delta U = -GM_E m(1/R_f - 1/R_i) = 2.38 \times 10^{10}$ J (an increase). Thus, the change in mechanical energy of the system is $\Delta E = \Delta K + \Delta U = 1.19 \times 10^{10}$ J, as we already calculated. The firing of the engine results in an increase in the total mechanical energy of the system. Because an increase in potential energy is accompanied by a decrease in kinetic energy, we conclude that the speed of an orbiting satellite decreases as its altitude increases.

Escape Speed

Suppose an object of mass m is projected vertically upward from the Earth's surface with an initial speed v_i, as illustrated in Figure 14.17. We can use energy considerations to find the minimum value of the initial speed needed to allow the object to escape the Earth's gravitational field. Equation 14.17 gives the total energy of the object at any point. At the surface of the Earth, $v = v_i$ and $r = r_i = R_E$. When the object reaches its maximum altitude, $v = v_f = 0$ and $r = r_f = r_{max}$. Because the total energy of the system is constant, substituting these conditions into Equation 14.20 gives

$$\frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{max}}$$

Solving for v_i^2 gives

$$v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right) \quad \textbf{(14.21)}$$

Therefore, if the initial speed is known, this expression can be used to calculate the maximum altitude h because we know that

$$h = r_{max} - R_E$$

We are now in a position to calculate **escape speed,** which is the minimum speed the object must have at the Earth's surface in order to escape from the influence of the Earth's gravitational field. Traveling at this minimum speed, the object continues to move farther and farther away from the Earth as its speed asymptotically approaches zero. Letting $r_{max} \to \infty$ in Equation 14.21 and taking $v_i = v_{esc}$, we obtain

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} \quad \textbf{(14.22)}$$

Escape speed

Note that this expression for v_{esc} is independent of the mass of the object. In other words, a spacecraft has the same escape speed as a molecule. Furthermore, the result is independent of the direction of the velocity and ignores air resistance.

If the object is given an initial speed equal to v_{esc}, its total energy is equal to zero. This can be seen by noting that when $r \to \infty$, the object's kinetic energy and its potential energy are both zero. If v_i is greater than v_{esc}, the total energy is greater than zero and the object has some residual kinetic energy as $r \to \infty$.

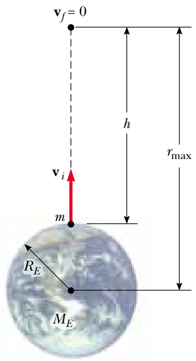


Figure 14.17 An object of mass m projected upward from the Earth's surface with an initial speed v_i reaches a maximum altitude h.

EXAMPLE 14.8 Escape Speed of a Rocket

Calculate the escape speed from the Earth for a 5 000-kg spacecraft, and determine the kinetic energy it must have at the Earth's surface in order to escape the Earth's gravitational field.

Solution Using Equation 14.22 gives

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{6.37 \times 10^6\ \text{m}}}$$

$$= 1.12 \times 10^4\ \text{m/s}$$

This corresponds to about 25 000 mi/h.

The kinetic energy of the spacecraft is

$$K = \tfrac{1}{2}mv_{esc}^2 = \tfrac{1}{2}(5.00 \times 10^3\ \text{kg})(1.12 \times 10^4\ \text{m/s})^2$$

$$= 3.14 \times 10^{11}\ \text{J}$$

This is equivalent to about 2 300 gal of gasoline.

TABLE 14.3 Escape Speeds from the Surfaces of the Planets, Moon, and Sun

Body	v_{esc} (km/s)
Mercury	4.3
Venus	10.3
Earth	11.2
Moon	2.3
Mars	5.0
Jupiter	60
Saturn	36
Uranus	22
Neptune	24
Pluto	1.1
Sun	618

Equations 14.21 and 14.22 can be applied to objects projected from any planet. That is, in general, the escape speed from the surface of any planet of mass M and radius R is

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

Escape speeds for the planets, the Moon, and the Sun are provided in Table 14.3. Note that the values vary from 1.1 km/s for Pluto to about 618 km/s for the Sun. These results, together with some ideas from the kinetic theory of gases (see Chapter 21), explain why some planets have atmospheres and others do not. As we shall see later, a gas molecule has an average kinetic energy that depends on the temperature of the gas. Hence, lighter molecules, such as hydrogen and helium, have a higher average speed than heavier species at the same temperature. When the average speed of the lighter molecules is not much less than the escape speed of a planet, a significant fraction of them have a chance to escape from the planet.

This mechanism also explains why the Earth does not retain hydrogen molecules and helium atoms in its atmosphere but does retain heavier molecules, such as oxygen and nitrogen. On the other hand, the very large escape speed for Jupiter enables that planet to retain hydrogen, the primary constituent of its atmosphere.

Quick Quiz 14.2

If you were a space prospector and discovered gold on an asteroid, it probably would not be a good idea to jump up and down in excitement over your find. Why?

Quick Quiz 14.3

Figure 14.18 is a drawing by Newton showing the path of a stone thrown from a mountaintop. He shows the stone landing farther and farther away when thrown at higher and higher speeds (at points D, E, F, and G), until finally it is thrown all the way around the Earth. Why didn't Newton show the stone landing at B and A before it was going fast enough to complete an orbit?

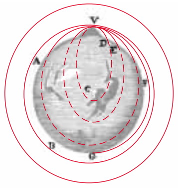


Figure 14.18 "The greater the velocity . . . with which [a stone] is projected, the farther it goes before it falls to the Earth. We may therefore suppose the velocity to be so increased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the Earth, till at last, exceeding the limits of the Earth, it should pass into space without touching." Sir Isaac Newton, *System of the World.*

Optional Section

14.9 THE GRAVITATIONAL FORCE BETWEEN AN EXTENDED OBJECT AND A PARTICLE

We have emphasized that the law of universal gravitation given by Equation 14.3 is valid only if the interacting objects are treated as particles. In view of this, how can we calculate the force between a particle and an object having finite dimensions? This is accomplished by treating the extended object as a collection of particles and making use of integral calculus. We first evaluate the potential energy function, and then calculate the gravitational force from that function.

We obtain the potential energy associated with a system consisting of a particle of mass m and an extended object of mass M by dividing the object into many elements, each having a mass ΔM_i (Fig. 14.19). The potential energy associated with the system consisting of any one element and the particle is $U = -Gm\,\Delta M_i/r_i$, where r_i is the distance from the particle to the element ΔM_i. The total potential energy of the overall system is obtained by taking the sum over all elements as $\Delta M_i \rightarrow 0$. In this limit, we can express U in integral form as

$$U = -Gm \int \frac{dM}{r} \qquad \textbf{(14.23)}$$

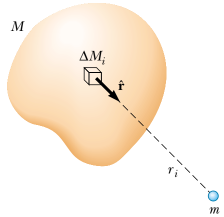


Figure 14.19 A particle of mass m interacting with an extended object of mass M. The total gravitational force exerted by the object on the particle can be obtained by dividing the object into numerous elements, each having a mass ΔM_i, and then taking a vector sum over the forces exerted by all elements.

Once U has been evaluated, we obtain the force exerted by the extended object on the particle by taking the negative derivative of this scalar function (see Section 8.6). If the extended object has spherical symmetry, the function U depends only on r, and the force is given by $-dU/dr$. We treat this situation in Section 14.10. In principle, one can evaluate U for any geometry; however, the integration can be cumbersome.

An alternative approach to evaluating the gravitational force between a particle and an extended object is to perform a vector sum over all mass elements of the object. Using the procedure outlined in evaluating U and the law of universal gravitation in the form shown in Equation 14.3, we obtain, for the total force exerted on the particle

Total force exerted on a particle by an extended object

$$\mathbf{F}_g = -Gm \int \frac{dM}{r^2}\,\hat{\mathbf{r}} \qquad \textbf{(14.24)}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from the element dM toward the particle (see Fig. 14.19) and the minus sign indicates that the direction of the force is opposite that of $\hat{\mathbf{r}}$. This procedure is not always recommended because working with a vector function is more difficult than working with the scalar potential energy function. However, if the geometry is simple, as in the following example, the evaluation of $\mathbf{F}$ can be straightforward.

EXAMPLE 14.9 Gravitational Force Between a Particle and a Bar

The left end of a homogeneous bar of length L and mass M is at a distance h from a particle of mass m (Fig. 14.20). Calculate the total gravitational force exerted by the bar on the particle.

Solution The arbitrary segment of the bar of length dx has a mass dM. Because the mass per unit length is constant, it follows that the ratio of masses dM/M is equal to the ratio of lengths dx/L, and so $dM = (M/L)\,dx$. In this problem, the variable r in Equation 14.24 is the distance x shown in Figure 14.20, the unit vector $\hat{\mathbf{r}}$ is $\hat{\mathbf{r}} = -\mathbf{i}$, and the force acting on the particle is to the right; therefore, Equation 14.24 gives us

$$\mathbf{F}_g = -Gm \int_h^{h+L} \frac{M dx}{L} \frac{1}{x^2}(-\mathbf{i}) = Gm\frac{M}{L}\int_h^{h+L}\frac{dx}{x^2}\,\mathbf{i}$$

$$\mathbf{F}_g = \frac{GmM}{L}\left[-\frac{1}{x}\right]_h^{h+L}\mathbf{i} = \boxed{\frac{GmM}{h(h+L)}\,\mathbf{i}}$$

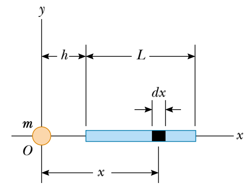


Figure 14.20 The gravitational force exerted by the bar on the particle is directed to the right. Note that the bar is *not* equivalent to a particle of mass M located at the center of mass of the bar.

We see that the force exerted on the particle is in the positive x direction, which is what we expect because the gravitational force is attractive.

Note that in the limit $L \rightarrow 0$, the force varies as $1/h^2$, which is what we expect for the force between two point masses. Furthermore, if $h \gg L$, the force also varies as $1/h^2$. This can be seen by noting that the denominator of the expression for $\mathbf{F}_g$ can be expressed in the form $h^2(1 + L/h)$, which is approximately equal to h^2 when $h \gg L$. Thus, when bodies are separated by distances that are great relative to their characteristic dimensions, they behave like particles.

Optional Section

14.10 THE GRAVITATIONAL FORCE BETWEEN A PARTICLE AND A SPHERICAL MASS

We have already stated that a large sphere attracts a particle outside it as if the total mass of the sphere were concentrated at its center. We now describe the force acting on a particle when the extended object is either a spherical shell or a solid sphere, and then apply these facts to some interesting systems.

Spherical Shell

Case 1. If a particle of mass m is located outside a spherical shell of mass M at, for instance, point P in Figure 14.21a, the shell attracts the particle as though the mass of the shell were concentrated at its center. We can show this, as Newton did, with integral calculus. Thus, as far as the gravitational force acting on a particle outside the shell is concerned, a spherical shell acts no differently from the solid spherical distributions of mass we have seen.

Case 2. If the particle is located inside the shell (at point P in Fig. 14.21b), the gravitational force acting on it can be shown to be zero.

We can express these two important results in the following way:

Force on a particle due to a spherical shell

$$\mathbf{F}_g = -\frac{GMm}{r^2}\,\hat{\mathbf{r}} \quad \text{for } r \geq R \qquad \textbf{(14.25a)}$$

$$\mathbf{F}_g = 0 \quad \text{for } r < R \qquad \textbf{(14.25b)}$$

The gravitational force as a function of the distance r is plotted in Figure 14.21c.

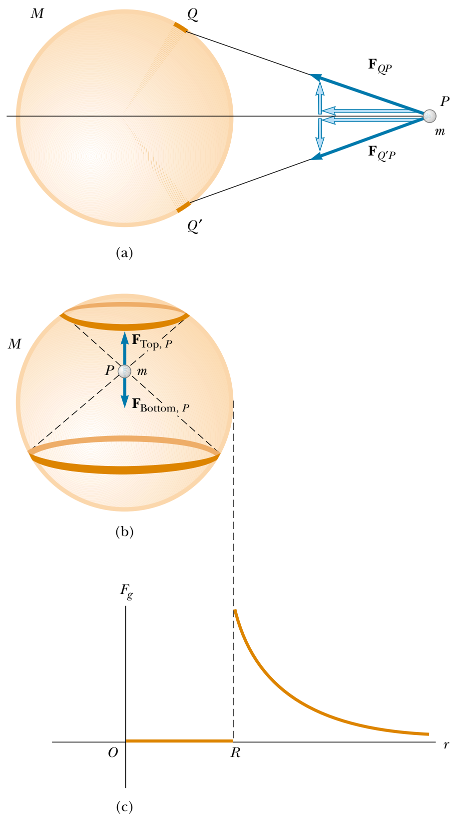


Figure 14.21 (a) The nonradial components of the gravitational forces exerted on a particle of mass m located at point P outside a spherical shell of mass M cancel out. (b) The spherical shell can be broken into rings. Even though point P is closer to the top ring than to the bottom ring, the bottom ring is larger, and the gravitational forces exerted on the particle at P by the matter in the two rings cancel each other. Thus, for a particle located at any point P inside the shell, there is no gravitational force exerted on the particle by the mass M of the shell. (c) The magnitude of the gravitational force versus the radial distance r from the center of the shell.

The shell does not act as a gravitational shield, which means that a particle inside a shell may experience forces exerted by bodies outside the shell.

Solid Sphere

Case 1. If a particle of mass m is located outside a homogeneous solid sphere of mass M (at point P in Fig. 14.22), the sphere attracts the particle as though the

mass of the sphere were concentrated at its center. We have used this notion at several places in this chapter already, and we can argue it from Equation 14.25a. A solid sphere can be considered to be a collection of concentric spherical shells. The masses of all of the shells can be interpreted as being concentrated at their common center, and the gravitational force is equivalent to that due to a particle of mass M located at that center.

Case 2. If a particle of mass m is located inside a homogeneous solid sphere of mass M (at point Q in Fig. 14.22), the gravitational force acting on it is due *only* to the mass M' contained within the sphere of radius $r < R$, shown in Figure 14.22. In other words,

Force on a particle due to a solid sphere

$$\mathbf{F}_g = -\frac{GmM}{r^2}\,\hat{\mathbf{r}} \qquad \text{for } r \geq R \tag{14.26a}$$

$$\mathbf{F}_g = -\frac{GmM'}{r^2}\,\hat{\mathbf{r}} \qquad \text{for } r < R \tag{14.26b}$$

This also follows from spherical-shell Case 1 because the part of the sphere that is

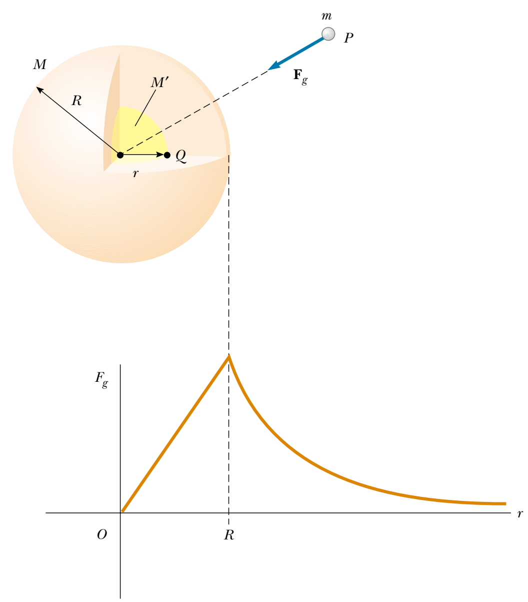


Figure 14.22 The gravitational force acting on a particle when it is outside a uniform solid sphere is GMm/r^2 and is directed toward the center of the sphere. The gravitational force acting on the particle when it is inside such a sphere is proportional to r and goes to zero at the center.

farther from the center than Q can be treated as a series of concentric spherical shells that do not exert a net force on the particle because the particle is inside them. Because the sphere is assumed to have a uniform density, it follows that the ratio of masses M'/M is equal to the ratio of volumes V'/V, where V is the total volume of the sphere and V' is the volume within the sphere of radius r only:

$$\frac{M'}{M} = \frac{V'}{V} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \frac{r^3}{R^3}$$

Solving this equation for M' and substituting the value obtained into Equation 14.26b, we have

$$\mathbf{F}_g = -\frac{GmM}{R^3} r\hat{\mathbf{r}} \qquad \text{for } r < R \tag{14.27}$$

This equation tells us that at the center of the solid sphere, where $r = 0$, the gravitational force goes to zero, as we intuitively expect. The force as a function of r is plotted in Figure 14.22.

Case 3. If a particle is located inside a solid sphere having a density ρ that is spherically symmetric but not uniform, then M' in Equation 14.26b is given by an integral of the form $M' = \int \rho\, dV$, where the integration is taken over the volume contained within the sphere of radius r in Figure 14.22. We can evaluate this integral if the radial variation of ρ is given. In this case, we take the volume element dV as the volume of a spherical shell of radius r and thickness dr, and thus $dV = 4\pi r^2\, dr$. For example, if $\rho = Ar$, where A is a constant, it is left to a problem (Problem 63) to show that $M' = \pi A r^4$. Hence, we see from Equation 14.26b that F is proportional to r^2 in this case and is zero at the center.

Quick Quiz 14.4

A particle is projected through a small hole into the interior of a spherical shell. Describe the motion of the particle inside the shell.

EXAMPLE 14.10 A Free Ride, Thanks to Gravity

An object of mass m moves in a smooth, straight tunnel dug between two points on the Earth's surface (Fig. 14.23). Show that the object moves with simple harmonic motion, and find the period of its motion. Assume that the Earth's density is uniform.

Solution The gravitational force exerted on the object acts toward the Earth's center and is given by Equation 14.27:

$$\mathbf{F}_g = -\frac{GmM}{R^3} r\hat{\mathbf{r}}$$

We receive our first indication that this force should result in simple harmonic motion by comparing it to Hooke's law, first seen in Section 7.3. Because the gravitational force on the object is linearly proportional to the displacement, the object experiences a Hooke's law force.

The y component of the gravitational force on the object is balanced by the normal force exerted by the tunnel wall, and the x component is

$$F_x = -\frac{GmM_E}{R_E^3} r\cos\theta$$

Because the x coordinate of the object is $x = r\cos\theta$, we can write

$$F_x = -\frac{GmM_E}{R_E^3} x$$

Applying Newton's second law to the motion along the x direction gives

$$F_x = -\frac{GmM_E}{R_E^3} x = ma_x$$

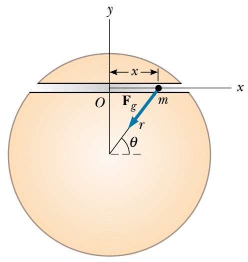


Figure 14.23 An object moves along a tunnel dug through the Earth. The component of the gravitational force $\mathbf{F}_g$ along the x axis is the driving force for the motion. Note that this component always acts toward O.

Solving for a_x, we obtain

$$a_x = -\frac{GM_E}{R_E^3} x$$

If we use the symbol ω^2 for the coefficient of x—$GM_E/R_E^3 = \omega^2$— we see that

$$(1) \qquad a_x = -\omega^2 x$$

an expression that matches the mathematical form of Equation 13.9, which gives the acceleration of a particle in simple harmonic motion: $a_x = -\omega^2 x$. Therefore, Equation (1), which we have derived for the acceleration of our object in the tunnel, is the acceleration equation for simple harmonic motion at angular speed ω with

$$\omega = \sqrt{\frac{GM_E}{R_E^3}}$$

Thus, the object in the tunnel moves in the same way as a block hanging from a spring! The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^3}{GM_E}}$$

$$= 2\pi\sqrt{\frac{(6.37 \times 10^6 \text{ m})^3}{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})}}$$

$$= 5.06 \times 10^3 \text{ s} = \boxed{84.3 \text{ min}}$$

This period is the same as that of a satellite traveling in a circular orbit just above the Earth's surface (ignoring any trees, buildings, or other objects in the way). Note that the result is independent of the length of the tunnel.

A proposal has been made to operate a mass-transit system between any two cities, using the principle described in this example. A one-way trip would take about 42 min. A more precise calculation of the motion must account for the fact that the Earth's density is not uniform. More important, there are many practical problems to consider. For instance, it would be impossible to achieve a frictionless tunnel, and so some auxiliary power source would be required. Can you think of other problems?

SUMMARY

Newton's law of universal gravitation states that the gravitational force of attraction between any two particles of masses m_1 and m_2 separated by a distance r has the magnitude

$$F_g = G\frac{m_1 m_2}{r^2} \tag{14.1}$$

where $G = 6.673 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$ is the universal gravitational constant. This equation enables us to calculate the force of attraction between masses under a wide variety of circumstances.

An object at a distance h above the Earth's surface experiences a gravitational force of magnitude mg', where g' is the free-fall acceleration at that elevation:

$$g' = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \tag{14.6}$$

In this expression, M_E is the mass of the Earth and R_E is its radius. Thus, the weight of an object decreases as the object moves away from the Earth's surface.

Kepler's laws of planetary motion state that

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

Kepler's third law can be expressed as

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)r^3 \qquad \textbf{(14.7)}$$

where M_S is the mass of the Sun and r is the orbital radius. For elliptical orbits, Equation 14.7 is valid if r is replaced by the semimajor axis a. Most planets have nearly circular orbits around the Sun.

The **gravitational field** at a point in space equals the gravitational force experienced by any test particle located at that point divided by the mass of the test particle:

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} \qquad \textbf{(14.10)}$$

The gravitational force is conservative, and therefore a potential energy function can be defined. The **gravitational potential energy** associated with two particles separated by a distance r is

$$U = -\frac{Gm_1m_2}{r} \qquad \textbf{(14.15)}$$

where U is taken to be zero as $r \rightarrow \infty$. The total potential energy for a system of particles is the sum of energies for all pairs of particles, with each pair represented by a term of the form given by Equation 14.15.

If an isolated system consists of a particle of mass m moving with a speed v in the vicinity of a massive body of mass M, the total energy E of the system is the sum of the kinetic and potential energies:

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \qquad \textbf{(14.17)}$$

The total energy is a constant of the motion. If the particle moves in a circular orbit of radius r around the massive body and if $M \gg m$, the total energy of the system is

$$E = -\frac{GMm}{2r} \qquad \textbf{(14.19)}$$

The total energy is negative for any bound system.

The **escape speed** for an object projected from the surface of the Earth is

$$v_{\text{esc}} = \sqrt{\frac{2GM_E}{R_E}} \qquad \textbf{(14.22)}$$

QUESTIONS

1. Use Kepler's second law to convince yourself that the Earth must move faster in its orbit during December, when it is closest to the Sun, than during June, when it is farthest from the Sun.
2. The gravitational force that the Sun exerts on the Moon is about twice as great as the gravitational force that the Earth exerts on the Moon. Why doesn't the Sun pull the Moon away from the Earth during a total eclipse of the Sun?
3. If a system consists of five particles, how many terms appear in the expression for the total potential energy? How many terms appear if the system consists of N particles?
4. Is it possible to calculate the potential energy function associated with a particle and an extended body without knowing the geometry or mass distribution of the extended body?
5. Does the escape speed of a rocket depend on its mass? Explain.
6. Compare the energies required to reach the Moon for a 10^5-kg spacecraft and a 10^3-kg satellite.
7. Explain why it takes more fuel for a spacecraft to travel from the Earth to the Moon than for the return trip. Estimate the difference.
8. Why don't we put a geosynchronous weather satellite in orbit around the 45th parallel? Wouldn't this be more useful for the United States than such a satellite in orbit around the equator?
9. Is the potential energy associated with the Earth–Moon system greater than, less than, or equal to the kinetic energy of the Moon relative to the Earth?
10. Explain why no work is done on a planet as it moves in a circular orbit around the Sun, even though a gravitational force is acting on the planet. What is the net work done on a planet during each revolution as it moves around the Sun in an elliptical orbit?
11. Explain why the force exerted on a particle by a uniform sphere must be directed toward the center of the sphere. Would this be the case if the mass distribution of the sphere were not spherically symmetric?
12. Neglecting the density variation of the Earth, what would be the period of a particle moving in a smooth hole dug between opposite points on the Earth's surface, passing through its center?
13. At what position in its elliptical orbit is the speed of a planet a maximum? At what position is the speed a minimum?
14. If you were given the mass and radius of planet X, how would you calculate the free-fall acceleration on the surface of this planet?
15. If a hole could be dug to the center of the Earth, do you think that the force on a mass m would still obey Equation 14.1 there? What do you think the force on m would be at the center of the Earth?
16. In his 1798 experiment, Cavendish was said to have "weighed the Earth." Explain this statement.
17. The gravitational force exerted on the *Voyager* spacecraft by Jupiter accelerated it toward escape speed from the Sun. How is this possible?
18. How would you find the mass of the Moon?
19. The *Apollo 13* spaceship developed trouble in the oxygen system about halfway to the Moon. Why did the spaceship continue on around the Moon and then return home, rather than immediately turn back to Earth?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 14.1 Newton's Law of Universal Gravitation
Section 14.2 Measuring the Gravitational Constant
Section 14.3 Free-Fall Acceleration and the Gravitational Force

1. Determine the order of magnitude of the gravitational force that you exert on another person 2 m away. In your solution, state the quantities that you measure or estimate and their values.
2. A 200-kg mass and a 500-kg mass are separated by 0.400 m. (a) Find the net gravitational force exerted by these masses on a 50.0-kg mass placed midway between them. (b) At what position (other than infinitely remote ones) can the 50.0-kg mass be placed so as to experience a net force of zero?
3. Three equal masses are located at three corners of a square of edge length ℓ, as shown in Figure P14.3. Find the gravitational field $\mathbf{g}$ at the fourth corner due to these masses.
4. Two objects attract each other with a gravitational force of magnitude 1.00×10^{-8} N when separated by 20.0 cm. If the total mass of the two objects is 5.00 kg, what is the mass of each?
5. Three uniform spheres of masses 2.00 kg, 4.00 kg, and 6.00 kg are placed at the corners of a right triangle, as illustrated in Figure P14.5. Calculate the resultant gravi-

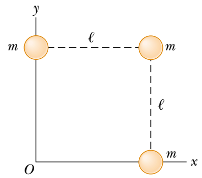


Figure P14.3

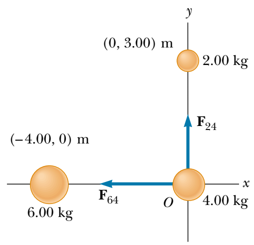


Figure P14.5

tational force on the 4.00-kg mass, assuming that the spheres are isolated from the rest of the Universe.

6. The free-fall acceleration on the surface of the Moon is about one-sixth that on the surface of the Earth. If the radius of the Moon is about $0.250R_E$, find the ratio of their average densities, $\rho_{\text{Moon}}/\rho_{\text{Earth}}$.

7. During a solar eclipse, the Moon, Earth, and Sun all lie on the same line, with the Moon between the Earth and the Sun. (a) What force is exerted by the Sun on the Moon? (b) What force is exerted by the Earth on the Moon? (c) What force is exerted by the Sun on the Earth?

8. The center-to-center distance between the Earth and the Moon is 384 400 km. The Moon completes an orbit in 27.3 days. (a) Determine the Moon's orbital speed. (b) If gravity were switched off, the Moon would move along a straight line tangent to its orbit, as described by Newton's first law. In its actual orbit in 1.00 s, how far does the Moon fall below the tangent line and toward the Earth?

WEB **9.** When a falling meteoroid is at a distance above the Earth's surface of 3.00 times the Earth's radius, what is its acceleration due to the Earth's gravity?

10. Two ocean liners, each with a mass of 40 000 metric tons, are moving on parallel courses, 100 m apart. What is the magnitude of the acceleration of one of the liners toward the other due to their mutual gravitational attraction? (Treat the ships as point masses.)

11. A student proposes to measure the gravitational constant G by suspending two spherical masses from the ceiling of a tall cathedral and measuring the deflection of the cables from the vertical. Draw a free-body diagram of one of the masses. If two 100.0-kg masses are suspended at the end of 45.00-m-long cables, and the cables are attached to the ceiling 1.000 m apart, what is the separation of the masses?

12. On the way to the Moon, the Apollo astronauts reached a point where the Moon's gravitational pull became stronger than the Earth's. (a) Determine the distance of this point from the center of the Earth. (b) What is the acceleration due to the Earth's gravity at this point?

Section 14.4 Kepler's Laws

Section 14.5 The Law of Gravity and the Motion of Planets

13. A particle of mass m moves along a straight line with constant speed in the x direction, a distance b from the x axis (Fig. P14.13). Show that Kepler's second law is satisfied by demonstrating that the two shaded triangles in the figure have the same area when $t_4 - t_3 = t_2 - t_1$.

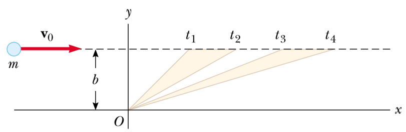


Figure P14.13

14. A communications satellite in geosynchronous orbit remains above a single point on the Earth's equator as the planet rotates on its axis. (a) Calculate the radius of its orbit. (b) The satellite relays a radio signal from a transmitter near the north pole to a receiver, also near the north pole. Traveling at the speed of light, how long is the radio wave in transit?

15. Plaskett's binary system consists of two stars that revolve in a circular orbit about a center of mass midway between them. This means that the masses of the two stars are equal (Fig. P14.15). If the orbital velocity of each star is 220 km/s and the orbital period of each is 14.4 days, find the mass M of each star. (For comparison, the mass of our Sun is 1.99×10^{30} kg.)

16. Plaskett's binary system consists of two stars that revolve in a circular orbit about a center of gravity midway between them. This means that the masses of the two stars are equal (see Fig. P14.15). If the orbital speed of each star is v and the orbital period of each is T, find the mass M of each star.

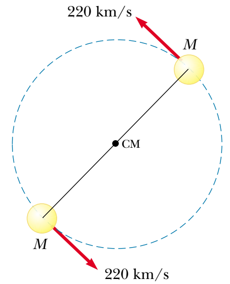


Figure P14.15 Problems 15 and 16.

17. The *Explorer VIII* satellite, placed into orbit November 3, 1960, to investigate the ionosphere, had the following orbit parameters: perigee, 459 km; apogee, 2 289 km (both distances above the Earth's surface); and period, 112.7 min. Find the ratio v_p/v_a of the speed at perigee to that at apogee.

18. Comet Halley (Fig. P14.18) approaches the Sun to within 0.570 AU, and its orbital period is 75.6 years (AU is the symbol for astronomical unit, where 1 AU = 1.50×10^{11} m is the mean Earth–Sun distance). How far from the Sun will Halley's comet travel before it starts its return journey?

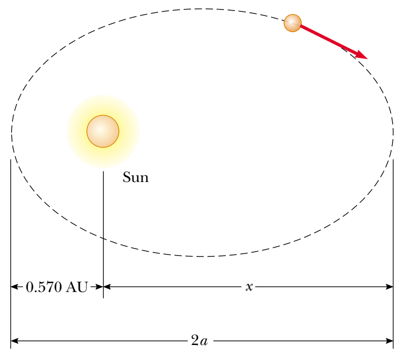


Figure P14.18

WEB **19.** Io, a satellite of Jupiter, has an orbital period of 1.77 days and an orbital radius of 4.22×10^5 km. From these data, determine the mass of Jupiter.

20. Two planets, X and Y, travel counterclockwise in circular orbits about a star, as shown in Figure P14.20. The radii of their orbits are in the ratio 3:1. At some time, they are aligned as in Figure P14.20a, making a straight line with the star. During the next five years, the angular displacement of planet X is 90.0°, as shown in Figure P14.20b. Where is planet Y at this time?

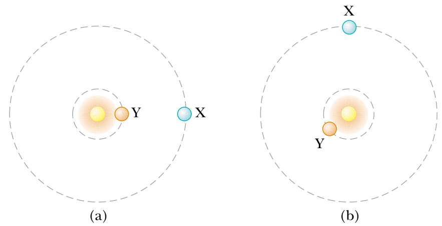


Figure P14.20

21. A synchronous satellite, which always remains above the same point on a planet's equator, is put in orbit around Jupiter so that scientists can study the famous red spot. Jupiter rotates once every 9.84 h. Use the data in Table 14.2 to find the altitude of the satellite.

22. Neutron stars are extremely dense objects that are formed from the remnants of supernova explosions. Many rotate very rapidly. Suppose that the mass of a certain spherical neutron star is twice the mass of the Sun and that its radius is 10.0 km. Determine the greatest possible angular speed it can have for the matter at the surface of the star on its equator to be just held in orbit by the gravitational force.

23. The Solar and Heliospheric Observatory (SOHO) spacecraft has a special orbit, chosen so that its view of the Sun is never eclipsed and it is always close enough to the Earth to transmit data easily. It moves in a near-circle around the Sun that is smaller than the Earth's circular orbit. Its period, however, is not less than 1 yr but is just equal to 1 yr. It is always located between the Earth and the Sun along the line joining them. Both objects exert gravitational forces on the observatory. Show that the spacecraft's distance from the Earth must be between 1.47×10^9 m and 1.48×10^9 m. In 1772 Joseph Louis Lagrange determined theoretically the special location that allows this orbit. The SOHO spacecraft took this position on February 14, 1996. (*Hint:* Use data that are precise to four digits. The mass of the Earth is 5.983×10^{24} kg.)

Section 14.6 The Gravitational Field

24. A spacecraft in the shape of a long cylinder has a length of 100 m, and its mass with occupants is 1 000 kg. It has

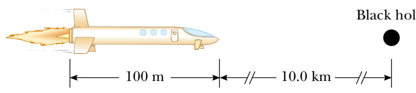


Figure P14.24

strayed too close to a 1.0-m-radius black hole having a mass 100 times that of the Sun (Fig. P14.24). The nose of the spacecraft is pointing toward the center of the black hole, and the distance between the nose and the black hole is 10.0 km. (a) Determine the total force on the spacecraft. (b) What is the difference in the gravitational fields acting on the occupants in the nose of the ship and on those in the rear of the ship, farthest from the black hole?

25. Compute the magnitude and direction of the gravitational field at a point P on the perpendicular bisector of two equal masses separated by a distance $2a$, as shown in Figure P14.25.

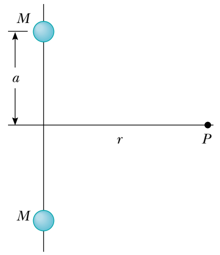


Figure P14.25

26. Find the gravitational field at a distance r along the axis of a thin ring of mass M and radius a.

Section 14.7 Gravitational Potential Energy

Note: Assume that $U = 0$ as $r \to \infty$.

27. A satellite of the Earth has a mass of 100 kg and is at an altitude of 2.00×10^6 m. (a) What is the potential energy of the satellite–Earth system? (b) What is the magnitude of the gravitational force exerted by the Earth on the satellite? (c) What force does the satellite exert on the Earth?

28. How much energy is required to move a 1 000-kg mass from the Earth's surface to an altitude twice the Earth's radius?

29. After our Sun exhausts its nuclear fuel, its ultimate fate may be to collapse to a *white-dwarf* state, in which it has approximately the same mass it has now but a radius equal to the radius of the Earth. Calculate (a) the average density of the white dwarf, (b) the acceleration due to gravity at its surface, and (c) the gravitational potential energy associated with a 1.00-kg object at its surface.

30. At the Earth's surface a projectile is launched straight up at a speed of 10.0 km/s. To what height will it rise? Ignore air resistance.

31. A system consists of three particles, each of mass 5.00 g, located at the corners of an equilateral triangle with sides of 30.0 cm. (a) Calculate the potential energy of the system. (b) If the particles are released simultaneously, where will they collide?

32. How much work is done by the Moon's gravitational field as a 1 000-kg meteor comes in from outer space and impacts the Moon's surface?

Section 14.8 Energy Considerations in Planetary and Satellite Motion

33. A 500-kg satellite is in a circular orbit at an altitude of 500 km above the Earth's surface. Because of air friction, the satellite is eventually brought to the Earth's surface, and it hits the Earth with a speed of 2.00 km/s. How much energy was transformed to internal energy by means of friction?

34. (a) What is the minimum speed, relative to the Sun, that is necessary for a spacecraft to escape the Solar System if it starts at the Earth's orbit? (b) *Voyager 1* achieved a maximum speed of 125 000 km/h on its way to photograph Jupiter. Beyond what distance from the Sun is this speed sufficient for a spacecraft to escape the Solar System?

35. A satellite with a mass of 200 kg is placed in Earth orbit at a height of 200 km above the surface. (a) Assuming a circular orbit, how long does the satellite take to complete one orbit? (b) What is the satellite's speed? (c) What is the minimum energy necessary to place this satellite in orbit (assuming no air friction)?

36. A satellite of mass m is placed in Earth orbit at an altitude h. (a) Assuming a circular orbit, how long does the satellite take to complete one orbit? (b) What is the satellite's speed? (c) What is the minimum energy necessary to place this satellite in orbit (assuming no air friction)?

WEB **37.** A spaceship is fired from the Earth's surface with an initial speed of 2.00×10^4 m/s. What will its speed be when it is very far from the Earth? (Neglect friction.)

38. A 1 000-kg satellite orbits the Earth at a constant altitude of 100 km. How much energy must be added to the system to move the satellite into a circular orbit at an altitude of 200 km?

39. A "treetop satellite" moves in a circular orbit just above the surface of a planet, which is assumed to offer no air resistance. Show that its orbital speed v and the escape speed from the planet are related by the expression $v_{esc} = \sqrt{2}v$.

40. The planet Uranus has a mass about 14 times the Earth's mass, and its radius is equal to about 3.7 Earth radii. (a) By setting up ratios with the corresponding Earth values, find the acceleration due to gravity at the cloud tops of Uranus. (b) Ignoring the rotation of the planet, find the minimum escape speed from Uranus.

41. Determine the escape velocity for a rocket on the far side of Ganymede, the largest of Jupiter's moons. The radius of Ganymede is 2.64×10^6 m, and its mass is 1.495×10^{23} kg. The mass of Jupiter is 1.90×10^{27} kg, and the distance between Jupiter and Ganymede is 1.071×10^9 m. Be sure to include the gravitational effect due to Jupiter, but you may ignore the motions of Jupiter and Ganymede as they revolve about their center of mass (Fig. P14.41).

Figure P14.41

42. In Robert Heinlein's *The Moon is a Harsh Mistress*, the colonial inhabitants of the Moon threaten to launch rocks down onto the Earth if they are not given independence (or at least representation). Assuming that a rail gun could launch a rock of mass m at twice the lunar escape speed, calculate the speed of the rock as it enters the Earth's atmosphere. (By *lunar escape speed* we mean the speed required to escape entirely from a stationary Moon alone in the Universe.)

43. Derive an expression for the work required to move an Earth satellite of mass m from a circular orbit of radius $2R_E$ to one of radius $3R_E$.

(Optional) Section 14.9 The Gravitational Force Between an Extended Object and a Particle

44. Consider two identical uniform rods of length L and mass m lying along the same line and having their closest points separated by a distance d (Fig. P14.44). Show that the mutual gravitational force between these rods has a magnitude

$$F = \frac{Gm^2}{L^2} \ln\left(\frac{(L+d)^2}{d(2L+d)}\right)$$

45. A uniform rod of mass M is in the shape of a semicircle of radius R (Fig. P14.45). Calculate the force on a point mass m placed at the center of the semicircle.

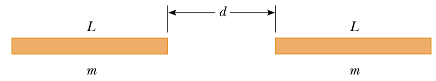


Figure P14.44

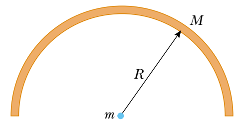


Figure P14.45

(Optional) Section 14.10 The Gravitational Force Between a Particle and a Spherical Mass

46. (a) Show that the period calculated in Example 14.10 can be written as

$$T = 2\pi\sqrt{\frac{R_E}{g}}$$

where g is the free-fall acceleration on the surface of the Earth. (b) What would this period be if tunnels were made through the Moon? (c) What practical problem regarding these tunnels on Earth would be removed if they were built on the Moon?

47. A 500-kg uniform solid sphere has a radius of 0.400 m. Find the magnitude of the gravitational force exerted by the sphere on a 50.0-g particle located (a) 1.50 m from the center of the sphere, (b) at the surface of the sphere, and (c) 0.200 m from the center of the sphere.

48. A uniform solid sphere of mass m_1 and radius R_1 is inside and concentric with a spherical shell of mass m_2 and radius R_2 (Fig. P14.48). Find the gravitational force exerted by the spheres on a particle of mass m located at (a) $r = a$, (b) $r = b$, and (c) $r = c$, where r is measured from the center of the spheres.

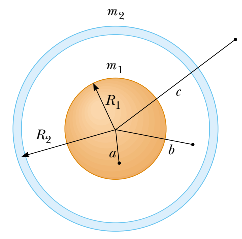


Figure P14.48

ADDITIONAL PROBLEMS

49. Let Δg_M represent the difference in the gravitational fields produced by the Moon at the points on the Earth's surface nearest to and farthest from the Moon. Find the fraction $\Delta g_M/g$, where g is the Earth's gravitational field. (This difference is responsible for the occurrence of the *lunar tides* on the Earth.)

50. Two spheres having masses M and $2M$ and radii R and $3R$, respectively, are released from rest when the distance between their centers is $12R$. How fast will each sphere be moving when they collide? Assume that the two spheres interact only with each other.

51. In Larry Niven's science-fiction novel *Ringworld,* a rigid ring of material rotates about a star (Fig. P14.51). The rotational speed of the ring is 1.25×10^6 m/s, and its radius is 1.53×10^{11} m. (a) Show that the centripetal acceleration of the inhabitants is 10.2 m/s^2. (b) The inhabitants of this ring world experience a normal contact force **n**. Acting alone, this normal force would produce an inward acceleration of 9.90 m/s^2. Additionally, the star at the center of the ring exerts a gravitational force on the ring and its inhabitants. The difference between the total acceleration and the acceleration provided by the normal force is due to the gravitational attraction of the central star. Show that the mass of the star is approximately 10^{32} kg.

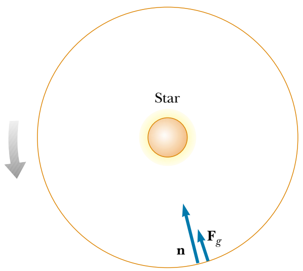


Figure P14.51

52. (a) Show that the rate of change of the free-fall acceleration with distance above the Earth's surface is

$$\frac{dg}{dr} = -\frac{2GM_E}{R_E^{\,3}}$$

This rate of change over distance is called a *gradient.* (b) If h is small compared to the radius of the Earth, show that the difference in free-fall acceleration between two points separated by vertical distance h is

$$|\Delta g| = \frac{2GM_E h}{R_E^{\,3}}$$

(c) Evaluate this difference for $h = 6.00$ m, a typical height for a two-story building.

53. A particle of mass m is located inside a uniform solid sphere of radius R and mass M, at a distance r from its center. (a) Show that the gravitational potential energy of the system is $U = (GmM/2R^3)r^2 - 3GmM/2R$. (b) Write an expression for the amount of work done by the gravitational force in bringing the particle from the surface of the sphere to its center.

54. *Voyagers 1* and *2* surveyed the surface of Jupiter's moon Io and photographed active volcanoes spewing liquid sulfur to heights of 70 km above the surface of this moon. Find the speed with which the liquid sulfur left the volcano. Io's mass is 8.9×10^{22} kg, and its radius is 1 820 km.

55. As an astronaut, you observe a small planet to be spherical. After landing on the planet, you set off, walking always straight ahead, and find yourself returning to your spacecraft from the opposite side after completing a lap of 25.0 km. You hold a hammer and a falcon feather at a height of 1.40 m, release them, and observe that they fall together to the surface in 29.2 s. Determine the mass of the planet.

56. A cylindrical habitat in space, 6.00 km in diameter and 30 km long, was proposed by G. K. O'Neill in 1974. Such a habitat would have cities, land, and lakes on the inside surface and air and clouds in the center. All of these would be held in place by the rotation of the cylinder about its long axis. How fast would the cylinder have to rotate to imitate the Earth's gravitational field at the walls of the cylinder?

WEB **57.** In introductory physics laboratories, a typical Cavendish balance for measuring the gravitational constant G uses lead spheres with masses of 1.50 kg and 15.0 g whose centers are separated by about 4.50 cm. Calculate the gravitational force between these spheres, treating each as a point mass located at the center of the sphere.

58. Newton's law of universal gravitation is valid for distances covering an enormous range, but it is thought to fail for very small distances, where the structure of space itself is uncertain. The crossover distance, far less than the diameter of an atomic nucleus, is called the *Planck length.* It is determined by a combination of the constants G, c, and h, where c is the speed of light in vacuum and h is Planck's constant (introduced briefly in Chapter 11 and discussed in greater detail in Chapter 40) with units of angular momentum. (a) Use dimensional analysis to find a combination of these three universal constants that has units of length. (b) Determine the order of magnitude of the Planck length. (*Hint:* You will need to consider noninteger powers of the constants.)

59. Show that the escape speed from the surface of a planet of uniform density is directly proportional to the radius of the planet.

60. (a) Suppose that the Earth (or another object) has density $\rho(r)$, which can vary with radius but is spherically symmetric. Show that at any particular radius r inside the Earth, the gravitational field strength $g(r)$ will increase as r increases, if and only if the density there exceeds 2/3 the average density of the portion of the Earth inside the radius r. (b) The Earth as a whole has an average density of 5.5 g/cm^3, while the density at the surface is 1.0 g/cm^3 on the oceans and about 3 g/cm^3 on land. What can you infer from this?

WEB **61.** Two hypothetical planets of masses m_1 and m_2 and radii r_1 and r_2, respectively, are nearly at rest when they are an infinite distance apart. Because of their gravitational attraction, they head toward each other on a collision course. (a) When their center-to-center separation is d, find expressions for the speed of each planet and their *relative* velocity. (b) Find the kinetic energy of each planet *just* before they collide, if $m_1 = 2.00 \times 10^{24}$ kg, $m_2 = 8.00 \times 10^{24}$ kg, $r_1 = 3.00 \times 10^6$ m, and $r_2 = 5.00 \times 10^6$ m. (*Hint:* Both energy and momentum are conserved.)

62. The maximum distance from the Earth to the Sun (at our aphelion) is 1.521×10^{11} m, and the distance of closest approach (at perihelion) is 1.471×10^{11} m. If the Earth's orbital speed at perihelion is 30.27 km/s, determine (a) the Earth's orbital speed at aphelion, (b) the kinetic and potential energies at perihelion, and (c) the kinetic and potential energies at aphelion. Is the total energy constant? (Neglect the effect of the Moon and other planets.)

63. A sphere of mass M and radius R has a nonuniform density that varies with r, the distance from its center, according to the expression $\rho = Ar$, for $0 \le r \le R$. (a) What is the constant A in terms of M and R? (b) Determine an expression for the force exerted on a particle of mass m placed outside the sphere. (c) Determine an expression for the force exerted on the particle if it is inside the sphere. (*Hint:* See Section 14.10 and note that the distribution is spherically symmetric.)

64. (a) Determine the amount of work (in joules) that must be done on a 100-kg payload to elevate it to a height of 1 000 km above the Earth's surface. (b) Determine the amount of additional work that is required to put the payload into circular orbit at this elevation.

65. X-ray pulses from Cygnus X-1, a celestial x-ray source, have been recorded during high-altitude rocket flights. The signals can be interpreted as originating when a blob of ionized matter orbits a black hole with a period of 5.0 ms. If the blob is in a circular orbit about a black hole whose mass is $20M_{\text{Sun}}$, what is the orbital radius?

66. Studies of the relationship of the Sun to its galaxy—the Milky Way—have revealed that the Sun is located near the outer edge of the galactic disk, about 30 000 lightyears from the center. Furthermore, it has been found that the Sun has an orbital speed of approximately 250 km/s around the galactic center. (a) What is the period of the Sun's galactic motion? (b) What is the order of magnitude of the mass of the Milky Way galaxy? Suppose that the galaxy is made mostly of stars, of which the Sun is typical. What is the order of magnitude of the number of stars in the Milky Way?

67. The oldest artificial satellite in orbit is *Vanguard I,* launched March 3, 1958. Its mass is 1.60 kg. In its initial orbit, its minimum distance from the center of the Earth was 7.02 Mm, and its speed at this perigee point was 8.23 km/s. (a) Find its total energy. (b) Find the magnitude of its angular momentum. (c) Find its speed at apogee and its maximum (apogee) distance from the center of the Earth. (d) Find the semimajor axis of its orbit. (e) Determine its period.

68. A rocket is given an initial speed vertically upward of $v_i = 2\sqrt{Rg}$ at the surface of the Earth, which has radius R and surface free-fall acceleration g. The rocket motors are quickly cut off, and thereafter the rocket coasts under the action of gravitational forces only. (Ignore atmospheric friction and the Earth's rotation.) Derive an expression for the subsequent speed v as a function of the distance r from the center of the Earth in terms of g, R, and r.

69. Two stars of masses M and m, separated by a distance d, revolve in circular orbits about their center of mass (Fig. P14.69). Show that each star has a period given by

$$T^2 = \frac{4\pi^2 d^3}{G(M + m)}$$

(*Hint:* Apply Newton's second law to each star, and note that the center-of-mass condition requires that $Mr_2 = mr_1$, where $r_1 + r_2 = d$.)

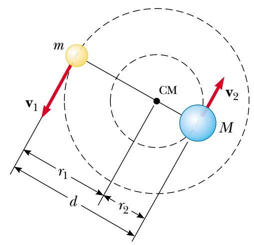


Figure P14.69

70. (a) A 5.00-kg mass is released 1.20×10^7 m from the center of the Earth. It moves with what acceleration relative to the Earth? (b) A 2.00×10^{24} kg mass is released 1.20×10^7 m from the center of the Earth. It moves with what acceleration relative to the Earth? Assume that the objects behave as pairs of particles, isolated from the rest of the Universe.

71. The acceleration of an object moving in the gravitational field of the Earth is

$$\mathbf{a} = -\frac{GM_E}{r^3}\mathbf{r}$$

where **r** is the position vector directed from the center of the Earth to the object. Choosing the origin at the center of the Earth and assuming that the small object is moving in the *xy* plane, we find that the rectangular (cartesian) components of its acceleration are

$$a_x = -\frac{GM_E x}{(x^2 + y^2)^{3/2}} \qquad a_y = -\frac{GM_E y}{(x^2 + y^2)^{3/2}}$$

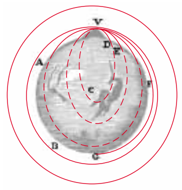
Use a computer to set up and carry out a numerical prediction of the motion of the object, according to Euler's method. Assume that the initial position of the object is $x = 0$ and $y = 2R_E$, where R_E is the radius of the Earth. Give the object an initial velocity of 5 000 m/s in the *x* direction. The time increment should be made as small as practical. Try 5 s. Plot the *x* and *y* coordinates of the object as time goes on. Does the object hit the Earth? Vary the initial velocity until you find a circular orbit.

Answers to Quick Quizzes

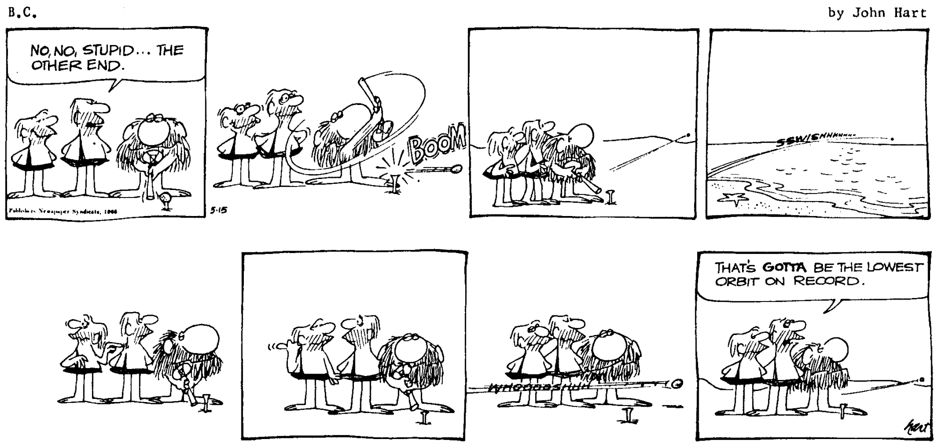
14.1 Kepler's third law (Eq. 14.7), which applies to all the planets, tells us that the period of a planet is proportional to $r^{3/2}$. Because Saturn and Jupiter are farther from the Sun than the Earth is, they have longer periods. The Sun's gravitational field is much weaker at Saturn and Jupiter than it is at the Earth. Thus, these planets experience much less centripetal acceleration than the Earth does, and they have correspondingly longer periods.

14.2 The mass of the asteroid might be so small that you would be able to exceed escape velocity by leg power alone. You would jump up, but you would never come back down!

14.3 Kepler's first law applies not only to planets orbiting the Sun but also to any relatively small object orbiting another under the influence of gravity. Any elliptical path that does not touch the Earth before reaching point *G* will continue around the other side to point *V* in a complete orbit (see figure in next column).



14.4 The gravitational force is zero inside the shell (Eq. 14.25b). Because the force on it is zero, the particle moves with constant velocity in the direction of its original motion outside the shell until it hits the wall opposite the entry hole. Its path thereafter depends on the nature of the collision and on the particle's original direction.



PUZZLER

Have you ever wondered why a tennis ball is fuzzy and why a golf ball has dimples? A "spitball" is an illegal baseball pitch because it makes the ball act too much like the fuzzy tennis ball or the dimpled golf ball. What principles of physics govern the behavior of these three pieces of sporting equipment (and also keep airplanes in the sky)? *(George Semple)*

chapter 15

Fluid Mechanics

Chapter Outline

Matter is normally classified as being in one of three states: solid, liquid, or gas. From everyday experience, we know that a solid has a definite volume and shape. A brick maintains its familiar shape and size day in and day out. We also know that a liquid has a definite volume but no definite shape. Finally, we know that an unconfined gas has neither a definite volume nor a definite shape. These definitions help us picture the states of matter, but they are somewhat artificial. For example, asphalt and plastics are normally considered solids, but over long periods of time they tend to flow like liquids. Likewise, most substances can be a solid, a liquid, or a gas (or a combination of any of these), depending on the temperature and pressure. In general, the time it takes a particular substance to change its shape in response to an external force determines whether we treat the substance as a solid, as a liquid, or as a gas.

A **fluid** is a collection of molecules that are randomly arranged and held together by weak cohesive forces and by forces exerted by the walls of a container. Both liquids and gases are fluids.

In our treatment of the mechanics of fluids, we shall see that we do not need to learn any new physical principles to explain such effects as the buoyant force acting on a submerged object and the dynamic lift acting on an airplane wing. First, we consider the mechanics of a fluid at rest—that is, *fluid statics*—and derive an expression for the pressure exerted by a fluid as a function of its density and depth. We then treat the mechanics of fluids in motion—that is, *fluid dynamics*. We can describe a fluid in motion by using a model in which we make certain simplifying assumptions. We use this model to analyze some situations of practical importance. An analysis leading to *Bernoulli's equation* enables us to determine relationships between the pressure, density, and velocity at every point in a fluid.

15.1 PRESSURE

Fluids do not sustain shearing stresses or tensile stresses; thus, the only stress that can be exerted on an object submerged in a fluid is one that tends to compress the object. In other words, the force exerted by a fluid on an object is always perpendicular to the surfaces of the object, as shown in Figure 15.1.

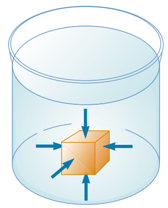


Figure 15.1 At any point on the surface of a submerged object, the force exerted by the fluid is perpendicular to the surface of the object. The force exerted by the fluid on the walls of the container is perpendicular to the walls at all points.

The pressure in a fluid can be measured with the device pictured in Figure 15.2. The device consists of an evacuated cylinder that encloses a light piston connected to a spring. As the device is submerged in a fluid, the fluid presses on the top of the piston and compresses the spring until the inward force exerted by the fluid is balanced by the outward force exerted by the spring. The fluid pressure can be measured directly if the spring is calibrated in advance. If F is the magnitude of the force exerted on the piston and A is the surface area of the piston,

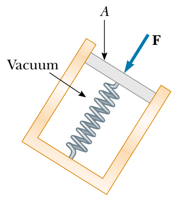


Figure 15.2 A simple device for measuring the pressure exerted by a fluid.

then the **pressure** P of the fluid at the level to which the device has been submerged is defined as the ratio F/A:

Definition of pressure

$$P \equiv \frac{F}{A} \qquad (15.1)$$

Note that pressure is a scalar quantity because it is proportional to the magnitude of the force on the piston.

To define the pressure at a specific point, we consider a fluid acting on the device shown in Figure 15.2. If the force exerted by the fluid over an infinitesimal surface element of area dA containing the point in question is dF, then the pressure at that point is

$$P = \frac{dF}{dA} \qquad (15.2)$$

Snowshoes keep you from sinking into soft snow because they spread the downward force you exert on the snow over a large area, reducing the pressure on the snow's surface.

As we shall see in the next section, the pressure exerted by a fluid varies with depth. Therefore, to calculate the total force exerted on a flat wall of a container, we must integrate Equation 15.2 over the surface area of the wall.

Because pressure is force per unit area, it has units of newtons per square meter (N/m^2) in the SI system. Another name for the SI unit of pressure is **pascal** (Pa):

$$1 \text{ Pa} \equiv 1 \text{ N/m}^2 \qquad (15.3)$$

Quick Quiz 15.1

Suppose you are standing directly behind someone who steps back and accidentally stomps on your foot with the heel of one shoe. Would you be better off if that person were a professional basketball player wearing sneakers or a petite woman wearing spike-heeled shoes? Explain.

QuickLab

Place a tack between your thumb and index finger, as shown in the figure. Now very gently squeeze the tack and note the sensation. The pointed end of the tack causes pain, and the blunt end does not. According to Newton's third law, the force exerted by the tack on the thumb is equal in magnitude and opposite in direction to the force exerted by the tack on the index finger. However, the pressure at the pointed end of the tack is much greater than the pressure at the blunt end. (Remember that pressure is force per unit area.)

Quick Quiz 15.2

After a long lecture, the daring physics professor stretches out for a nap on a bed of nails, as shown in Figure 15.3. How is this possible?

Figure 15.3

Example 15.1 The Water Bed

The mattress of a water bed is 2.00 m long by 2.00 m wide and 30.0 cm deep. (a) Find the weight of the water in the mattress.

Solution The density of water is 1 000 kg/m^3 (Table 15.1), and so the mass of the water is

$$M = \rho V = (1\,000 \text{ kg/m}^3)(1.20 \text{ m}^3) = 1.20 \times 10^3 \text{ kg}$$

and its weight is

$$Mg = (1.20 \times 10^3 \text{ kg})(9.80 \text{ m/s}^2) = 1.18 \times 10^4 \text{ N}$$

This is approximately 2 650 lb. (A regular bed weighs approximately 300 lb.) Because this load is so great, such a water bed is best placed in the basement or on a sturdy, well-supported floor.

(b) Find the pressure exerted by the water on the floor when the bed rests in its normal position. Assume that the entire lower surface of the bed makes contact with the floor.

Solution When the bed is in its normal position, the cross-sectional area is 4.00 m^2; thus, from Equation 15.1, we find that

$$P = \frac{1.18 \times 10^4 \text{ N}}{4.00 \text{ m}^2} = 2.95 \times 10^3 \text{ Pa}$$

TABLE 15.1 Densities of Some Common Substances at Standard Temperature (0°C) and Pressure (Atmospheric)

Substance	ρ (kg/m^3)	Substance	ρ (kg/m^3)
Air	1.29	Ice	0.917×10^3
Aluminum	2.70×10^3	Iron	7.86×10^3
Benzene	0.879×10^3	Lead	11.3×10^3
Copper	8.92×10^3	Mercury	13.6×10^3
Ethyl alcohol	0.806×10^3	Oak	0.710×10^3
Fresh water	1.00×10^3	Oxygen gas	1.43
Glycerine	1.26×10^3	Pine	0.373×10^3
Gold	19.3×10^3	Platinum	21.4×10^3
Helium gas	1.79×10^{-1}	Seawater	1.03×10^3
Hydrogen gas	8.99×10^{-2}	Silver	10.5×10^3

15.2 VARIATION OF PRESSURE WITH DEPTH

As divers well know, water pressure increases with depth. Likewise, atmospheric pressure decreases with increasing altitude; it is for this reason that aircraft flying at high altitudes must have pressurized cabins.

We now show how the pressure in a liquid increases linearly with depth. As Equation 1.1 describes, the *density* of a substance is defined as its mass per unit volume: $\rho \equiv m/V$. Table 15.1 lists the densities of various substances. These values vary slightly with temperature because the volume of a substance is temperature dependent (as we shall see in Chapter 19). Note that under standard conditions (at 0°C and at atmospheric pressure) the densities of gases are about 1/1 000 the densities of solids and liquids. This difference implies that the average molecular spacing in a gas under these conditions is about ten times greater than that in a solid or liquid.

Now let us consider a fluid of density ρ at rest and open to the atmosphere, as shown in Figure 15.4. We assume that ρ is constant; this means that the fluid is incompressible. Let us select a sample of the liquid contained within an imaginary cylinder of cross-sectional area A extending from the surface to a depth h. The

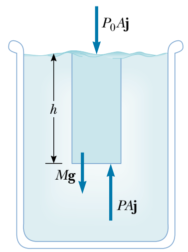


Figure 15.4 How pressure varies with depth in a fluid. The net force exerted on the volume of water within the darker region must be zero.

QuickLab

Poke two holes in the side of a paper or polystyrene cup—one near the top and the other near the bottom. Fill the cup with water and watch the water flow out of the holes. Why does water exit from the bottom hole at a higher speed than it does from the top hole?

pressure exerted by the outside liquid on the bottom face of the cylinder is P, and the pressure exerted on the top face of the cylinder is the atmospheric pressure P_0. Therefore, the upward force exerted by the outside fluid on the bottom of the cylinder is PA, and the downward force exerted by the atmosphere on the top is P_0A. The mass of liquid in the cylinder is $M = \rho V = \rho Ah$; therefore, the weight of the liquid in the cylinder is $Mg = \rho Ahg$. Because the cylinder is in equilibrium, the net force acting on it must be zero. Choosing upward to be the positive y direction, we see that

$$\sum F_y = PA - P_0A - Mg = 0$$

or

$$PA - P_0A - \rho Ahg = 0$$

$$PA - P_0A = \rho Ahg$$

Variation of pressure with depth

$$P = P_0 + \rho gh \qquad \textbf{(15.4)}$$

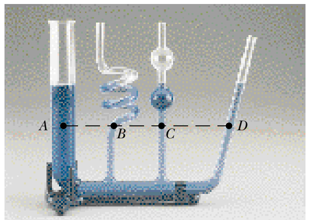
That is, **the pressure P at a depth h below the surface of a liquid open to the atmosphere is *greater* than atmospheric pressure by an amount ρgh.** In our calculations and working of end-of-chapter problems, we usually take atmospheric pressure to be

$$P_0 = 1.00 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$$

Equation 15.4 implies that the pressure is the same at all points having the same depth, independent of the shape of the container.

Quick Quiz 15.3

In the derivation of Equation 15.4, why were we able to ignore the pressure that the liquid exerts on the sides of the cylinder?



This arrangement of interconnected tubes demonstrates that the pressure in a liquid is the same at all points having the same elevation. For example, the pressure is the same at points A, B, C, and D.

In view of the fact that the pressure in a fluid depends on depth and on the value of P_0, any increase in pressure at the surface must be transmitted to every other point in the fluid. This concept was first recognized by the French scientist Blaise Pascal (1623–1662) and is called **Pascal's law: A change in the pressure applied to a fluid is transmitted undiminished to every point of the fluid and to the walls of the container.**

An important application of Pascal's law is the hydraulic press illustrated in Figure 15.5a. A force of magnitude F_1 is applied to a small piston of surface area A_1. The pressure is transmitted through a liquid to a larger piston of surface area A_2. Because the pressure must be the same on both sides, $P = F_1/A_1 = F_2/A_2$. Therefore, the force F_2 is greater than the force F_1 by a factor A_2/A_1, which is called the *force-multiplying factor*. Because liquid is neither added nor removed, the volume pushed down on the left as the piston moves down a distance d_1 equals the volume pushed up on the right as the right piston moves up a distance d_2. That is, $A_1d_1 = A_2d_2$; thus, the force-multiplying factor can also be written as d_1/d_2. Note that $F_1d_1 = F_2d_2$. Hydraulic brakes, car lifts, hydraulic jacks, and forklifts all make use of this principle (Fig. 15.5b).

Quick Quiz 15.4

A grain silo has many bands wrapped around its perimeter (Fig. 15.6). Why is the spacing between successive bands smaller at the lower portions of the silo, as shown in the photograph?

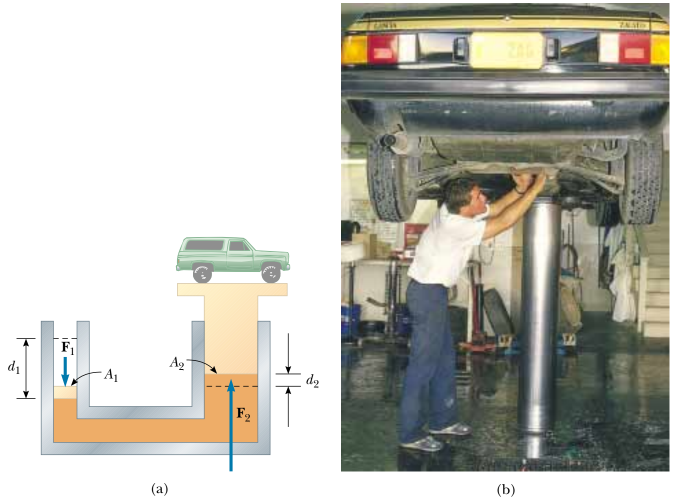


Figure 15.5 (a) Diagram of a hydraulic press. Because the increase in pressure is the same on the two sides, a small force $\mathbf{F}_1$ at the left produces a much greater force $\mathbf{F}_2$ at the right. (b) A vehicle undergoing repair is supported by a hydraulic lift in a garage.

Figure 15.6

EXAMPLE 15.2 The Car Lift

In a car lift used in a service station, compressed air exerts a force on a small piston that has a circular cross section and a radius of 5.00 cm. This pressure is transmitted by a liquid to a piston that has a radius of 15.0 cm. What force must the compressed air exert to lift a car weighing 13 300 N? What air pressure produces this force?

Solution Because the pressure exerted by the compressed air is transmitted undiminished throughout the liquid, we have

$$F_1 = \left(\frac{A_1}{A_2}\right)F_2 = \frac{\pi(5.00 \times 10^{-2}\text{ m})^2}{\pi(15.0 \times 10^{-2}\text{ m})^2}(1.33 \times 10^4 \text{ N})$$
$$= 1.48 \times 10^3 \text{ N}$$

The air pressure that produces this force is

$$P = \frac{F_1}{A_1} = \frac{1.48 \times 10^3 \text{ N}}{\pi(5.00 \times 10^{-2}\text{ m})^2} = \boxed{1.88 \times 10^5 \text{ Pa}}$$

This pressure is approximately twice atmospheric pressure.

The input work (the work done by $\mathbf{F}_1$) is equal to the output work (the work done by $\mathbf{F}_2$), in accordance with the principle of conservation of energy.

EXAMPLE 15.3 A Pain in the Ear

Estimate the force exerted on your eardrum due to the water above when you are swimming at the bottom of a pool that is 5.0 m deep.

Solution First, we must find the unbalanced pressure on the eardrum; then, after estimating the eardrum's surface area, we can determine the force that the water exerts on it.

The air inside the middle ear is normally at atmospheric pressure P_0. Therefore, to find the net force on the eardrum, we must consider the difference between the total pressure at the bottom of the pool and atmospheric pressure:

$$\begin{aligned} P_{bot} - P_0 &= \rho g h \\ &= (1.00 \times 10^3 \text{ kg/m}^3)(9.80 \text{ m/s}^2)(5.0 \text{ m}) \\ &= 4.9 \times 10^4 \text{ Pa} \end{aligned}$$

We estimate the surface area of the eardrum to be approximately $1 \text{ cm}^2 = 1 \times 10^{-4} \text{ m}^2$. This means that the force on it is $F = (P_{bot} - P_0)A \approx 5$ N. Because a force on the eardrum of this magnitude is extremely uncomfortable, swimmers often “pop their ears” while under water, an action that pushes air from the lungs into the middle ear. Using this technique equalizes the pressure on the two sides of the eardrum and relieves the discomfort.

EXAMPLE 15.4 The Force on a Dam

Water is filled to a height H behind a dam of width w (Fig. 15.7). Determine the resultant force exerted by the water on the dam.

Solution Because pressure varies with depth, we cannot calculate the force simply by multiplying the area by the pressure. We can solve the problem by finding the force dF exerted on a narrow horizontal strip at depth h and then integrating the expression to find the total force. Let us imagine a vertical y axis, with $y = 0$ at the bottom of the dam and our strip a distance y above the bottom.

We can use Equation 15.4 to calculate the pressure at the depth h; we omit atmospheric pressure because it acts on both sides of the dam:

$$P = \rho g h = \rho g(H - y)$$

Using Equation 15.2, we find that the force exerted on the shaded strip of area $dA = w\,dy$ is

$$dF = P\,dA = \rho g(H - y)w\,dy$$

Therefore, the total force on the dam is

$$F = \int P\,dA = \int_0^H \rho g(H - y)w\,dy = \tfrac{1}{2}\rho g w H^2$$

Note that the thickness of the dam shown in Figure 15.7 increases with depth. This design accounts for the greater and greater pressure that the water exerts on the dam at greater depths.

Exercise Find an expression for the average pressure on the dam from the total force exerted by the water on the dam.

Answer $\tfrac{1}{2}\rho g H$.

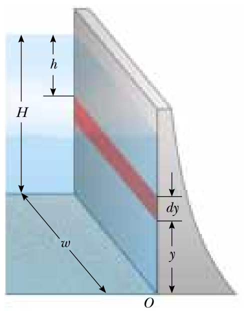


Figure 15.7 Because pressure varies with depth, the total force exerted on a dam must be obtained from the expression $F = \int P\,dA$, where dA is the area of the dark strip.

15.3 PRESSURE MEASUREMENTS

One simple device for measuring pressure is the open-tube manometer illustrated in Figure 15.8a. One end of a U-shaped tube containing a liquid is open to the atmosphere, and the other end is connected to a system of unknown pressure P. The difference in pressure $P - P_0$ is equal to $\rho g h$; hence, $P = P_0 + \rho g h$. The pressure P is called the **absolute pressure,** and the difference $P - P_0$ is called the **gauge pressure.** The latter is the value that normally appears on a pressure gauge. For example, the pressure you measure in your bicycle tire is the gauge pressure.

Another instrument used to measure pressure is the common *barometer,* which was invented by Evangelista Torricelli (1608–1647). The barometer consists of a long, mercury-filled tube closed at one end and inverted into an open container of mercury (Fig. 15.8b). The closed end of the tube is nearly a vacuum, and so its pressure can be taken as zero. Therefore, it follows that $P_0 = \rho g h$, where h is the height of the mercury column.

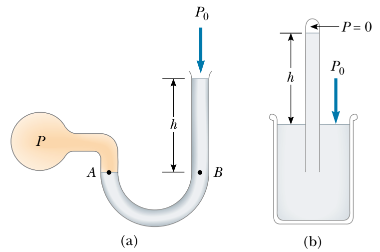


Figure 15.8 Two devices for measuring pressure: (a) an open-tube manometer and (b) a mercury barometer.

One atmosphere ($P_0 = 1$ atm) of pressure is defined as the pressure that causes the column of mercury in a barometer tube to be exactly 0.760 0 m in height at 0°C, with $g = 9.806\,65 \text{ m/s}^2$. At this temperature, mercury has a density of $13.595 \times 10^3 \text{ kg/m}^3$; therefore,

$$\begin{aligned} P_0 &= \rho g h = (13.595 \times 10^3 \text{ kg/m}^3)(9.806\,65 \text{ m/s}^2)(0.760\,0 \text{ m}) \\ &= 1.013 \times 10^5 \text{ Pa} = 1 \text{ atm} \end{aligned}$$

Quick Quiz 15.5

Other than the obvious problem that occurs with freezing, why don’t we use water in a barometer in the place of mercury?

15.4 BUOYANT FORCES AND ARCHIMEDES’S PRINCIPLE

Have you ever tried to push a beach ball under water? This is extremely difficult to do because of the large upward force exerted by the water on the ball. The upward force exerted by water on any immersed object is called a **buoyant force.** We can determine the magnitude of a buoyant force by applying some logic and Newton’s second law. Imagine that, instead of air, the beach ball is filled with water. If you were standing on land, it would be difficult to hold the water-filled ball in your arms. If you held the ball while standing neck deep in a pool, however, the force you would need to hold it would almost disappear. In fact, the required force would be zero if we were to ignore the thin layer of plastic of which the beach ball is made. Because the water-filled ball is in equilibrium while it is submerged, the magnitude of the upward buoyant force must equal its weight.

If the submerged ball were filled with air rather than water, then the upward buoyant force exerted by the surrounding water would still be present. However, because the weight of the water is now replaced by the much smaller weight of that volume of air, the net force is upward and quite great; as a result, the ball is pushed to the surface.

Archimedes's principle

Archimedes (c. 287–212 B.C.)
Archimedes, a Greek mathematician, physicist, and engineer, was perhaps the greatest scientist of antiquity. He was the first to compute accurately the ratio of a circle's circumference to its diameter, and he showed how to calculate the volume and surface area of spheres, cylinders, and other geometric shapes. He is well known for discovering the nature of the buoyant force.

Archimedes was also a gifted inventor. One of his practical inventions, still in use today, is Archimedes's screw—an inclined, rotating, coiled tube originally used to lift water from the holds of ships. He also invented the catapult and devised systems of levers, pulleys, and weights for raising heavy loads. Such inventions were successfully used to defend his native city Syracuse during a two-year siege by the Romans.

The manner in which buoyant forces act is summarized by **Archimedes's principle,** which states that **the magnitude of the buoyant force always equals the weight of the fluid displaced by the object.** The buoyant force acts vertically upward through the point that was the center of gravity of the displaced fluid.

Note that Archimedes's principle does not refer to the makeup of the object experiencing the buoyant force. The object's composition is not a factor in the buoyant force. We can verify this in the following manner: Suppose we focus our attention on the indicated cube of liquid in the container illustrated in Figure 15.9. This cube is in equilibrium as it is acted on by two forces. One of these forces is the gravitational force $\mathbf{F}_g$. What cancels this downward force? Apparently, the rest of the liquid in the container is holding the cube in equilibrium. Thus, the magnitude of the buoyant force **B** exerted on the cube is exactly equal to the magnitude of $\mathbf{F}_g$, which is the weight of the liquid inside the cube:

$$B = F_g$$

Now imagine that the cube of liquid is replaced by a cube of steel of the same dimensions. What is the buoyant force acting on the steel? The liquid surrounding a cube behaves in the same way no matter what the cube is made of. Therefore, **the buoyant force acting on the steel cube is the same as the buoyant force acting on a cube of liquid of the same dimensions.** In other words, the magnitude of the buoyant force is the same as the weight of the *liquid* cube, not the steel cube. Although mathematically more complicated, this same principle applies to submerged objects of any shape, size, or density.

Although we have described the magnitude and direction of the buoyant force, we still do not know its origin. Why would a fluid exert such a strange force, almost as if the fluid were trying to expel a foreign body? To understand why, look again at Figure 15.9. The pressure at the bottom of the cube is greater than the pressure at the top by an amount $\rho g h$, where h is the length of any side of the cube. The pressure difference ΔP between the bottom and top faces of the cube is equal to the buoyant force per unit area of those faces—that is, $\Delta P = B/A$. Therefore, $B = (\Delta P)A = (\rho g h)A = \rho g V$, where V is the volume of the cube. Because the mass of the fluid in the cube is $M = \rho V$, we see that

$$B = F_g = \rho V g = Mg \qquad \textbf{(15.5)}$$

where Mg is the weight of the fluid in the cube. Thus, the buoyant force is a result of the pressure differential on a submerged or partly submerged object.

Before we proceed with a few examples, it is instructive for us to compare the forces acting on a totally submerged object with those acting on a floating (partly submerged) object.

Case 1: Totally Submerged Object When an object is totally submerged in a fluid of density ρ_f, the magnitude of the upward buoyant force is $B = \rho_f V_o g$, where V_o is the volume of the object. If the object has a mass M and density ρ_o, its weight is equal to $F_g = Mg = \rho_o V_o g$, and the net force on it is $B - F_g = (\rho_f - \rho_o)V_o g$. Hence, if the density of the object is less than the density of the fluid, then the downward force of gravity is less than the buoyant force, and the unconstrained object accelerates upward (Fig. 15.10a). If the density of the object is greater than the density of the fluid, then the upward buoyant force is less than the downward force of gravity, and the unsupported object sinks (Fig. 15.10b).

Case 2: Floating Object Now consider an object of volume V_o in static equilibrium floating on a fluid—that is, an object that is only partially submerged. In this

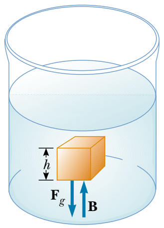


Figure 15.9 The external forces acting on the cube of liquid are the force of gravity $\mathbf{F}_g$ and the buoyant force **B**. Under equilibrium conditions, $B = F_g$.

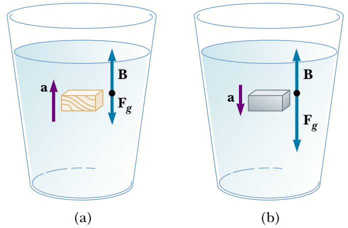


Figure 15.10 (a) A totally submerged object that is less dense than the fluid in which it is submerged experiences a net upward force. (b) A totally submerged object that is denser than the fluid sinks.

case, the upward buoyant force is balanced by the downward gravitational force acting on the object. If V_f is the volume of the fluid displaced by the object (this volume is the same as the volume of that part of the object that is beneath the fluid level), the buoyant force has a magnitude $B = \rho_f V_f g$. Because the weight of the object is $F_g = Mg = \rho_o V_o g$, and because $F_g = B$, we see that $\rho_f V_f g = \rho_o V_o g$, or

$$\frac{\rho_o}{\rho_f} = \frac{V_f}{V_o} \qquad \textbf{(15.6)}$$

Under normal conditions, the average density of a fish is slightly greater than the density of water. It follows that the fish would sink if it did not have some mechanism for adjusting its density. The fish accomplishes this by internally regulating the size of its air-filled swim bladder to balance the change in the magnitude of the buoyant force acting on it. In this manner, fish are able to swim to various depths. Unlike a fish, a scuba diver cannot achieve neutral buoyancy (at which the buoyant force just balances the weight) by adjusting the magnitude of the buoyant force B. Instead, the diver adjusts F_g by manipulating lead weights.

Hot-air balloons. Because hot air is less dense than cold air, a net upward force acts on the balloons.

Quick Quiz 15.6

Steel is much denser than water. In view of this fact, how do steel ships float?

Quick Quiz 15.7

A glass of water contains a single floating ice cube (Fig. 15.11). When the ice melts, does the water level go up, go down, or remain the same?

Quick Quiz 15.8

When a person in a rowboat in a small pond throws an anchor overboard, does the water level of the pond go up, go down, or remain the same?

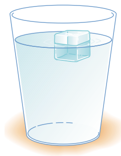


Figure 15.11

EXAMPLE 15.5 Eureka!

Archimedes supposedly was asked to determine whether a crown made for the king consisted of pure gold. Legend has it that he solved this problem by weighing the crown first in air and then in water, as shown in Figure 15.12. Suppose the scale read 7.84 N in air and 6.86 N in water. What should Archimedes have told the king?

Solution When the crown is suspended in air, the scale reads the true weight $T_1 = F_g$ (neglecting the buoyancy of air). When it is immersed in water, the buoyant force $\mathbf{B}$ reduces the scale reading to an apparent weight of $T_2 = F_g - B$. Hence, the buoyant force exerted on the crown is the difference between its weight in air and its weight in water:

$$B = F_g - T_2 = 7.84 \text{ N} - 6.86 \text{ N} = 0.98 \text{ N}$$

Because this buoyant force is equal in magnitude to the weight of the displaced water, we have $\rho_w g V_w = 0.98$ N, where V_w is the volume of the displaced water and ρ_w is its density. Also, the volume of the crown V_c is equal to the volume of the displaced water because the crown is completely submerged. Therefore,

$$V_c = V_w = \frac{0.98 \text{ N}}{g\rho_w} = \frac{0.98 \text{ N}}{(9.8 \text{ m/s}^2)(1\,000 \text{ kg/m}^3)}$$
$$= 1.0 \times 10^{-4} \text{ m}^3$$

Finally, the density of the crown is

$$\rho_c = \frac{m_c}{V_c} = \frac{m_c g}{V_c g} = \frac{7.84 \text{ N}}{(1.0 \times 10^{-4} \text{ m}^3)(9.8 \text{ m/s}^2)}$$
$$= 8.0 \times 10^3 \text{ kg/m}^3$$

From Table 15.1 we see that the density of gold is 19.3×10^3 kg/m^3. Thus, Archimedes should have told the king that he had been cheated. Either the crown was hollow, or it was not made of pure gold.

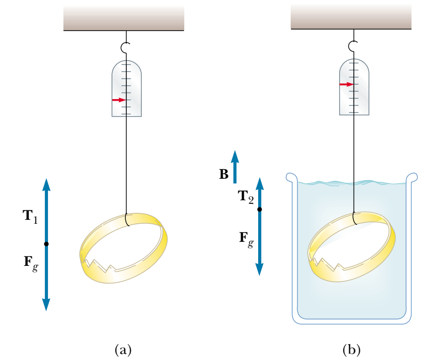


Figure 15.12 (a) When the crown is suspended in air, the scale reads its true weight $T_1 = F_g$ (the buoyancy of air is negligible). (b) When the crown is immersed in water, the buoyant force $\mathbf{B}$ reduces the scale reading to the apparent weight $T_2 = F_g - B$.

EXAMPLE 15.6 A Titanic Surprise

An iceberg floating in seawater, as shown in Figure 15.13a, is extremely dangerous because much of the ice is below the surface. This hidden ice can damage a ship that is still a considerable distance from the visible ice. What fraction of the iceberg lies below the water level?

Solution This problem corresponds to Case 2. The weight of the iceberg is $F_{gi} = \rho_i V_i g$, where $\rho_i = 917$ kg/m^3 and V_i is the volume of the whole iceberg. The magnitude of the upward buoyant force equals the weight of the displaced water: $B = \rho_w V_w g$, where V_w, the volume of the displaced water, is equal to the volume of the ice beneath the water (the shaded region in Fig. 15.13b) and ρ_w is the density of seawater, $\rho_w = 1\,030$ kg/m^3. Because $\rho_i V_i g = \rho_w V_w g$, the fraction of ice beneath the water's surface is

$$f = \frac{V_w}{V_i} = \frac{\rho_i}{\rho_w} = \frac{917 \text{ kg/m}^3}{1\,030 \text{ kg/m}^3} = 0.890 \quad \text{or} \quad 89.0\%$$

Figure 15.13 (a) Much of the volume of this iceberg is beneath the water.
(b) A ship can be damaged even when it is not near the exposed ice.

15.5 FLUID DYNAMICS

Thus far, our study of fluids has been restricted to fluids at rest. We now turn our attention to fluids in motion. Instead of trying to study the motion of each particle of the fluid as a function of time, we describe the properties of a moving fluid at each point as a function of time.

Flow Characteristics

When fluid is in motion, its flow can be characterized as being one of two main types. The flow is said to be **steady,** or **laminar,** if each particle of the fluid follows a smooth path, such that the paths of different particles never cross each other, as shown in Figure 15.14. In steady flow, the velocity of the fluid at any point remains constant in time.

Above a certain critical speed, fluid flow becomes **turbulent;** turbulent flow is irregular flow characterized by small whirlpool-like regions, as shown in Figure 15.15.

The term **viscosity** is commonly used in the description of fluid flow to characterize the degree of internal friction in the fluid. This internal friction, or *viscous force,* is associated with the resistance that two adjacent layers of fluid have to moving relative to each other. Viscosity causes part of the kinetic energy of a fluid to be converted to internal energy. This mechanism is similar to the one by which an object sliding on a rough horizontal surface loses kinetic energy.

Because the motion of real fluids is very complex and not fully understood, we make some simplifying assumptions in our approach. In our model of an **ideal fluid,** we make the following four assumptions:

Properties of an ideal fluid

1. **The fluid is nonviscous.** In a nonviscous fluid, internal friction is neglected. An object moving through the fluid experiences no viscous force.
2. **The flow is steady.** In steady (laminar) flow, the velocity of the fluid at each point remains constant.

Figure 15.14 Laminar flow around an automobile in a test wind tunnel.

Figure 15.15 Hot gases from a cigarette made visible by smoke particles. The smoke first moves in laminar flow at the bottom and then in turbulent flow above.

3. **The fluid is incompressible.** The density of an incompressible fluid is constant.
4. **The flow is irrotational.** In irrotational flow, the fluid has no angular momentum about any point. If a small paddle wheel placed anywhere in the fluid does not rotate about the wheel's center of mass, then the flow is irrotational.

15.6 STREAMLINES AND THE EQUATION OF CONTINUITY

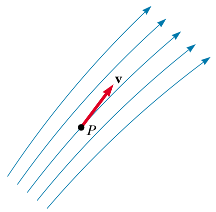


Figure 15.16 A particle in laminar flow follows a streamline, and at each point along its path the particle's velocity is tangent to the streamline.

The path taken by a fluid particle under steady flow is called a **streamline.** The velocity of the particle is always tangent to the streamline, as shown in Figure 15.16. A set of streamlines like the ones shown in Figure 15.16 form a *tube of flow.* Note that fluid particles cannot flow into or out of the sides of this tube; if they could, then the streamlines would cross each other.

Consider an ideal fluid flowing through a pipe of nonuniform size, as illustrated in Figure 15.17. The particles in the fluid move along streamlines in steady flow. In a time t, the fluid at the bottom end of the pipe moves a distance $\Delta x_1 = v_1 t$. If A_1 is the cross-sectional area in this region, then the mass of fluid contained in the left shaded region in Figure 15.17 is $m_1 = \rho A_1 \Delta x_1 = \rho A_1 v_1 t$, where ρ is the (nonchanging) density of the ideal fluid. Similarly, the fluid that moves through the upper end of the pipe in the time t has a mass $m_2 = \rho A_2 v_2 t$. However, because *mass is conserved* and because the flow is steady, the mass that crosses A_1 in a time t must equal the mass that crosses A_2 in the time t. That is, $m_1 = m_2$, or $\rho A_1 v_1 t = \rho A_2 v_2 t$; this means that

Equation of continuity

$$A_1 v_1 = A_2 v_2 = \text{constant} \tag{15.7}$$

This expression is called the **equation of continuity.** It states that

the product of the area and the fluid speed at all points along the pipe is a constant for an incompressible fluid.

This equation tells us that the speed is high where the tube is constricted (small A) and low where the tube is wide (large A). The product Av, which has the dimensions of volume per unit time, is called either the *volume flux* or the *flow rate.* The condition Av = constant is equivalent to the statement that the volume of fluid that enters one end of a tube in a given time interval equals the volume leaving the other end of the tube in the same time interval if no leaks are present.

Figure 15.18

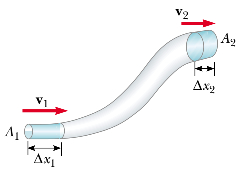


Figure 15.17 A fluid moving with steady flow through a pipe of varying cross-sectional area. The volume of fluid flowing through area A_1 in a time interval t must equal the volume flowing through area A_2 in the same time interval. Therefore, $A_1 v_1 = A_2 v_2$.

Quick Quiz 15.9

As water flows from a faucet, as shown in Figure 15.18, why does the stream of water become narrower as it descends?

EXAMPLE 15.7 Niagara Falls

Each second, 5 525 m^3 of water flows over the 670-m-wide cliff of the Horseshoe Falls portion of Niagara Falls. The water is approximately 2 m deep as it reaches the cliff. What is its speed at that instant?

Solution The cross-sectional area of the water as it reaches the edge of the cliff is $A = (670 \text{ m})(2 \text{ m}) = 1\,340 \text{ m}^2$. The flow rate of 5 525 m^3/s is equal to Av. This gives

$$v = \frac{5\,525 \text{ m}^3/\text{s}}{A} = \frac{5\,525 \text{ m}^3/\text{s}}{1\,340 \text{ m}^2} = 4 \text{ m/s}$$

Note that we have kept only one significant figure because our value for the depth has only one significant figure.

Exercise A barrel floating along in the river plunges over the Falls. How far from the base of the cliff is the barrel when it reaches the water 49 m below?

Answer 13 m ≈ 10 m.

15.7 BERNOULLI'S EQUATION

When you press your thumb over the end of a garden hose so that the opening becomes a small slit, the water comes out at high speed, as shown in Figure 15.19. Is the water under greater pressure when it is inside the hose or when it is out in the air? You can answer this question by noting how hard you have to push your thumb against the water inside the end of the hose. The pressure inside the hose is definitely greater than atmospheric pressure.

The relationship between fluid speed, pressure, and elevation was first derived in 1738 by the Swiss physicist Daniel Bernoulli. Consider the flow of an ideal fluid through a nonuniform pipe in a time t, as illustrated in Figure 15.20. Let us call the lower shaded part section 1 and the upper shaded part section 2. The force exerted by the fluid in section 1 has a magnitude $P_1 A_1$. The work done by this force in a time t is $W_1 = F_1 \Delta x_1 = P_1 A_1 \Delta x_1 = P_1 V$, where V is the volume of section 1. In a similar manner, the work done by the fluid in section 2 in the same time t is $W_2 = -P_2 A_2 \Delta x_2 = -P_2 V$. (The volume that passes through section 1 in a time t equals the volume that passes through section 2 in the same time.) This work is negative because the fluid force opposes the displacement. Thus, the net work done by these forces in the time t is

$$W = (P_1 - P_2)V$$

Daniel Bernoulli (1700–1782) Daniel Bernoulli, a Swiss physicist and mathematician, made important discoveries in fluid dynamics. Born into a family of mathematicians, he was the only member of the family to make a mark in physics.

Bernoulli's most famous work, *Hydrodynamica,* was published in 1738; it is both a theoretical and a practical study of equilibrium, pressure, and speed in fluids. He showed that as the speed of a fluid increases, its pressure decreases.

In *Hydrodynamica* Bernoulli also attempted the first explanation of the behavior of gases with changing pressure and temperature; this was the beginning of the kinetic theory of gases, a topic we study in Chapter 21. *(Corbis–Bettmann)*

Figure 15.19 The speed of water spraying from the end of a hose increases as the size of the opening is decreased with the thumb.

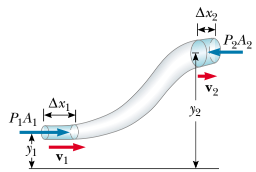


Figure 15.20 A fluid in laminar flow through a constricted pipe. The volume of the shaded section on the left is equal to the volume of the shaded section on the right.

QuickLab

Place two soda cans on their sides approximately 2 cm apart on a table. Align your mouth at table level and with the space between the cans. Blow a horizontal stream of air through this space. What do the cans do? Is this what you expected? Compare this with the force acting on a car parked close to the edge of a road when a big truck goes by. How does the outcome relate to Equation 15.9?

Part of this work goes into changing the kinetic energy of the fluid, and part goes into changing the gravitational potential energy. If m is the mass that enters one end and leaves the other in a time t, then the change in the kinetic energy of this mass is

$$\Delta K = \tfrac{1}{2}mv_2^{\,2} - \tfrac{1}{2}mv_1^{\,2}$$

The change in gravitational potential energy is

$$\Delta U = mgy_2 - mgy_1$$

We can apply Equation 8.13, $W = \Delta K + \Delta U$, to this volume of fluid to obtain

$$(P_1 - P_2)V = \tfrac{1}{2}mv_2^{\,2} - \tfrac{1}{2}mv_1^{\,2} + mgy_2 - mgy_1$$

If we divide each term by V and recall that $\rho = m/V$, this expression reduces to

$$P_1 - P_2 = \tfrac{1}{2}\rho v_2^{\,2} - \tfrac{1}{2}\rho v_1^{\,2} + \rho g y_2 - \rho g y_1$$

Rearranging terms, we obtain

$$P_1 + \tfrac{1}{2}\rho v_1^{\,2} + \rho g y_1 = P_2 + \tfrac{1}{2}\rho v_2^{\,2} + \rho g y_2 \qquad \textbf{(15.8)}$$

This is **Bernoulli's equation** as applied to an ideal fluid. It is often expressed as

Bernoulli's equation

$$P + \tfrac{1}{2}\rho v^2 + \rho g y = \text{constant} \qquad \textbf{(15.9)}$$

This expression specifies that, in laminar flow, the sum of the pressure (P), kinetic energy per unit volume ($\tfrac{1}{2}\rho v^2$), and gravitational potential energy per unit volume ($\rho g y$) has the same value at all points along a streamline.

When the fluid is at rest, $v_1 = v_2 = 0$ and Equation 15.8 becomes

$$P_1 - P_2 = \rho g(y_2 - y_1) = \rho g h$$

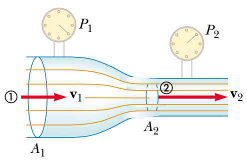
This is in agreement with Equation 15.4.

EXAMPLE 15.8 The Venturi Tube

The horizontal constricted pipe illustrated in Figure 15.21, known as a *Venturi tube,* can be used to measure the flow speed of an incompressible fluid. Let us determine the flow speed at point 2 if the pressure difference $P_1 - P_2$ is known.

Solution Because the pipe is horizontal, $y_1 = y_2$, and applying Equation 15.8 to points 1 and 2 gives

$$(1) \qquad P_1 + \tfrac{1}{2}\rho v_1^{\,2} = P_2 + \tfrac{1}{2}\rho v_2^{\,2}$$



(a)


(b)

Figure 15.21 (a) Pressure P_1 is greater than pressure P_2 because $v_1 < v_2$. This device can be used to measure the speed of fluid flow. (b) A Venturi tube.

From the equation of continuity, $A_1v_1 = A_2v_2$, we find that

$$(2) \qquad v_1 = \frac{A_2}{A_1}v_2$$

Substituting this expression into equation (1) gives

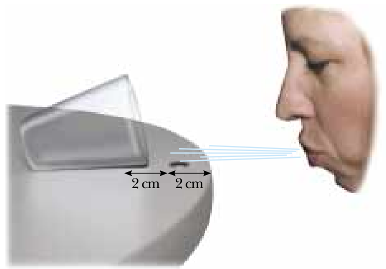
$$P_1 + \tfrac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^{\,2} = P_2 + \tfrac{1}{2}\rho v_2^{\,2}$$

$$v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^{\,2} - A_2^{\,2})}}$$

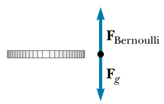
We can use this result and the continuity equation to obtain an expression for v_1. Because $A_2 < A_1$, Equation (2) shows us that $v_2 > v_1$. This result, together with equation (1), indicates that $P_1 > P_2$. In other words, the pressure is reduced in the constricted part of the pipe. This result is somewhat analogous to the following situation: Consider a very crowded room in which people are squeezed together. As soon as a door is opened and people begin to exit, the squeezing (pressure) is least near the door, where the motion (flow) is greatest.

EXAMPLE 15.9 A Good Trick

It is possible to blow a dime off a table and into a tumbler. Place the dime about 2 cm from the edge of the table. Place the tumbler on the table horizontally with its open edge about 2 cm from the dime, as shown in Figure 15.22a. If you blow forcefully across the top of the dime, it will rise, be caught in the airstream, and end up in the tumbler. The mass of a dime is $m = 2.24$ g, and its surface area is $A = 2.50 \times 10^{-4}\ \text{m}^2$. How hard are you blowing when the dime rises and travels into the tumbler?



(a)



(b)

Figure 15.22

Solution Figure 15.22b indicates we must calculate the upward force acting on the dime. First, note that a thin stationary layer of air is present between the dime and the table. When you blow across the dime, it deflects most of the moving air from your breath across its top, so that the air above the dime has a greater speed than the air beneath it. This fact, together with Bernoulli's equation, demonstrates that the air moving across the top of the dime is at a lower pressure than the air beneath the dime. If we neglect the small thickness of the dime, we can apply Equation 15.8 to obtain

$$P_{\text{above}} + \tfrac{1}{2}\rho v^2_{\text{above}} = P_{\text{beneath}} + \tfrac{1}{2}\rho v^2_{\text{beneath}}$$

Because the air beneath the dime is almost stationary, we can neglect the last term in this expression and write the difference as $P_{\text{beneath}} - P_{\text{above}} = \tfrac{1}{2}\rho v^2_{\text{above}}$. If we multiply this pressure difference by the surface area of the dime, we obtain the upward force acting on the dime. At the very least, this upward force must balance the gravitational force acting on the dime, and so, taking the density of air from Table 15.1, we can state that

$$F_g = mg = (P_{\text{beneath}} - P_{\text{above}})A = (\tfrac{1}{2}\rho v^2_{\text{above}})A$$

$$v_{\text{above}} = \sqrt{\frac{2mg}{\rho A}} = \sqrt{\frac{2(2.24 \times 10^{-3}\ \text{kg})(9.80\ \text{m/s}^2)}{(1.29\ \text{kg/m}^3)(2.50 \times 10^{-4}\ \text{m}^2)}}$$

$$v_{\text{above}} = 11.7\ \text{m/s}$$

The air you blow must be moving faster than this if the upward force is to exceed the weight of the dime. Practice this trick a few times and then impress all your friends!

EXAMPLE 15.10 Torricelli's Law

An enclosed tank containing a liquid of density ρ has a hole in its side at a distance y_1 from the tank's bottom (Fig. 15.23). The hole is open to the atmosphere, and its diameter is much smaller than the diameter of the tank. The air above the liquid is maintained at a pressure P. Determine the speed at which the liquid leaves the hole when the liquid's level is a distance h above the hole.

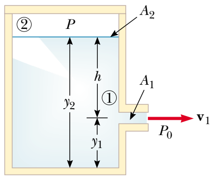


Figure 15.23 When P is much larger than atmospheric pressure P_0, the liquid speed as the liquid passes through the hole in the side of the container is given approximately by $v_1 = \sqrt{2(P - P_0)/\rho}$.

Solution Because $A_2 \gg A_1$, the liquid is approximately at rest at the top of the tank, where the pressure is P. Applying Bernoulli's equation to points 1 and 2 and noting that at the hole P_1 is equal to atmospheric pressure P_0, we find that

$$P_0 + \tfrac{1}{2}\rho v_1^2 + \rho g y_1 = P + \rho g y_2$$

But $y_2 - y_1 = h$; thus, this expression reduces to

$$v_1 = \sqrt{\frac{2(P - P_0)}{\rho} + 2gh}$$

When P is much greater than P_0 (so that the term $2gh$ can be neglected), the exit speed of the water is mainly a function of P. If the tank is open to the atmosphere, then $P = P_0$ and $v_1 = \sqrt{2gh}$. In other words, for an open tank, the speed of liquid coming out through a hole a distance h below the surface is equal to that acquired by an object falling freely through a vertical distance h. This phenomenon is known as **Torricelli's law.**

Optional Section

15.8 OTHER APPLICATIONS OF BERNOULLI'S EQUATION

The lift on an aircraft wing can be explained, in part, by the Bernoulli effect. Airplane wings are designed so that the air speed above the wing is greater than that below the wing. As a result, the air pressure above the wing is less than the pressure below, and a net upward force on the wing, called *lift,* results.

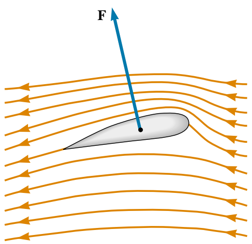
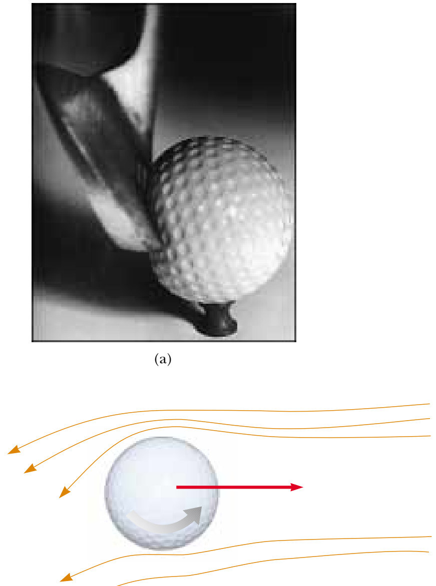


Figure 15.24 Streamline flow around an airplane wing. The pressure above the wing is less than the pressure below, and a dynamic lift upward results.

Another factor influencing the lift on a wing is shown in Figure 15.24. The wing has a slight upward tilt that causes air molecules striking its bottom to be deflected downward. This deflection means that the wing is exerting a downward force on the air. According to Newton's third law, the air must exert an equal and opposite force on the wing.

Finally, turbulence also has an effect. If the wing is tilted too much, the flow of air across the upper surface becomes turbulent, and the pressure difference across the wing is not as great as that predicted by Bernoulli's equation. In an extreme case, this turbulence may cause the aircraft to stall.

In general, an object moving through a fluid experiences lift as the result of any effect that causes the fluid to change its direction as it flows past the object. Some factors that influence lift are the shape of the object, its orientation with respect to the fluid flow, any spinning motion it might have, and the texture of its surface. For example, a golf ball struck with a club is given a rapid backspin, as shown in Figure 15.25a. The dimples on the ball help "entrain" the air to follow the curvature of the ball's surface. This effect is most pronounced on the top half of the ball, where the ball's surface is moving in the same direction as the air flow. Figure 15.25b shows a thin layer of air wrapping part way around the ball and being deflected downward as a result. Because the ball pushes the air down, the air must push up on the ball. Without the dimples, the air is not as well entrained, and the golf ball does not travel as far. For the same reason, a tennis ball's fuzz helps the spinning ball "grab" the air rushing by and helps deflect it.



(a)

(b)

Figure 15.25 (a) A golf ball is made to spin when struck by the club. (b) The spinning ball experiences a lifting force that allows it to travel much farther than it would if it were not spinning.

QuickLab

You can easily demonstrate the effect of changing fluid direction by lightly holding the back of a spoon against a stream of water coming from a faucet. You will see the stream "attach" itself to the curvature of the spoon and be deflected sideways. You will also feel the third-law force exerted by the water on the spoon.

A number of devices operate by means of the pressure differentials that result from differences in a fluid's speed. For example, a stream of air passing over one end of an open tube, the other end of which is immersed in a liquid, reduces the pressure above the tube, as illustrated in Figure 15.26. This reduction in pressure causes the liquid to rise into the air stream. The liquid is then dispersed into a fine spray of droplets. You might recognize that this so-called atomizer is used in perfume bottles and paint sprayers. The same principle is used in the carburetor of a gasoline engine. In this case, the low-pressure region in the carburetor is produced by air drawn in by the piston through the air filter. The gasoline vaporizes in that region, mixes with the air, and enters the cylinder of the engine, where combustion occurs.

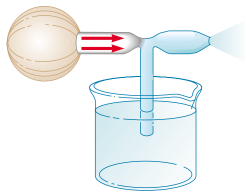


Figure 15.26 A stream of air passing over a tube dipped into a liquid causes the liquid to rise in the tube.

Quick Quiz 15.10

People in buildings threatened by a tornado are often told to open the windows to minimize damage. Why?

SUMMARY

The **pressure** P in a fluid is the force per unit area exerted by the fluid on a surface:

$$P \equiv \frac{F}{A} \quad \textbf{(15.1)}$$

In the SI system, pressure has units of newtons per square meter (N/m^2), and $1\ N/m^2 = 1$ pascal (Pa).

The pressure in a fluid at rest varies with depth h in the fluid according to the expression

$$P = P_0 + \rho g h \quad \textbf{(15.4)}$$

where P_0 is atmospheric pressure ($= 1.013 \times 10^5\ N/m^2$) and ρ is the density of the fluid, assumed uniform.

Pascal's law states that when pressure is applied to an enclosed fluid, the pressure is transmitted undiminished to every point in the fluid and to every point on the walls of the container.

When an object is partially or fully submerged in a fluid, the fluid exerts on the object an upward force called the **buoyant force.** According to **Archimedes's principle,** the magnitude of the buoyant force is equal to the weight of the fluid displaced by the object. Be sure you can apply this principle to a wide variety of situations, including sinking objects, floating ones, and neutrally buoyant ones.

You can understand various aspects of a fluid's dynamics by assuming that the fluid is nonviscous and incompressible and that the fluid's motion is a steady flow with no rotation.

Two important concepts regarding ideal fluid flow through a pipe of nonuniform size are as follows:

1. The flow rate (volume flux) through the pipe is constant; this is equivalent to stating that the product of the cross-sectional area A and the speed v at any point is a constant. This result is expressed in the **equation of continuity:**

$$A_1 v_1 = A_2 v_2 = \text{constant} \quad \textbf{(15.7)}$$

   You can use this expression to calculate how the velocity of a fluid changes as the fluid is constricted or as it flows into a more open area.
2. The sum of the pressure, kinetic energy per unit volume, and gravitational potential energy per unit volume has the same value at all points along a streamline. This result is summarized in **Bernoulli's equation:**

$$P + \tfrac{1}{2}\rho v^2 + \rho g y = \text{constant} \quad \textbf{(15.9)}$$

QUESTIONS

1. Two drinking glasses of the same weight but of different shape and different cross-sectional area are filled to the same level with water. According to the expression $P = P_0 + \rho g h$, the pressure at the bottom of both glasses is the same. In view of this, why does one glass weigh more than the other?
2. If the top of your head has a surface area of $100\ cm^2$, what is the weight of the air above your head?
3. When you drink a liquid through a straw, you reduce the pressure in your mouth and let the atmosphere move the liquid. Explain why this is so. Can you use a straw to sip a drink on the Moon?
4. A helium-filled balloon rises until its density becomes the same as that of the surrounding air. If a sealed submarine begins to sink, will it go all the way to the bottom of the ocean or will it stop when its density becomes the same as that of the surrounding water?
5. A fish rests on the bottom of a bucket of water while the bucket is being weighed. When the fish begins to swim around, does the weight change?

6. Does a ship ride higher in the water of an inland lake or in the ocean? Why?
7. Lead has a greater density than iron, and both metals are denser than water. Is the buoyant force on a lead object greater than, less than, or equal to the buoyant force on an iron object of the same volume?
8. The water supply for a city is often provided by reservoirs built on high ground. Water flows from the reservoir, through pipes, and into your home when you turn the tap on your faucet. Why is the flow of water more rapid out of a faucet on the first floor of a building than it is in an apartment on a higher floor?
9. Smoke rises in a chimney faster when a breeze is blowing than when there is no breeze at all. Use Bernoulli's equation to explain this phenomenon.
10. If a Ping–Pong ball is above a hair dryer, the ball can be suspended in the air column emitted by the dryer. Explain.
11. When ski jumpers are airborne (Fig. Q15.11), why do they bend their bodies forward and keep their hands at their sides?

Figure Q15.11

12. Explain why a sealed bottle partially filled with a liquid can float.
13. When is the buoyant force on a swimmer greater—after exhaling or after inhaling?
14. A piece of unpainted wood barely floats in a container partly filled with water. If the container is sealed and then pressurized above atmospheric pressure, does the wood rise, sink, or remain at the same level? (*Hint:* Wood is porous.)
15. A flat plate is immersed in a liquid at rest. For what orientation of the plate is the pressure on its flat surface uniform?
16. Because atmospheric pressure is about $10^5\ N/m^2$ and the area of a person's chest is about $0.13\ m^2$, the force of the atmosphere on one's chest is around 13 000 N. In view of this enormous force, why don't our bodies collapse?
17. How would you determine the density of an irregularly shaped rock?
18. Why do airplane pilots prefer to take off into the wind?
19. If you release a ball while inside a freely falling elevator, the ball remains in front of you rather than falling to the floor because the ball, the elevator, and you all experience the same downward acceleration **g**. What happens if you repeat this experiment with a helium-filled balloon? (This one is tricky.)
20. Two identical ships set out to sea. One is loaded with a cargo of Styrofoam, and the other is empty. Which ship is more submerged?
21. A small piece of steel is tied to a block of wood. When the wood is placed in a tub of water with the steel on top, half of the block is submerged. If the block is inverted so that the steel is underwater, does the amount of the block submerged increase, decrease, or remain the same? What happens to the water level in the tub when the block is inverted?
22. Prairie dogs (Fig. Q15.22) ventilate their burrows by building a mound over one entrance, which is open to a stream of air. A second entrance at ground level is open to almost stagnant air. How does this construction create an air flow through the burrow?

Figure Q15.22

23. An unopened can of diet cola floats when placed in a tank of water, whereas a can of regular cola of the same brand sinks in the tank. What do you suppose could explain this phenomenon?
24. Figure Q15.24 shows a glass cylinder containing four liquids of different densities. From top to bottom, the liquids are oil (orange), water (yellow), salt water (green), and mercury (silver). The cylinder also contains, from top to bottom, a Ping–Pong ball, a piece of wood, an egg, and a steel ball. (a) Which of these liquids has the lowest density, and which has the greatest? (b) What can you conclude about the density of each object?

Figure Q15.24

25. In Figure Q15.25, an air stream moves from right to left through a tube that is constricted at the middle. Three Ping–Pong balls are levitated in equilibrium above the vertical columns through which the air escapes. (a) Why is the ball at the right higher than the one in the middle?

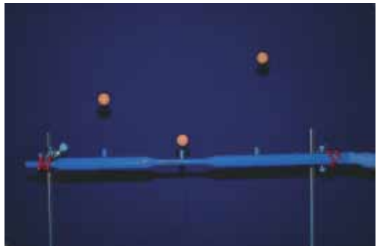


Figure Q15.25

(b) Why is the ball at the left lower than the ball at the right even though the horizontal tube has the same dimensions at these two points?

26. You are a passenger on a spacecraft. For your comfort, the interior contains air just like that at the surface of the Earth. The craft is coasting through a very empty region of space. That is, a nearly perfect vacuum exists just outside the wall. Suddenly a meteoroid pokes a hole, smaller than the palm of your hand, right through the wall next to your seat. What will happen? Is there anything you can or should do about it?

PROBLEMS

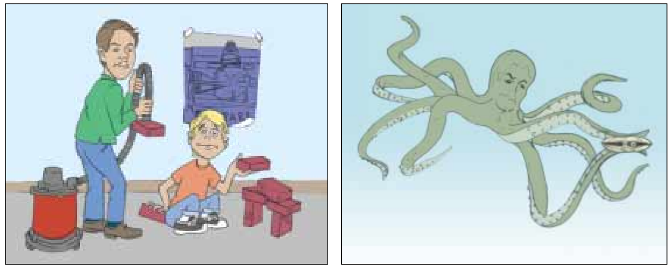
1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 15.1 Pressure

1. Calculate the mass of a solid iron sphere that has a diameter of 3.00 cm.
2. Find the order of magnitude of the density of the *nucleus* of an atom. What does this result suggest concerning the structure of matter? (Visualize a nucleus as protons and neutrons closely packed together. Each has mass 1.67×10^{-27} kg and radius on the order of 10^{-15} m.)
3. A 50.0-kg woman balances on one heel of a pair of high-heeled shoes. If the heel is circular and has a radius of 0.500 cm, what pressure does she exert on the floor?
4. The four tires of an automobile are inflated to a gauge pressure of 200 kPa. Each tire has an area of 0.024 0 m^2 in contact with the ground. Determine the weight of the automobile.
5. What is the total mass of the Earth's atmosphere? (The radius of the Earth is 6.37×10^6 m, and atmospheric pressure at the Earth's surface is 1.013×10^5 N/m^2.)

Section 15.2 Variation of Pressure with Depth

6. (a) Calculate the absolute pressure at an ocean depth of 1 000 m. Assume the density of seawater is 1 024 kg/m^3 and that the air above exerts a pressure of 101.3 kPa. (b) At this depth, what force must the frame around a circular submarine porthole having a diameter of 30.0 cm exert to counterbalance the force exerted by the water?
7. The spring of the pressure gauge shown in Figure 15.2 has a force constant of 1 000 N/m, and the piston has a diameter of 2.00 cm. When the gauge is lowered into water, at what depth does the piston move in by 0.500 cm?
8. The small piston of a hydraulic lift has a cross-sectional area of 3.00 cm^2, and its large piston has a cross-sectional area of 200 cm^2 (see Fig. 15.5a). What force must be applied to the small piston for it to raise a load of 15.0 kN? (In service stations, this force is usually generated with the use of compressed air.)



(a) (b)

Figure P15.10

WEB 9. What must be the contact area between a suction cup (completely exhausted) and a ceiling if the cup is to support the weight of an 80.0-kg student?

10. (a) A very powerful vacuum cleaner has a hose 2.86 cm in diameter. With no nozzle on the hose, what is the weight of the heaviest brick that the cleaner can lift (Fig. P15.10)? (b) A very powerful octopus uses one sucker of diameter 2.86 cm on each of the two shells of a clam in an attempt to pull the shells apart. Find the greatest force that the octopus can exert in salt water 32.3 m in depth. (*Caution:* Experimental verification can be interesting, but do not drop a brick on your foot. Do not overheat the motor of a vacuum cleaner. Do not get an octopus mad at you.)

11. For the cellar of a new house, a hole with vertical sides descending 2.40 m is dug in the ground. A concrete foundation wall is built all the way across the 9.60-m width of the excavation. This foundation wall is 0.183 m away from the front of the cellar hole. During a rainstorm, drainage from the street fills up the space in front of the concrete wall but not the cellar behind the wall. The water does not soak into the clay soil. Find the force that the water causes on the foundation wall. For comparison, the weight of the water is given by

$$2.40 \text{ m} \times 9.60 \text{ m} \times 0.183 \text{ m} \times 1\,000 \text{ kg/m}^3 \times 9.80 \text{ m/s}^2 = 41.3 \text{ kN}$$

12. A swimming pool has dimensions 30.0 m × 10.0 m and a flat bottom. When the pool is filled to a depth of 2.00 m with fresh water, what is the force caused by the water on the bottom? On each end? On each side?

13. A sealed spherical shell of diameter d is rigidly attached to a cart that is moving horizontally with an acceleration a, as shown in Figure P15.13. The sphere is nearly filled with a fluid having density ρ and also contains one small bubble of air at atmospheric pressure. Find an expression for the pressure P at the center of the sphere.

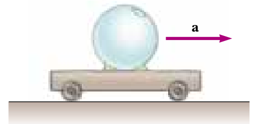


Figure P15.13

14. The tank shown in Figure P15.14 is filled with water to a depth of 2.00 m. At the bottom of one of the side walls is a rectangular hatch 1.00 m high and 2.00 m wide. The hatch is hinged at its top. (a) Determine the force that the water exerts on the hatch. (b) Find the torque exerted about the hinges.

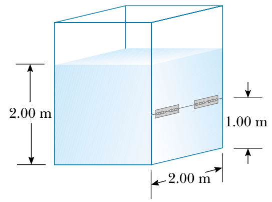


Figure P15.14

15. **Review Problem.** A solid copper ball with a diameter of 3.00 m at sea level is placed at the bottom of the ocean (at a depth of 10.0 km). If the density of seawater is 1 030 kg/m^3, by how much (approximately) does the diameter of the ball decrease when it reaches bottom? Take the bulk modulus of copper as 14.0×10^{10} N/m^2.

Section 15.3 Pressure Measurements

16. Normal atmospheric pressure is 1.013×10^5 Pa. The approach of a storm causes the height of a mercury barometer to drop by 20.0 mm from the normal height. What is the atmospheric pressure? (The density of mercury is 13.59 g/cm^3.)

WEB **17.** Blaise Pascal duplicated Torricelli's barometer, using a red Bordeaux wine, of density 984 kg/m^3, as the working liquid (Fig. P15.17). What was the height h of the wine column for normal atmospheric pressure? Would you expect the vacuum above the column to be as good as that for mercury?

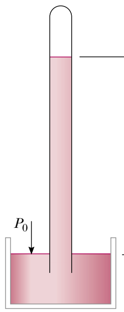


Figure P15.17

18. Mercury is poured into a U-tube, as shown in Figure P15.18a. The left arm of the tube has a cross-sectional area A_1 of 10.0 cm^2, and the right arm has a cross-sectional area A_2 of 5.00 cm^2. One-hundred grams of water are then poured into the right arm, as shown in Figure P15.18b. (a) Determine the length of the water column in the right arm of the U-tube. (b) Given that the density of mercury is 13.6 g/cm^3, what distance h does the mercury rise in the left arm?

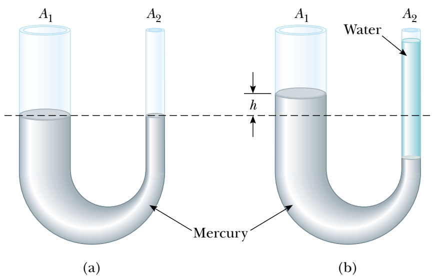


Figure P15.18

19. A U-tube of uniform cross-sectional area and open to the atmosphere is partially filled with mercury. Water is then poured into both arms. If the equilibrium configuration of the tube is as shown in Figure P15.19, with $h_2 = 1.00$ cm, determine the value of h_1.

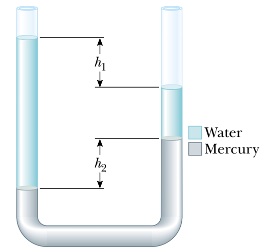


Figure P15.19

Section 15.4 Buoyant Forces and Archimedes's Principle

20. (a) A light balloon is filled with 400 m^3 of helium. At 0°C, what is the mass of the payload that the balloon can lift? (b) In Table 15.1, note that the density of hydrogen is nearly one-half the density of helium. What load can the balloon lift if it is filled with hydrogen?

21. A Styrofoam slab has a thickness of 10.0 cm and a density of 300 kg/m^3. When a 75.0-kg swimmer is resting on it, the slab floats in fresh water with its top at the same level as the water's surface. Find the area of the slab.

22. A Styrofoam slab has thickness h and density ρ_S. What is the area of the slab if it floats with its upper surface just awash in fresh water, when a swimmer of mass m is on top?

23. A piece of aluminum with mass 1.00 kg and density 2 700 kg/m^3 is suspended from a string and then completely immersed in a container of water (Fig. P15.23). Calculate the tension in the string (a) before and (b) after the metal is immersed.

24. A 10.0-kg block of metal measuring 12.0 cm × 10.0 cm × 10.0 cm is suspended from a scale and immersed in water, as shown in Figure P15.23b. The 12.0-cm dimension is vertical, and the top of the block is 5.00 cm from the surface of the water. (a) What are the forces acting on the top and on the bottom of the block? (Take $P_0 = 1.013\,0 \times 10^5$ N/m^2.) (b) What is the reading of the spring scale? (c) Show that the buoyant force equals the difference between the forces at the top and bottom of the block.

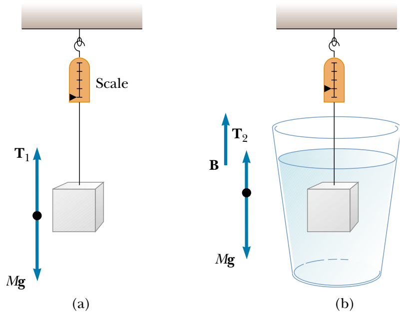


Figure P15.23 Problems 23 and 24.

WEB **25.** A cube of wood having a side dimension of 20.0 cm and a density of 650 kg/m^3 floats on water. (a) What is the distance from the horizontal top surface of the cube to the water level? (b) How much lead weight must be placed on top of the cube so that its top is just level with the water?

26. To an order of magnitude, how many helium-filled toy balloons would be required to lift you? Because helium is an irreplaceable resource, develop a theoretical answer rather than an experimental answer. In your solution, state what physical quantities you take as data and the values you measure or estimate for them.

27. A plastic sphere floats in water with 50.0% of its volume submerged. This same sphere floats in glycerin with 40.0% of its volume submerged. Determine the densities of the glycerin and the sphere.

28. A frog in a hemispherical pod finds that he just floats without sinking into a sea of blue-green ooze having a density of 1.35 g/cm^3 (Fig. P15.28). If the pod has a radius of 6.00 cm and a negligible mass, what is the mass of the frog?

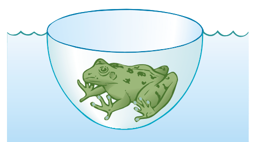


Figure P15.28

29. How many cubic meters of helium are required to lift a balloon with a 400-kg payload to a height of 8 000 m? (Take $\rho_{He} = 0.180$ kg/m^3.) Assume that the balloon maintains a constant volume and that the density of air decreases with the altitude z according to the expression $\rho_{air} = \rho_0 e^{-z/8\,000}$, where z is in meters and $\rho_0 = 1.25$ kg/m^3 is the density of air at sea level.

30. **Review Problem.** A long cylindrical tube of radius r is weighted on one end so that it floats upright in a fluid having a density ρ. It is pushed downward a distance x from its equilibrium position and then released. Show that the tube will execute simple harmonic motion if the resistive effects of the water are neglected, and determine the period of the oscillations.

31. A bathysphere used for deep-sea exploration has a radius of 1.50 m and a mass of 1.20×10^4 kg. To dive, this submarine takes on mass in the form of seawater. Determine the amount of mass that the submarine must take on if it is to descend at a constant speed of 1.20 m/s, when the resistive force on it is 1 100 N in the upward direction. Take 1.03×10^3 kg/m^3 as the density of seawater.

32. The United States possesses the eight largest warships in the world—aircraft carriers of the *Nimitz* class—and it is building one more. Suppose that one of the ships bobs up to float 11.0 cm higher in the water when 50 fighters take off from it at a location where $g =$ 9.78 m/s^2. The planes have an average mass of 29 000 kg. Find the horizontal area enclosed by the waterline of the ship. (By comparison, its flight deck has an area of 18 000 m^2.)

Section 15.5 Fluid Dynamics

Section 15.6 Streamlines and the Equation of Continuity

Section 15.7 Bernoulli's Equation

33. (a) A water hose 2.00 cm in diameter is used to fill a 20.0-L bucket. If it takes 1.00 min to fill the bucket, what is the speed v at which water moves through the hose? (*Note:* 1 L = 1 000 cm^3.) (b) If the hose has a nozzle 1.00 cm in diameter, find the speed of the water at the nozzle.

34. A horizontal pipe 10.0 cm in diameter has a smooth reduction to a pipe 5.00 cm in diameter. If the pressure of the water in the larger pipe is 8.00×10^4 Pa and the pressure in the smaller pipe is 6.00×10^4 Pa, at what rate does water flow through the pipes?

35. A large storage tank, open at the top and filled with water, develops a small hole in its side at a point 16.0 m below the water level. If the rate of flow from the leak is 2.50×10^{-3} m^3/min, determine (a) the speed at which the water leaves the hole and (b) the diameter of the hole.

36. Through a pipe of diameter 15.0 cm, water is pumped from the Colorado River up to Grand Canyon Village, located on the rim of the canyon. The river is at an elevation of 564 m, and the village is at an elevation of 2 096 m. (a) What is the minimum pressure at which the water must be pumped if it is to arrive at the village?

(b) If 4 500 m^3 are pumped per day, what is the speed of the water in the pipe? (c) What additional pressure is necessary to deliver this flow? (*Note:* You may assume that the acceleration due to gravity and the density of air are constant over this range of elevations.)

37. Water flows through a fire hose of diameter 6.35 cm at a rate of 0.012 0 m^3/s. The fire hose ends in a nozzle with an inner diameter of 2.20 cm. What is the speed at which the water exits the nozzle?

38. Old Faithful Geyser in Yellowstone National Park erupts at approximately 1-h intervals, and the height of the water column reaches 40.0 m (Fig. P15.38). (a) Consider the rising stream as a series of separate drops. Analyze the free-fall motion of one of these drops to determine the speed at which the water leaves the ground. (b) Treating the rising stream as an ideal fluid in streamline flow, use Bernoulli's equation to determine the speed of the water as it leaves ground level. (c) What is the pressure (above atmospheric) in the heated underground chamber if its depth is 175 m? You may assume that the chamber is large compared with the geyser's vent.

Figure P15.38

(Optional)

Section 15.8 Other Applications of Bernoulli's Equation

39. An airplane has a mass of 1.60×10^4 kg, and each wing has an area of 40.0 m^2. During level flight, the pressure on the lower wing surface is 7.00×10^4 Pa. Determine the pressure on the upper wing surface.

40. A Venturi tube may be used as a fluid flow meter (see Fig. 15.21). If the difference in pressure is $P_1 - P_2 = 21.0$ kPa, find the fluid flow rate in cubic meters per second, given that the radius of the outlet tube is 1.00 cm, the radius of the inlet tube is 2.00 cm, and the fluid is gasoline ($\rho = 700$ kg/m^3).

41. A Pitot tube can be used to determine the velocity of air flow by measuring the difference between the total pressure and the static pressure (Fig. P15.41). If the fluid in the tube is mercury, whose density is $\rho_{Hg} =$ 13 600 kg/m^3, and if $\Delta h = 5.00$ cm, find the speed of air flow. (Assume that the air is stagnant at point A, and take $\rho_{air} = 1.25$ kg/m^3.)

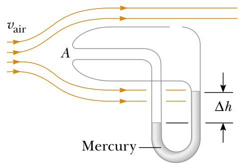


Figure P15.41

42. An airplane is cruising at an altitude of 10 km. The pressure outside the craft is 0.287 atm; within the passenger compartment, the pressure is 1.00 atm and the temperature is 20°C. A small leak occurs in one of the window seals in the passenger compartment. Model the air as an ideal fluid to find the speed of the stream of air flowing through the leak.

43. A siphon is used to drain water from a tank, as illustrated in Figure P15.43. The siphon has a uniform diameter. Assume steady flow without friction. (a) If the distance $h = 1.00$ m, find the speed of outflow at the end of the siphon. (b) What is the limitation on the height of the top of the siphon above the water surface? (For the flow of liquid to be continuous, the pressure must not drop below the vapor pressure of the liquid.)

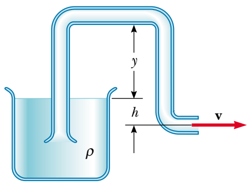


Figure P15.43

44. A hypodermic syringe contains a medicine with the density of water (Fig. P15.44). The barrel of the syringe has a cross-sectional area $A = 2.50 \times 10^{-5}$ m^2, and the needle has a cross-sectional area $a = 1.00 \times 10^{-8}$ m^2. In the absence of a force on the plunger, the pressure everywhere is 1 atm. A force **F** of magnitude 2.00 N acts on the plunger, making the medicine squirt horizontally from the needle. Determine the speed of the medicine as it leaves the needle's tip.

WEB **45.** A large storage tank is filled to a height h_0. The tank is punctured at a height h above the bottom of the tank (Fig. P15.45). Find an expression for how far from the tank the exiting stream lands.

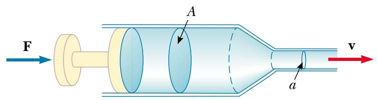


Figure P15.44

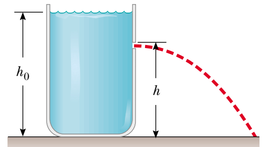


Figure P15.45 Problems 45 and 46.

46. A hole is punched at a height h in the side of a container of height h_0. The container is full of water, as shown in Figure P15.45. If the water is to shoot as far as possible horizontally, (a) how far from the bottom of the container should the hole be punched? (b) Neglecting frictional losses, how far (initially) from the side of the container will the water land?

ADDITIONAL PROBLEMS

47. A Ping–Pong ball has a diameter of 3.80 cm and an average density of 0.084 0 g/cm^3. What force would be required to hold it completely submerged under water?

48. Figure P15.48 shows a tank of water with a valve at the bottom. If this valve is opened, what is the maximum height attained by the water stream exiting the right side of the tank? Assume that $h = 10.0$ m, $L = 2.00$ m, and $\theta = 30.0°$, and that the cross-sectional area at point A is very large compared with that at point B.

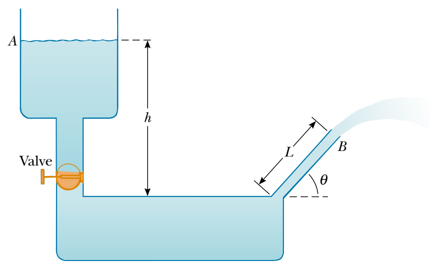


Figure P15.48

49. A helium-filled balloon is tied to a 2.00-m-long, 0.050 0-kg uniform string. The balloon is spherical with a radius of 0.400 m. When released, the balloon lifts a length h of string and then remains in equilibrium, as shown in Figure P15.49. Determine the value of h. The envelope of the balloon has a mass of 0.250 kg.

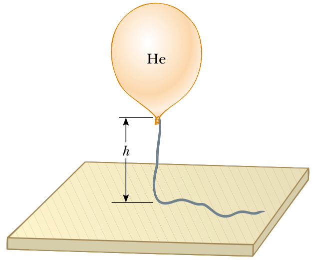


Figure P15.49

50. Water is forced out of a fire extinguisher by air pressure, as shown in Figure P15.50. How much gauge air pressure in the tank (above atmospheric) is required for the water jet to have a speed of 30.0 m/s when the water level is 0.500 m below the nozzle?

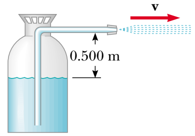


Figure P15.50

51. The true weight of an object is measured in a vacuum, where buoyant forces are absent. An object of volume V is weighed in air on a balance with the use of weights of density ρ. If the density of air is ρ_{air} and the balance reads F_g', show that the true weight F_g is

$$F_g = F_g' + \left(V - \frac{F_g'}{\rho g}\right)\rho_{air} g$$

52. Evangelista Torricelli was the first to realize that we live at the bottom of an ocean of air. He correctly surmised that the pressure of our atmosphere is attributable to the weight of the air. The density of air at 0°C at the Earth's surface is 1.29 kg/m^3. The density decreases with increasing altitude (as the atmosphere thins). On the other hand, if we assume that the density is constant

(1.29 kg/m^3) up to some altitude h, and zero above that altitude, then h would represent the thickness of our atmosphere. Use this model to determine the value of h that gives a pressure of 1.00 atm at the surface of the Earth. Would the peak of Mt. Everest rise above the surface of such an atmosphere?

53. A wooden dowel has a diameter of 1.20 cm. It floats in water with 0.400 cm of its diameter above water level (Fig. P15.53). Determine the density of the dowel.

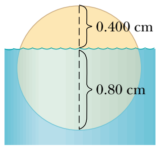


Figure P15.53

54. A light spring of constant $k = 90.0$ N/m rests vertically on a table (Fig. P15.54a). A 2.00-g balloon is filled with helium (density $= 0.180 \text{ kg/m}^3$) to a volume of 5.00 m^3 and is then connected to the spring, causing it to stretch as shown in Figure P15.54b. Determine the extension distance L when the balloon is in equilibrium.

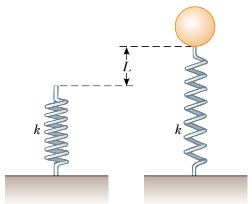


Figure P15.54

55. A 1.00-kg beaker containing 2.00 kg of oil (density $= 916.0 \text{ kg/m}^3$) rests on a scale. A 2.00-kg block of iron is suspended from a spring scale and completely submerged in the oil, as shown in Figure P15.55. Determine the equilibrium readings of both scales.

56. A beaker of mass m_b containing oil of mass m_0 (density $= \rho_0$) rests on a scale. A block of iron of mass m_{Fe} is suspended from a spring scale and completely submerged in the oil, as shown in Figure P15.55. Determine the equilibrium readings of both scales.

WEB **57.** **Review Problem.** With reference to Figure 15.7, show that the total torque exerted by the water behind the dam about an axis through O is $\frac{1}{6}\rho g w H^3$. Show that the effective line of action of the total force exerted by the water is at a distance $\frac{1}{3}H$ above O.

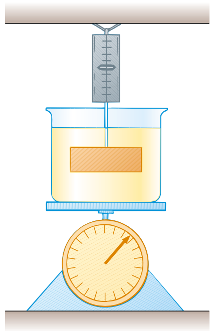


Figure P15.55 Problems 55 and 56.

58. In about 1657 Otto von Guericke, inventor of the air pump, evacuated a sphere made of two brass hemispheres. Two teams of eight horses each could pull the hemispheres apart only on some trials, and then "with greatest difficulty," with the resulting sound likened to a cannon firing (Fig. P15.58). (a) Show that the force F required to pull the evacuated hemispheres apart is $\pi R^2(P_0 - P)$, where R is the radius of the hemispheres and P is the pressure inside the hemispheres, which is much less than P_0. (b) Determine the force if $P = 0.100P_0$ and $R = 0.300$ m.

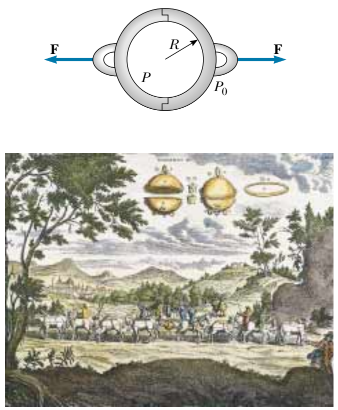


Figure P15.58 The colored engraving, dated 1672, illustrates Otto von Guericke's demonstration of the force due to air pressure as performed before Emperor Ferdinand III in 1657. (

59. In 1983 the United States began coining the cent piece out of copper-clad zinc rather than pure copper. The mass of the old copper cent is 3.083 g, whereas that of the new cent is 2.517 g. Calculate the percent of zinc (by volume) in the new cent. The density of copper is 8.960 g/cm^3, and that of zinc is 7.133 g/cm^3. The new and old coins have the same volume.

60. A thin spherical shell with a mass of 4.00 kg and a diameter of 0.200 m is filled with helium (density $= 0.180 \text{ kg/m}^3$). It is then released from rest on the bottom of a pool of water that is 4.00 m deep. (a) Neglecting frictional effects, show that the shell rises with constant acceleration and determine the value of that acceleration. (b) How long does it take for the top of the shell to reach the water's surface?

61. An incompressible, nonviscous fluid initially rests in the vertical portion of the pipe shown in Figure P15.61a, where $L = 2.00$ m. When the valve is opened, the fluid flows into the horizontal section of the pipe. What is the speed of the fluid when all of it is in the horizontal section, as in Figure P15.61b? Assume that the cross-sectional area of the entire pipe is constant.

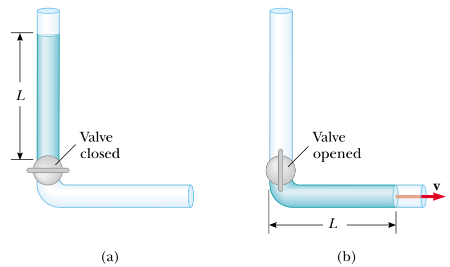


Figure P15.61

62. **Review Problem.** A uniform disk with a mass of 10.0 kg and a radius of 0.250 m spins at 300 rev/min on a low-friction axle. It must be brought to a stop in 1.00 min by a brake pad that makes contact with the disk at an average distance of 0.220 m from the axis. The coefficient of friction between the pad and the disk is 0.500. A piston in a cylinder with a diameter of 5.00 cm presses the brake pad against the disk. Find the pressure that the brake fluid in the cylinder must have.

63. Figure P15.63 shows Superman attempting to drink water through a very long straw. With his great strength, he achieves maximum possible suction. The walls of the tubular straw do not collapse. (a) Find the maximum height through which he can lift the water. (b) Still thirsty, the Man of Steel repeats his attempt on the Moon, which has no atmosphere. Find the difference between the water levels inside and outside the straw.

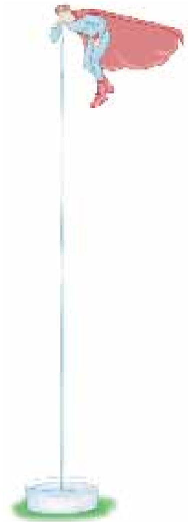


Figure P15.63

64. Show that the variation of atmospheric pressure with altitude is given by $P = P_0 e^{-\alpha h}$, where $\alpha = \rho_0 g/P_0$, P_0 is atmospheric pressure at some reference level $y = 0$, and ρ_0 is the atmospheric density at this level. Assume that the decrease in atmospheric pressure with increasing altitude is given by Equation 15.4, so that $dP/dy = -\rho g$, and assume that the density of air is proportional to the pressure.

65. A cube of ice whose edge measures 20.0 mm is floating in a glass of ice-cold water with one of its faces parallel to the water's surface. (a) How far below the water surface is the bottom face of the block? (b) Ice-cold ethyl alcohol is gently poured onto the water's surface to form a layer 5.00 mm thick above the water. The alcohol does not mix with the water. When the ice cube again attains hydrostatic equilibrium, what is the distance from the top of the water to the bottom face of the block? (c) Additional cold ethyl alcohol is poured onto the water's surface until the top surface of the alcohol coincides with the top surface of the ice cube (in hydrostatic equilibrium). How thick is the required layer of ethyl alcohol?

66. **Review Problem.** A light balloon filled with helium with a density of 0.180 kg/m^3 is tied to a light string of length $L = 3.00$ m. The string is tied to the ground, forming an "inverted" simple pendulum, as shown in Figure P15.66a. If the balloon is displaced slightly from its equilibrium position as shown in Figure P15.66b, (a) show that the ensuing motion is simple harmonic and (b) determine the period of the motion. Take the density of air to be 1.29 kg/m^3 and ignore any energy loss due to air friction.

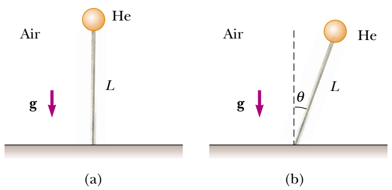


Figure P15.66

67. The water supply of a building is fed through a main 6.00-cm-diameter pipe. A 2.00-cm-diameter faucet tap located 2.00 m above the main pipe is observed to fill a 25.0-L container in 30.0 s. (a) What is the speed at which the water leaves the faucet? (b) What is the gauge pressure in the 6-cm main pipe? (Assume that the faucet is the only "leak" in the building.)

68. The *spirit-in-glass thermometer,* invented in Florence, Italy, around 1654, consists of a tube of liquid (the spirit) containing a number of submerged glass spheres with slightly different masses (Fig. P15.68). At sufficiently low temperatures, all the spheres float, but as the temperature rises, the spheres sink one after the other. The device is a crude but interesting tool for measuring temperature. Suppose that the tube is filled with ethyl alcohol, whose density is $0.789\ 45 \text{ g/cm}^3$ at 20.0°C and decreases to $0.780\ 97 \text{ g/cm}^3$ at 30.0°C. (a) If one of the spheres has a radius of 1.000 cm and is in equilibrium halfway up the tube at 20.0°C, determine its mass. (b) When the temperature increases to 30.0°C, what mass must a second sphere of the same radius have to be in equilibrium at the halfway point? (c) At 30.0°C the first sphere has fallen to the bottom of the tube. What upward force does the bottom of the tube exert on this sphere?

69. A U-tube open at both ends is partially filled with water (Fig. P15.69a). Oil having a density of 750 kg/m^3 is then poured into the right arm and forms a column $L = 5.00$ cm in height (Fig. P15.69b). (a) Determine

Figure P15.68

the difference h in the heights of the two liquid surfaces. (b) The right arm is shielded from any air motion while air is blown across the top of the left arm until the surfaces of the two liquids are at the same height (Fig. P15.69c). Determine the speed of the air being blown across the left arm. (Take the density of air as 1.29 kg/m^3.)

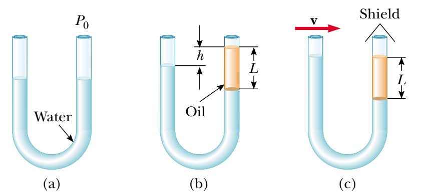


Figure P15.69

Answers to Quick Quizzes

15.1 You would be better off with the basketball player. Although weight is distributed over the larger surface area, equal to about half of the total surface area of the sneaker sole, the pressure (F/A) that he applies is relatively small. The woman's lesser weight is distributed over the very small cross-sectional area of the spiked heel. Some museums make women in high-heeled shoes wear slippers or special heel attachments so that they do not damage the wood floors.

15.2 If the professor were to try to support his entire weight on a single nail, the pressure exerted on his skin would be his entire weight divided by the very small surface area of the nail point. This extremely great pressure would cause the nail to puncture his skin. However, if the professor distributes his weight over several hundred nails, as shown in the photograph, the pressure exerted on his skin is considerably reduced because the surface area that supports his weight is now the total surface area of all the nail points. (Lying on the bed of nails is much more comfortable than sitting on the bed, and standing on the bed without shoes is definitely not recommended. Do not lie on a bed of nails unless you have been shown how to do so safely.)

15.3 Because the horizontal force exerted by the outside fluid on an element of the cylinder is equal and opposite the horizontal force exerted by the fluid on another element diametrically opposite the first, the net force on the cylinder in the horizontal direction is zero.

15.4 If you think of the grain stored in the silo as a fluid, then the pressure it exerts on the walls increases with increasing depth. The spacing between bands is smaller at the lower portions so that the greater outward forces acting on the walls can be overcome. The silo on the right shows another way of accomplishing the same thing: double banding at the bottom.

15.5 Because water is so much less dense than mercury, the column for a water barometer would have to be $h = P_0/\rho g = 10.3$ m high, and such a column is inconveniently tall.

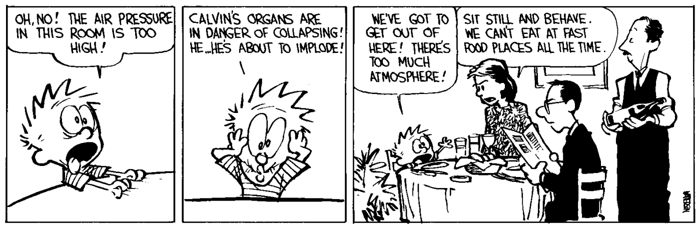
15.6 The entire hull of a ship is full of air, and the density of air is about one-thousandth the density of water. Hence, the total weight of the ship equals the weight of the volume of water that is displaced by the portion of the ship that is below sea level.

15.7 Remains the same. In effect, the ice creates a "hole" in the water, and the weight of the water displaced from the hole is the same as all the weight of the cube. When the cube changes from ice to water, the water just fills the hole.

15.8 Goes down because the anchor displaces more water while in the boat than it does in the pond. While it is in the boat, the anchor can be thought of as a floating object that displaces a volume of water weighing as much as it does. When the anchor is thrown overboard, it sinks and displaces a volume of water equal to its own volume. Because the density of the anchor is greater than that of water, the volume of water that weighs the same as the anchor is greater than the volume of the anchor.

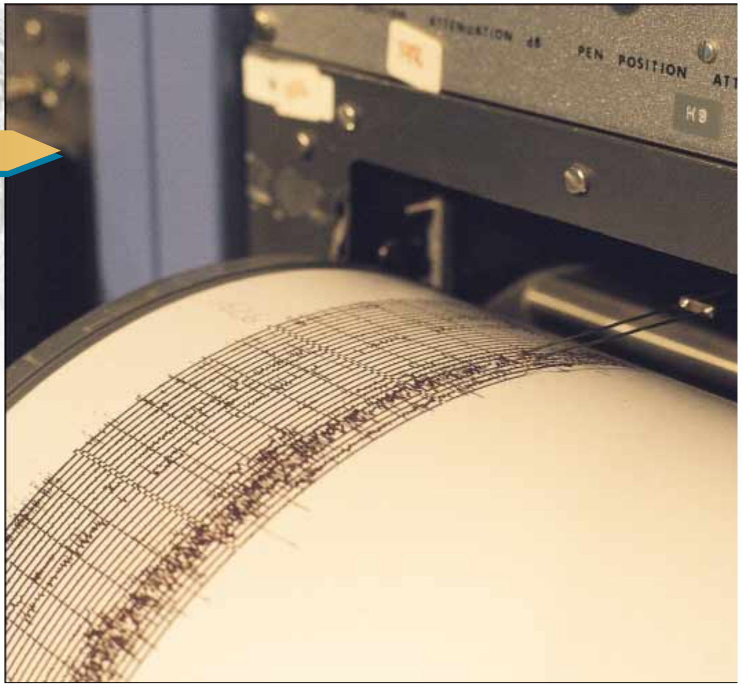
15.9 As the water falls, its speed increases. Because the flow rate Av must remain constant at all cross sections (see Eq. 15.7), the stream must become narrower as the speed increases.

15.10 The rapidly moving air characteristic of a tornado is at a pressure below atmospheric pressure. The stationary air inside the building remains at atmospheric pressure. The pressure difference results in an outward force on the roof and walls, and this force can be great enough to lift the roof off the building. Opening the windows helps to equalize the inside and outside pressures.



PUZZLER

A simple seismograph can be constructed with a spring-suspended pen that draws a line on a slowly unrolling strip of paper. The paper is mounted on a structure attached to the ground. During an earthquake, the pen remains nearly stationary while the paper shakes beneath it. How can a few jagged lines on a piece of paper allow scientists at a seismograph station to determine the distance to the origin of an earthquake? *(Ken M. Johns/Photo Researchers, Inc.)*



chapter

16 Wave Motion

Chapter Outline

Most of us experienced waves as children when we dropped a pebble into a pond. At the point where the pebble hits the water's surface, waves are created. These waves move outward from the creation point in expanding circles until they reach the shore. If you were to examine carefully the motion of a leaf floating on the disturbed water, you would see that the leaf moves up, down, and sideways about its original position but does not undergo any net displacement away from or toward the point where the pebble hit the water. The water molecules just beneath the leaf, as well as all the other water molecules on the pond's surface, behave in the same way. That is, the water *wave* moves from the point of origin to the shore, but the water is not carried with it.

An excerpt from a book by Einstein and Infeld gives the following remarks concerning wave phenomena:[1]

> A bit of gossip starting in Washington reaches New York [by word of mouth] very quickly, even though not a single individual who takes part in spreading it travels between these two cities. There are two quite different motions involved, that of the rumor, Washington to New York, and that of the persons who spread the rumor. The wind, passing over a field of grain, sets up a wave which spreads out across the whole field. Here again we must distinguish between the motion of the wave and the motion of the separate plants, which undergo only small oscillations... The particles constituting the medium perform only small vibrations, but the whole motion is that of a progressive wave. The essentially new thing here is that for the first time we consider the motion of something which is not matter, but energy propagated through matter.

The world is full of waves, the two main types being *mechanical* waves and *electromagnetic* waves. We have already mentioned examples of mechanical waves: sound waves, water waves, and "grain waves." In each case, some physical medium is being disturbed—in our three particular examples, air molecules, water molecules, and stalks of grain. Electromagnetic waves do not require a medium to propagate; some examples of electromagnetic waves are visible light, radio waves, television signals, and x-rays. Here, in Part 2 of this book, we study only mechanical waves.

The wave concept is abstract. When we observe what we call a water wave, what we see is a rearrangement of the water's surface. Without the water, there would be no wave. A wave traveling on a string would not exist without the string. Sound waves could not travel through air if there were no air molecules. With mechanical waves, what we interpret as a wave corresponds to the propagation of a disturbance through a medium.

Interference patterns produced by outward-spreading waves from many drops of liquid falling into a body of water.

[1] A. Einstein and L. Infeld, *The Evolution of Physics,* New York, Simon & Schuster, 1961. Excerpt from "What Is a Wave?"

The mechanical waves discussed in this chapter require (1) some source of disturbance, (2) a medium that can be disturbed, and (3) some physical connection through which adjacent portions of the medium can influence each other. We shall find that all waves carry energy. The amount of energy transmitted through a medium and the mechanism responsible for that transport of energy differ from case to case. For instance, the power of ocean waves during a storm is much greater than the power of sound waves generated by a single human voice.

16.1 BASIC VARIABLES OF WAVE MOTION

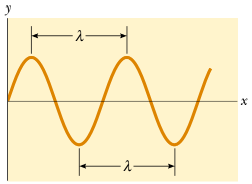


Figure 16.1 The wavelength λ of a wave is the distance between adjacent crests, adjacent troughs, or any other comparable adjacent identical points.

Imagine you are floating on a raft in a large lake. You slowly bob up and down as waves move past you. As you look out over the lake, you may be able to see the individual waves approaching. The point at which the displacement of the water from its normal level is highest is called the **crest** of the wave. The distance from one crest to the next is called the **wavelength** λ (Greek letter lambda). More generally, the wavelength is **the minimum distance between any two identical points (such as the crests) on adjacent waves,** as shown in Figure 16.1.

If you count the number of seconds between the arrivals of two adjacent waves, you are measuring the **period** T of the waves. In general, the period is **the time required for two identical points (such as the crests) of adjacent waves to pass by a point.**

The same information is more often given by the inverse of the period, which is called the **frequency** f. In general, the frequency of a periodic wave is **the number of crests (or troughs, or any other point on the wave) that pass a given point in a unit time interval.** The maximum displacement of a particle of the medium is called the **amplitude** A of the wave. For our water wave, this represents the highest distance of a water molecule above the undisturbed surface of the water as the wave passes by.

Waves travel with a specific speed, and this speed depends on the properties of the medium being disturbed. For instance, sound waves travel through room-temperature air with a speed of about 343 m/s (781 mi/h), whereas they travel through most solids with a speed greater than 343 m/s.

16.2 DIRECTION OF PARTICLE DISPLACEMENT

One way to demonstrate wave motion is to flick one end of a long rope that is under tension and has its opposite end fixed, as shown in Figure 16.2. In this manner, a single wave bump (called a *wave pulse*) is formed and travels along the rope with a definite speed. This type of disturbance is called a **traveling wave,** and Figure 16.2 represents four consecutive "snapshots" of the creation and propagation of the traveling wave. The rope is the medium through which the wave travels. Such a single pulse, in contrast to a train of pulses, has no frequency, no period, and no wavelength. However, the pulse does have definite amplitude and definite speed. As we shall see later, the properties of this particular medium that determine the speed of the wave are the tension in the rope and its mass per unit length. The shape of the wave pulse changes very little as it travels along the rope.[2]

As the wave pulse travels, each small segment of the rope, as it is disturbed, moves in a direction perpendicular to the wave motion. Figure 16.3 illustrates this point for one particular segment, labeled P. Note that no part of the rope ever moves in the direction of the wave.

[2] Strictly speaking, the pulse changes shape and gradually spreads out during the motion. This effect is called *dispersion* and is common to many mechanical waves, as well as to electromagnetic waves. We do not consider dispersion in this chapter.

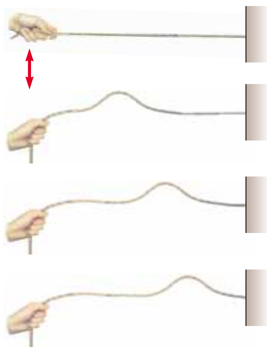


Figure 16.2 A wave pulse traveling down a stretched rope. The shape of the pulse is approximately unchanged as it travels along the rope.

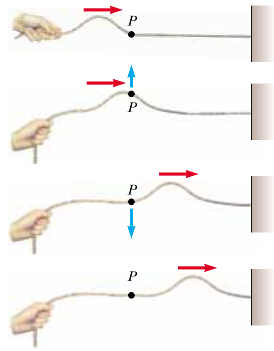


Figure 16.3 A pulse traveling on a stretched rope is a transverse wave. The direction of motion of any element P of the rope (blue arrows) is perpendicular to the direction of wave motion (red arrows).

Transverse wave

A traveling wave that causes the particles of the disturbed medium to move perpendicular to the wave motion is called a **transverse wave.**

Compare this with another type of wave—one moving down a long, stretched spring, as shown in Figure 16.4. The left end of the spring is pushed briefly to the right and then pulled briefly to the left. This movement creates a sudden compression of a region of the coils. The compressed region travels along the spring (to the right in Figure 16.4). The compressed region is followed by a region where the coils are extended. Notice that the direction of the displacement of the coils is *parallel* to the direction of propagation of the compressed region.

Longitudinal wave

A traveling wave that causes the particles of the medium to move parallel to the direction of wave motion is called a **longitudinal wave.**

Sound waves, which we shall discuss in Chapter 17, are another example of longitudinal waves. The disturbance in a sound wave is a series of high-pressure and low-pressure regions that travel through air or any other material medium.

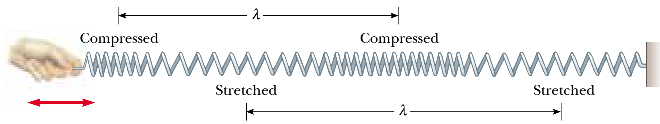


Figure 16.4 A longitudinal wave along a stretched spring. The displacement of the coils is in the direction of the wave motion. Each compressed region is followed by a stretched region.

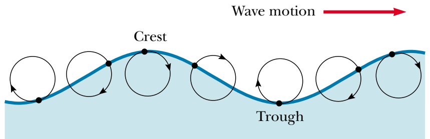


Figure 16.5 The motion of water molecules on the surface of deep water in which a wave is propagating is a combination of transverse and longitudinal displacements, with the result that molecules at the surface move in nearly circular paths. Each molecule is displaced both horizontally and vertically from its equilibrium position.

QuickLab

Make a "telephone" by poking a small hole in the bottom of two paper cups, threading a string through the holes, and tying knots in the ends of the string. If you speak into one cup while pulling the string taut, a friend can hear your voice in the other cup. What kind of wave is present in the string?

Some waves in nature exhibit a combination of transverse and longitudinal displacements. Surface water waves are a good example. When a water wave travels on the surface of deep water, water molecules at the surface move in nearly circular paths, as shown in Figure 16.5. Note that the disturbance has both transverse and longitudinal components. The transverse displacement is seen in Figure 16.5 as the variations in vertical position of the water molecules. The longitudinal displacement can be explained as follows: As the wave passes over the water's surface, water molecules at the crests move in the direction of propagation of the wave, whereas molecules at the troughs move in the direction opposite the propagation. Because the molecule at the labeled crest in Figure 16.5 will be at a trough after half a period, its movement in the direction of the propagation of the wave will be canceled by its movement in the opposite direction. This holds for every other water molecule disturbed by the wave. Thus, there is no net displacement of any water molecule during one complete cycle. Although the *molecules* experience no net displacement, the *wave* propagates along the surface of the water.

The three-dimensional waves that travel out from the point under the Earth's surface at which an earthquake occurs are of both types—transverse and longitudinal. The longitudinal waves are the faster of the two, traveling at speeds in the range of 7 to 8 km/s near the surface. These are called **P waves,** with "P" standing for *primary* because they travel faster than the transverse waves and arrive at a seismograph first. The slower transverse waves, called **S waves** (with "S" standing for *secondary*), travel through the Earth at 4 to 5 km/s near the surface. By recording the time interval between the arrival of these two sets of waves at a seismograph, the distance from the seismograph to the point of origin of the waves can be determined. A single such measurement establishes an imaginary sphere centered on the seismograph, with the radius of the sphere determined by the difference in arrival times of the P and S waves. The origin of the waves is located somewhere on that sphere. The imaginary spheres from three or more monitoring stations located far apart from each other intersect at one region of the Earth, and this region is where the earthquake occurred.

Quick Quiz 16.1

(a) In a long line of people waiting to buy tickets, the first person leaves and a pulse of motion occurs as people step forward to fill the gap. As each person steps forward, the gap moves through the line. Is the propagation of this gap transverse or longitudinal? (b) Consider the "wave" at a baseball game: people stand up and shout as the wave arrives at their location, and the resultant pulse moves around the stadium. Is this wave transverse or longitudinal?

16.3 ONE-DIMENSIONAL TRAVELING WAVES

Consider a wave pulse traveling to the right with constant speed v on a long, taut string, as shown in Figure 16.6. The pulse moves along the x axis (the axis of the string), and the transverse (vertical) displacement of the string (the medium) is measured along the y axis. Figure 16.6a represents the shape and position of the pulse at time $t = 0$. At this time, the shape of the pulse, whatever it may be, can be represented as $y = f(x)$. That is, y, which is the vertical position of any point on the string, is some definite function of x. The displacement y, sometimes called the *wave function,* depends on both x and t. For this reason, it is often written $y(x, t)$, which is read "y as a function of x and t." Consider a particular point P on the string, identified by a specific value of its x coordinate. Before the pulse arrives at P, the y coordinate of this point is zero. As the wave passes P, the y coordinate of this point increases, reaches a maximum, and then decreases to zero. Therefore, **the wave function y represents the y coordinate of any point P of the medium at any time t.**

Because its speed is v, the wave pulse travels to the right a distance vt in a time t (see Fig. 16.6b). If the shape of the pulse does not change with time, we can represent the wave function y for all times after $t = 0$. Measured in a stationary reference frame having its origin at O, the wave function is

$$y = f(x - vt) \qquad \textbf{(16.1)}$$

Wave traveling to the right

If the wave pulse travels to the left, the string displacement is

$$y = f(x + vt) \qquad \textbf{(16.2)}$$

Wave traveling to the left

For any given time t, the wave function y as a function of x defines a curve representing the shape of the pulse at this time. This curve is equivalent to a "snapshot" of the wave at this time. For a pulse that moves without changing shape, the speed of the pulse is the same as that of any feature along the pulse, such as the crest shown in Figure 16.6. To find the speed of the pulse, we can calculate how far the crest moves in a short time and then divide this distance by the time interval. To follow the motion of the crest, we must substitute some particular value, say x_0, in Equation 16.1 for $x - vt$. Regardless of how x and t change individually, we must require that $x - vt = x_0$ in order to stay with the crest. This expression therefore represents the equation of motion of the crest. At $t = 0$, the crest is at $x = x_0$; at a

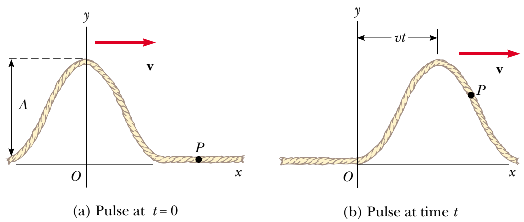


Figure 16.6 A one-dimensional wave pulse traveling to the right with a speed v. (a) At $t = 0$, the shape of the pulse is given by $y = f(x)$. (b) At some later time t, the shape remains unchanged and the vertical displacement of any point P of the medium is given by $y = f(x - vt)$.